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Prime Bi-Ternary Γ-Ideals in Ternary Γ-Semirings

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ABSTRACT

This paper is divided into four sections, in section 1, introduction of the paper. In section 2, we provided some preliminaries which are useful for further development of the paper. In section 3, we were introduced bi-ternary Γ-ideals, strongly bi-ternary Γ-ideals and semiprime bi-ternary Γ-ideals in ternary Γ-semirings. It is proved that (1) Every strongly prime bi-ternary Γ-ideal of a ternary Γ-semiring T is a prime bi-ternary Γ-ideal of T. (2) Every prime bi-ternary Γ-ideal of a ternary Γ-semiring T is a semiprime bi-ternary Γ-ideal of T. In section 4, the terms irreducible and strongly irreducible bi-ternary Γ-ideals in ternary Γ-semiring. It is proved that if B be a bi-ternary Γ-ideal of a ternary Γ-semiring T and $a \in T$ such that $a \notin B$. Then there exists an irreducible bi-ternary Γ-ideal I of T such that $B \subseteq I$ and $a \notin I$. Further it is proved that in a ternary **Γ**-semiring T, the following assertions are equivalent: (1) Every bi-ternary **Γ**-ideal of T is idempotent. (2) $B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2 = B_1 \cap B_2 \cap B_3$ for any bi-ternary **Γ**-ideals B_1, B_2, B_3 of T. (3) Each bi-ternary **Γ**-ideal of T is semiprime. (4) Each proper bi-ternary **Γ**-ideal of T is the intersection of irreducible semiprime bi-ternary **Γ**-ideals of T which contain it. Further it is also proved that in a ternary **Γ**-semiringT, the following assertions are equivalent: (1) The set of bi-ternary **Γ**-ideals of T is totally ordered by set inclusion. (2) Each bi-ternary **Γ**-ideal of T is strongly irreducible. (3) Each bi-ternary **Γ**-ideal of T is irreducible. It is proved that (1) Each bi-ternary **Γ**-ideal of a ternary **Γ**-semiring T is strongly prime if and only if each bi-ternary **Γ**-ideal of T is **Γ**-idempotent and the set of bi-ternary **Γ**-ideals of T is totally ordered by inclusion and (2) If the set of bi-ternary Γ-ideals of a ternary Γ-semiring T is totally ordered, then the concepts of primeness and strongly primeness coincide.

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INTRODUCTION

Algebraic structures play a very outstanding role in mathematics with classification in multifarious disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological areas etc. Ternary generalizations of algebraic structures are the very natural ways for further development and in depth comprehension of their basic traits.

Cayley for the first time pioneered and launched first ternary algebraic operations in the way back in the 19th century. Cayley's ideas expounded and developed nary generalizations of matrices and their determinants[9, 14] and general theory of n-ary algebras[3, 10] and ternary rings[11] (For physical apllications in Nambu mechanics, super symmetry, Yang-Baxter equations etc). Ternary structures and their generalizations create some hopes because of their possibility of applications in physics. A few important physical applications are listed in[1,2,6,7]. In pursuance of Lister's generalizations of ternary rings introduced in 1971, T. K. Dutta and S. Kar came up with the notion of ternary semirings.

T. K. Dutta and S. Kar initiated prime ideals and prime radical of ternary semirings in[4]. The same researchers launched semiprime ideals and irreducible ideals of ternary semirings in[5]. Furthermore S. Kar in[8] came up with the notion of quasi-ideals and bi-ideals in ternary semi rings. Similarly, M. Shabir and M. Bano floated prime bi-ideals in ternary semi groups in[13]. In the opening section of this paper, we assemble requisite material on prime, strongly prime and semi prime bi-ternary Γ-ideals in ternary Γ-semi rings. Deploying the taxonomic order, we define irreducible and strongly irreducible bi-ternary Γ-ideals in ternary Γ-semi rings and specify some classes of ternary Γ-semi rings by the characteristics of these ternary Γ-ideals.

PRELIMINARIES

Definition 1

Let T and Γ be two additive commutative semi groups. T is said to be a ***Ternary Γ-semiring*** if there exist a mapping from T ×Γ× T ×Γ× T to T which maps $(x_1, \alpha, x_2, \beta, x_3) \rightarrow [x_1\alpha x_2\beta x_3]$ satisfying the conditions :

i) $[[a\alpha b\beta c]\gamma d\delta e] = [a\alpha[b\beta c\gamma d]\delta e] = [a\alpha b\beta[c\gamma d\delta e]]$

ii) $[(a + b)\alpha c\beta d] = [a\alpha c\beta d] + [b\alpha c\beta d]$

iii) $[a\alpha(b + c)\beta d] = [a\alpha b\beta d] + [a\alpha c\beta d]$

iv) $[a\alpha b\beta(c + d)] = [a\alpha b\beta c] + [a\alpha b\beta d]$ for all $a, b, c, d \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Definition 2

An element 0of a ternary Γ-semiring T is said to be an ***absorbing zero*** of T provided $0 + x = x = x + 0$ and $0\alpha a\beta b = a\alpha 0\beta b = a\alpha b\beta 0 = 0 \; \forall a, b, x \in$ T and $\alpha, \beta \in \Gamma$.

Note 3

Throughout this paper, T will always denote a ternary Γ-semiring with zero and unless otherwise stated a ternary Γ-semiring means a ternary Γ-semiring with zero.

Definition 4

A ternary Γ-semiring T is said to be ***commutative ternary Γ-semiring*** provided $a\Gamma b\Gamma c = b\Gamma c\Gamma a = c\Gamma a\Gamma b = b\Gamma a\Gamma c = c\Gamma b\Gamma a = a\Gamma c\Gamma b$ for all $a, b, c \in$ T.

Note 5

For the convenience we write $x_1\alpha x_2\beta x_3$ instead of $[x_1\alpha x_2\beta x_3]$

Note 6

Let T be a ternary semiring. If A, B and C are three subsets of T, we shall denote the set AΓBΓC = $\{\Sigma a\alpha b\beta c : a \in A, b \in B, c \in C, \alpha, \beta \in \Gamma\}$.

Note 7

Let T be a ternary semiring. If A, B are two subsets of T, we shall denote the set A + B = $\{a+b : a \in A, b \in B\}$ and 2A = { $a + a$: $a \in$ A}.

Definition 8

Let T be ternary Γ-semiring. A non empty subset 'S' is said to be a ***ternary sub*Γ*-semiring*** of T if S is an additive subsemigroup of T and $a\alpha b\beta c \in$ S for all $a,b,c \in$ S and $\alpha, \beta \in \Gamma$.

Note 9

A non empty subset S of a ternary Γ-semiring T is a ternary sub Γ-semiring if and only if S + S ⊆ S and SΓSΓS ⊆ S.

Definition 10

A nonempty subset A of a ternary Γ-semiring T is said to be ***left ternary* Γ*-ideal*** of T if (1) $a, b \in$ A implies $a + b \in$ A. (2) $b, c \in$ T, $a \in$ A, $\alpha, \beta \in \Gamma$ implies $b\alpha c\beta a \in$ A.

Note 11

A nonempty subset A of a ternary Γ-semiring T is a left ternary Γ-ideal of T if and only if A is additive subsemi group of T and TΓTΓA ⊆ A.

Definition 12

A nonempty subset of a ternary Γ-semiring T is said to be a ***lateral ternary* Γ*-ideal*** of T if (1) $a, b \in$ A ⇒ $a + b \in$ A. (2) $b, c \in$ T, $a \in$ A, $\alpha, \beta \in \Gamma \Rightarrow b\alpha a\beta c \in$ A.

Note 13

A nonempty subset of A of a ternary Γ-semiring T is a lateral ternary Γ-ideal of T if and only if A is additive subsemi group of T and TΓAΓT ⊆ A.

Definition 14

A nonempty subset A of a ternary Γ-semiring T is a ***right ternary* Γ*-ideal*** of T if (1) $a, b \in$ A ⇒ $a + b \in$ A. (2) $b, c \in$ T , $a \in$ A, $\alpha, \beta \in \Gamma \Rightarrow a\alpha b\beta c \in$ A.

Note 15

A nonempty subset A of a ternary Γ-semiring T is a right ternary Γ-ideal of T if and only if A is additive subsemi group of T and AΓTΓT ⊆ A.

Definition 16

A nonempty subset A of a ternary Γ-semiring T is said to be ***ternary* Γ*-ideal*** of T if

(1) $a, b \in$ A ⇒ $a + b \in$ A

(2) $b, c \in$ T , $a \in$ A, $\alpha, \beta \in \Gamma \Rightarrow b\alpha c\beta a \in$ A, $b\alpha a\beta c \in$ A, $a\alpha b\beta c \in$ A.

Note 17

A nonempty subset A of a ternary Γ-semiring T is a ternary Γ-ideal of T if and only if it is left ternary Γ-ideal, lateral ternary Γ-ideal and right ternary Γ-ideal of T.

Definition 18

An element a of a ternary Γ-semiring. T is said to be ***regular*** if there exist $x, y \in$ T such that $a = a\alpha x\beta a\gamma y\delta a$ for all $\alpha, \beta, \gamma, \delta \in \Gamma$.

Definition 19

An element a of a ternary Γ-semiring T is said to be an ***α-idempotent*** element provided $a = a\alpha a\alpha a$.

Definition 20

A ternary Γ-semiring T is called α-idempotent ternary Γ-semiring if every element of T is α-idempotent.

Definition 21

An element *a* of a ternary Γ-semiring T is said to be an ***(α, β)-idempotent*** element provided $a = a\alpha a\beta a$ for all $\alpha, \beta \in \Gamma$.

Definition 22

A ternary Γ-subsemiring B of a ternary Γ-semiring T is called a ***bi-ternary Γ-ideal*** of T if BΓTΓBΓTB⊆B.

Theorem 22

The intersection of any family of bi-ternary Γ-ideals of a ternary Γ-semiring T is a bi-ternary Γ-ideal of T.

Proof

Let$\{B_i : i \in I\}$be any family of prime bi-ternary Γ-ideals of a ternary Γ-semiring T. Let $x, y \in \bigcap_{i\in I} B_i$. Then $x, y \in B_i$ for all $i \in I$.As each B_i is a bi-ternary Γ-ideal of T, we have $x + y \in B_i$ for all $i \in I$ implies $x + y \in \bigcap_{i\in I} B_i$. Now let $x, y, z \in \bigcap_{i\in I} B_i$. Then $x, y, z \in B_i$ for all $i \in I$.As each B_i is a bi-ternary Γ-ideal of T, we have $x\alpha y\beta z \in B_i$ for all $i \in I$ implies $x\alpha y\beta z \in \bigcap_{i\in I} B_i$. Now as $B_i\Gamma T\Gamma B_i\Gamma T\Gamma B_i \subseteq B_i$ and $\bigcap_{i\in I} B_i \subseteq B_i$ for all $i \in I$, we have

$\bigcap_{i\in I} B_i \Gamma T\Gamma \bigcap_{i\in I} B_i \Gamma T\Gamma \bigcap_{i\in I} B_i \subseteq B_i\Gamma T\Gamma B_i\Gamma T\Gamma B_i \subseteq B_i$for all $i \in I$.

This implies $\bigcap_{i\in I} B_i \Gamma T\Gamma \bigcap_{i\in I} B_i \Gamma T\Gamma \bigcap_{i\in I} B_i \subseteq \bigcap_{i\in I} B_i$. Thus $\bigcap_{i\in I} B_i$ is a bi-ternary Γ-ideal of T.

Theorem 23

Every left (right, lateral) ternary Γ-ideal of a ternary Γ-semiring A is a bi-ternary Γ-ideal of A.

Theorem 24

If B is a bi-ternary Γ-ideal of a regular ternary Γ-semiring T and X, Y, be any non-empty subsets of T, then BΓXΓY, XΓBΓY and XΓYΓB are bi-ternary Γ-ideals of T.

Corollary 25

If B_1,B_2,B_3 are any bi-ternary Γ-ideals of a regular ternary Γ-semiring T, then $B_1\Gamma B_2\Gamma B_3$ is a bi-ternary Γ-ideal of T.

PRIME, STRONGLY PRIME AND SEMI PRIME BI-TERNARY Γ-IDEAL OF TERNARY Γ-SEMIRING

Definition 1

A bi-ternary Γ-ideal B of a ternary Γ-semiring T is said to be a ***prime bi-ternary Γ-ideal*** if $B_1\Gamma B_2\Gamma B_3 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ternary Γ-ideals B_1,B_2,B_3 of T.

Definition 2

A bi-ternary Γ-ideal B of a ternary Γ-semiring T is said to be a ***strongly prime bi-ternary Γ-ideal*** of T if $B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ternary Γ-ideals B_1,B_2,B_3 of T.

Definition 3

A bi-ternary Γ-ideal B of a ternary Γ-semiring T is said to be a ***semi prime bi-ternary Γ-ideal*** of T if$(B_1\Gamma)^2 B_1 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ternary Γ-ideal B_1 of T.

Proposition 4

Every strongly prime bi-ternary Γ-ideal of a ternary Γ-semiring T is a prime bi-ternary Γ-ideal of T.

Proof

Let B be a strongly prime bi-ternary Γ-ideal of a ternary Γ-semiring A. Let B_1,B_2,B_3 be three bi-ternary Γ-ideals of T such that $B_1\Gamma B_2\Gamma B_3 \subseteq B$. This implies $B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2 \subseteq B$. Thus $B_1\subseteq B$ or $B_2\subseteq B$ or $B_3\subseteq B$. Hence B is a prime bi-ternary Γ-ideal of T.

Theorem 5

Every prime bi-ternary Γ-ideal of a ternary Γ-semiring T is a semi prime bi-ternary Γ-ideal of T.

Proof

Let B be a prime bi-ternary Γ-ideal of a ternary Γ-semiring A. Now let B_1 be any bi-ternary Γ-ideal of T such that $(B_1\Gamma)^2 B_1 \subseteq B$. Then $B_1\subseteq B$.Thus B is a semi prime bi-ternary Γ-ideal of A.

Note 6

A prime bi-ternary Γ-ideal of a ternary Γ-semiring is not necessarily a strongly prime bi-ternary Γ-ideal and a semi prime bi-ternary Γ-ideal of a ternary Γ-semiring is not necessarily a prime bi-ternary Γ-ideal. This fact is clear from the following examples:

Example 7

Let A $=\{\Phi,\{a\},\{b\},\{a, b\}\}$ and Γ= $\{,\{a\},\{b\},\{a, b\}\}$. Define addition and ternary multiplication on T as X+Y = XΔY = (X ∪Y)−(X ∩Y) and (XΓY)ΓZ = X∩Y∩Z for all X,Y,Z ∈ T. Then T isaternaryΓ-semiring. Bi-ideals of Tare $\{\Phi\}$, $\{\Phi,\{a\}\}$, $\{\Phi,\{b\}\}$ and $\{\Phi,\{a\},\{b\},\{a,b\}\}$. Since $(X\Gamma)^2X = X$ for all X ∈ T, so each bi-ternary Γ-ideal of T is semi prime. In particular $\{\Phi\}$ is a semi prime bi-ternary Γ-ideal of T but not a prime bi-ternary Γ-ideal of T, because $\{\Phi,\{a\}\}\Gamma\{\Phi,\{b\}\}\Gamma\{\Phi,\{a\},\{b\},\{a,b\}\} = \{\Phi,\{a\}\}\cap\{\Phi,\{b\}\}\cap\{\Phi,\{a\},\{b\},\{a, b\}\} = \{\Phi\}\subseteq\{\Phi\}$.

But none of $\{\Phi,\{a\}\}$, $\{\Phi,\{b\}\}$ and $\{\Phi,\{a\},\{b\},\{a,b\}\}$ is contained in $\{\Phi\}$.

Theorem 8

The intersection of any family of prime bi-ternary Γ-ideals of a ternary Γ-semiring T is a semi prime bi-ternary Γ-ideal of T.

Proof

Let$\{B_i : i\in I\}$be any family of prime bi-ternary Γ-ideals of a ternary Γ-semiring T. We have to show that $\bigcap_{i\in I} B_i$ is a semi prime bi-ternary Γ-ideal of T. By theorem 2.22, $\bigcap_{i\in I} B_i$ is a bi-ternary Γ-ideal of T. Now let B be any bi-ternary Γ-ideal of T such that $(B\Gamma)^2B\subseteq\bigcap_{i\in I} B_i$, implies $B\Gamma B\Gamma B = (B\Gamma)^2B\subseteq B_i$ for all $i\in I$. Thus $B \subseteq B_i$ for all $i\in I$, because each B_i is a prime bi-ternary Γ-ideal of T. This implies $B \subseteq\bigcap_{i\in I} B_i$. Hence $\bigcap_{i\in I} B_i$ is a semi prime bi-ternary Γ-ideal of T.

IRREDUCIBLE AND STRONGLY IRREDUCIBLE BI-TERNARY Γ-IDEAL OF TERNARY Γ-SEMIRING

Definition 1

A bi-ternary Γ-ideal B of a ternary Γ-semiring T is said to be an ***irreducible bi-ternary Γ-ideal*** of T if $B_1\cap B_2\cap B_3 = B$ implies either $B_1 = B$ or $B_2 = B$ or $B_3 = B$ for any bi-ternary Γ-ideals B_1,B_2,B_3 of T.

Definition 2

bi-ternary Γ-ideal B of a ternary Γ-semiring T is said to be a ***strongly irreducible bi-ternary Γ-ideal*** of T if

$B_1 \cap B_2 \cap B_3 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$ for any bi-ternary Γ-idealsB$_1$,B$_2$,B$_3$ of T.

Theorem 3

Every strongly irreducible semi prime bi-ternary Γ-ideal of a ternary Γ-semiring T is a strongly prime bi-ternary Γ-ideal of T.

Proof

Let B be a strongly irreducible semi prime bi-ternary Γ-ideal of a ternary Γ-semiring T. Let B$_1$,B$_2$,B$_3$ be three bi-ternary Γ-ideals of T such that

$B_1\Gamma B_2\Gamma B_3 \quad \cap B_2\Gamma B_3\Gamma B_1 \quad \cap B_3\Gamma B_1\Gamma B_2 \subseteq B \longrightarrow$(i)

Then we have to show that either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. As $B_1 \cap B_2 \cap B_3 \subseteq B_1$,

$B_1 \cap B_2 \cap B_3 \subseteq B_2$ and $B_1 \cap B_2 \cap B_3 \subseteq B_3$ implies $[(B_1 \cap B_2 \cap B_3)\Gamma]^2(B_1 \cap B_2 \cap B_3) \subseteq B_1\Gamma B_2\Gamma B_3$, $[(B_1 \cap B_2 \cap B_3)\Gamma]^2(B_1 \cap B_2 \cap B_3) \subseteq B_2\Gamma B_3\Gamma B_1$ and $[(B_1 \cap B_2 \cap B_3)\Gamma]^2(B_1 \cap B_2 \cap B_3) \subseteq B_3\Gamma B_1\Gamma B_2$.

Thus $[(B_1 \cap B_2 \cap B_3)\Gamma]^2(B_1 \cap B_2 \cap B_3) \subseteq B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2 \subseteq B$ (by using (i)).

This implies $B_1 \cap B_2 \cap B_3 \subseteq B$, because B is a semi prime bi-ternary Γ-ideal of T.

Thus $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$, because B is a strongly irreducible bi-ternary Γ-ideal of T. Hence B is a strongly prime bi-ternary Γ-ideal of A.

Theorem 4

Let B be a bi-ternary Γ-ideal of a ternary Γ-semiring T and $a \in$ T such that $a \notin$ B. Then there exists an irreducible bi-ternary Γ-ideal I of T such that B ⊆ I and $a \notin$I.

Proof

Let X be the collection of all bi-ternary Γ-ideals of T which contain B but do not contain a, that is X = {Y$_i$: Y$_i$ is a bi-ternary Γ-ideal of T, B ⊆Y$_i$ and $a \notin$Y$_i$}. Then X is non-empty as B ∈ X. The collection X is a partially ordered set under inclusion. If {Y$_i$: i∈I} is any totally ordered subset (chain) of X, then $\bigcup_{i \in I} Y_i$ = Y is also a bi-ternary Γ-ideal of T containing B and $a \notin$Y. So Y is an upper bound of {Y$_i$: i∈I}. Thus every chain in X has an upper bound in X. Hence by Zorn's lemma, there exists a maximal element I (say) in X. This implies B ⊆I and $a \notin$I. Now we show that I is an irreducible bi-ternary Γ-ideal of T. For this let C, D and E be three bi-ternary Γ-ideals of T such that I = C∩D∩E. If C, D and E properly contain I, then $a \in$C, $a \in$D and $a \in$E. Thus $a \in$C∩D∩E = I. Which is a contradiction to the fact that $a \notin$I. So either I = C or I = D or I = E. Hence I is an irreducible bi-ternary Γ-ideal of T.

Theorem 5

For a ternary Γ-semiring T, the following assertions are equivalent:

(1) Every bi-ternary Γ-ideal of T is idempotent.

(2) $B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2 = B_1 \cap B_2 \cap B_3$ for any bi-ternary Γ-ideals B$_1$,B$_2$,B$_3$ of T.

(3) Each bi-ternary Γ-ideal of T is semiprime.

(4) Each proper bi-ternary Γ-ideal of T is the intersection of irreducible semiprime bi-ternary Γ-ideals of T which contain it.

Proof

(1) ⇒ (2) Let B$_1$,B$_2$,B$_3$ be any three bi-ternary Γ-ideals of T. Then $B_1 \cap B_2 \cap B_3$ is also a bi-ternary Γ-ideal of T, by theorem 22. By hypothesis, we have

$B_1 \cap B_2 \cap B_3 = [(B_1 \cap B_2 \cap B_3)\Gamma]^2(B_1 \cap B_2 \cap B_3)$

= $(B_1 \cap B_2 \cap B_3)\Gamma(B_1 \cap B_2 \cap B_3)\Gamma(B_1 \cap B_2 \cap B_3) \subseteq B_1\Gamma B_2 \Gamma B3$.

Similarly

$B_1 \cap B_2 \cap B_3 \subseteq B_2\Gamma B_3 \Gamma B_1$ and $B_1 \cap B_2 \cap B_3 \subseteq B_3 \Gamma B_1 \Gamma B_2$.

So $B_1 \cap B_2 \cap B_3 \subseteq B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3 \Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2$.Now $B_1\Gamma B_2\Gamma B_3$,$B_2\Gamma B_3\Gamma B_1$ and$B_3\Gamma B_1\Gamma B_2$, being the products of three bi-ternary Γ-ideals of T, are bi-ternary Γ-ideals of T, by Corollary 2.25. Also $B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2$ is a bi-ternary Γ-ideal of T, by theorem 2.22. Then by hypothesis

$B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2 = [(B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2)\Gamma]^2 (B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2) \subseteq (B_1\Gamma B_2\Gamma B_3)\Gamma(B_3\Gamma B_1\Gamma B_2)\Gamma(B_2\Gamma B_3\Gamma B_1) \subseteq (B_1\Gamma T\Gamma T)\Gamma(T\Gamma B_1\Gamma T)\Gamma(T\Gamma T\Gamma B_1) = B_1\Gamma(T\Gamma T\Gamma T)\Gamma B_1\Gamma(T\Gamma T\Gamma T)\Gamma B_1 = B_1\Gamma T\Gamma B_1\Gamma T\Gamma B_1 \subseteq B_1$.

Similarly $B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2 \subseteq B_2$ and $B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2 \subseteq B_3$.Thus $B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2 \subseteq B_1 \cap B_2 \cap B_3$.

Hence $B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2 = B_1 \cap B_2 \cap B_3$.

(2) ⇒ (1) Let B be a bi-ternary Γ-ideal of T.

Then by hypothesis B = B∩B∩B =BΓBΓB ∩BΓBΓB∩BΓBΓB = BΓBΓB.

(1) ⇒ (3) Let B be a bi-ternary Γ-ideal of T such that $(B_1\Gamma)^2 B_1 \subseteq$ B for any bi-ternary Γ-ideal B_1 of T. Then by hypothesis, we have $B_1 = (B_1\Gamma)^2 B_1 \subseteq$B. Hence every bi-ternary Γ-ideal of T is a semiprime bi-ternary Γ-ideal of T.

(3) ⇒ (4) Let each bi-ternary Γ-ideal of T is semiprime. Now let B be a proper bi-ternary Γ-ideal of T. If $\bigcap_\alpha I_\alpha$ is the intersection of all bi-ternary Γ-ideals of T containing B, then B ⊆. If this inclusion is proper, then there exists $a \in \bigcap_\alpha I_\alpha$ such that $a \notin$ B.

This implies $a \in I_\alpha$ for all α. As $a \notin$ B, then by Theorem 4.4, there exists an irreducible bi-ternary Γ-ideal I (say) of T such that B ⊆ I and $a \notin$ I. Which is a contradiction to the fact that $a \in I_\alpha$ for all α. So B $= \bigcap_\alpha I_\alpha$. By hypothesis, each bi-ternary Γ-ideal of T is semiprime. Thus each proper bi-ternary Γ-ideal of T is the intersection of irreducible semiprime bi-ternary Γ-ideals of T which contain it.

(4)⇒(1) Let each proper bi-ternary Γ-ideal of T is the intersection of irreducible semi prime bi-ternary Γ-ideals of T which contain it. Now if B is a bi-ternary Γ-ideal of T, then $(B\Gamma)^2 B$ is also a bi-ternary Γ-ideal of T, by Corollary 2.25. If $(B\Gamma)^2 B =$ T (improper bi-ternary Γ-ideal), then A $\subseteq (B\Gamma)^2 B$. This implies B ⊆ A $\subseteq (B\Gamma)^2 B$. Also $(B\Gamma)^2 B \subseteq$ B.

So $(B\Gamma)^2 B =$ B for each bi-ternary Γ-ideal B of T. Now if $(B\Gamma)^2 B$ is a proper bi-ternary Γ-ideal of T, that is $(B\Gamma)^2 B \neq$T, then $(B\Gamma)^2 B = \cap\{B_\alpha$: B_α is an irreducible semiprime bi-ternary Γ-ideal of T such that $(B\Gamma)^2 B \subseteq B_\alpha$ for all $\alpha\}$. This implies B ⊆ B_α for all α, because each B_α is a semi prime bi-ternary Γ-ideal of T. Thus B $\subseteq \cap B_\alpha = (B\Gamma)^2 B$.

Also $(B\Gamma)^2 B \subseteq$ B, as B is closed under multiplication. Hence $(B\Gamma)^2 B =$ B for each bi-ternary Γ-ideal B of T.

Theorem 6

If each bi-ternary Γ-ideal of a ternary Γ-semiring T is idempotent then a bi-ternary Γ-ideal B of T is strongly irreducible if and only if B is strongly prime.

Proof

Let B be a strongly irreducible bi-ternary Γ-ideal of T. Then we have to show that B is a strongly prime bi-ternary Γ-ideal of T. For this let B_1, B_2, B_3 be any three bi-ternary Γ-ideals of T such that $B_1ΓB_2ΓB_3 \cap B_2ΓB_3ΓB_1 \cap B_3ΓB_1ΓB_2 \subseteq B$. By Theorem 4.5, we have $B_1 \cap B_2 \cap B_3 = B_1ΓB_2ΓB_3 \cap B_2ΓB_3ΓB_1 \cap B_3ΓB_1ΓB_2 \subseteq B$. But B is a strongly irreducible bi-ternary Γ-ideal of T. Thus we have $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is a strongly prime bi-ternary Γ-ideal of T.

Conversely suppose that B is a strongly prime bi-ternary Γ-ideal of T. To show that B is strongly irreducible bi-ternary Γ-ideal of T, let B_1, B_2, B_3 be any bi-ternary Γ-ideals of T such that $B_1 \cap B_2 \cap B_3 \subseteq B$. By Theorem 4.5, we have $B_1ΓB_2ΓB_3 \cap B_2ΓB_3ΓB_1 \cap B_3ΓB_1ΓB_2 = B_1 \cap B_2 \cap B_3 \subseteq B$. But B is a strongly prime bi-ternary Γ-ideal of T, we have $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is a strongly irreducible bi-ternary Γ-ideal of T.

Next we characterize those ternary Γ-semi rings in which each bi-ternary Γ-ideal is strongly prime and also those ternary Γ-semi rings in which each bi-ternary Γ-ideal is strongly irreducible.

Theorem 7

Each bi-ternary Γ-ideal of a ternary Γ-semiring T is strongly prime if and only if each bi-ternary Γ-ideal of T is Γ-idempotent and the set of bi-ternary Γ-ideals of T is totally ordered by inclusion.

Proof

Suppose that each bi-ternary Γ-ideal of T is strongly prime. This implies that each bi-ternary Γ-ideal of T is semiprime. Thus by Theorem 4.5, each bi- ternary Γ-ideal of T is idempotent. Now we show that the set of bi-ternary Γ-ideals of T is totally ordered by inclusion. For this, let B_1, B_2 be two bi-ternary Γ-ideals of T. Then by Theorem 4.5, we have

$B_1 \cap B_2 = B_1 \cap B_2 \cap T = B_1ΓB_2ΓT \cap B_2ΓTΓB_1 \cap AΓB_1ΓB_2$, $\Rightarrow B_1ΓB_2ΓT \cap B_2ΓTΓB_1 \cap AΓB_1ΓB_2 \subseteq B_1 \cap B_2$.

By hypothesis, B_1 and B_2 are strongly prime bi-ternary Γ-ideals of T, so is $B_1 \cap B_2$. Then $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$ or $T \subseteq B_1 \cap B_2$.Thus $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.Hence the set of bi-ternary Γ-ideals of T is totally ordered by inclusion.

Conversely, assume that each bi-ternary Γ-ideal of T is Γ-idempotent and the set of bi-ternary Γ-ideals of T is totally ordered by inclusion. We have to show that each bi-ternary Γ-ideal of T is strongly prime. For this, let B be an arbitrary bi-ternary Γ-ideal of T and B_1, B_2, B_3 be any bi-ternary Γ-ideals of T such that $B_1ΓB_2ΓB_3 \cap B_2ΓB_3ΓB_1 \cap B_3ΓB_1ΓB_2 \subseteq B$. By Theorem 4.5, we have $B_1 \cap B_2 \cap B_3 = B_1ΓB_2ΓB_3 \cap B_2ΓB_3ΓB_1 \cap B_3ΓB_1ΓB_2 \subseteq B \longrightarrow$ (i).

Since the set of bi-ternary Γ-ideals of T is totally ordered by inclusion, so for B_1, B_2, B_3 we have the following six possibilities:

(ii) $B_1 \subseteq B_2 \subseteq B_3$ (iii) $B_1 \subseteq B_3 \subseteq B_2$
(iv) $B_2 \subseteq B_3 \subseteq B_1$ (v) $B_2 \subseteq B_1 \subseteq B_3$
(vi) $B_3 \subseteq B_1 \subseteq B_2$ (vii) $B_3 \subseteq B_2 \subseteq B_1$.

In these cases we have

(ii) $B_1 \cap B_2 \cap B_3 = B_1$ (iii) $B_1 \cap B_2 \cap B_3 = B_1$
(iv) $B_1 \cap B_2 \cap B_3 = B_2$ (v) $B_1 \cap B_2 \cap B_3 = B_2$
(vi) $B_1 \cap B_2 \cap B_3 = B_3$ (vii) $B_1 \cap B_2 \cap B_3 = B_3$.

Thus (i) gives, either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence B is strongly prime.

Theorem 8

If the set of bi-ternary Γ-ideals of a ternary Γ-semiring T is totally ordered, then each bi-ternary Γ-ideal of T is Γ-idempotent if and only if each bi-ternary Γ-ideal of T is prime.

Proof

Suppose each bi-ternary Γ-ideal of T is Γ-idempotent and B is an arbitrary bi-ternary Γ-ideal of T and B_1, B_2, B_3 be any bi-ternary Γ-ideals of T such that $B_1\Gamma B_2\Gamma B_3\subseteq B$. As the set of bi-ternary Γ-ideals of T is totally ordered, then for B_1, B_2, B_3 we have the following six possibilities:

(i) $B_1\subseteq B_2\subseteq B_3$ (ii) $B_1\subseteq B_3\subseteq B_2$
(iii) $B_2\subseteq B_3\subseteq B_1$ (iv) $B_2\subseteq B_1\subseteq B_3$
(v) $B_3\subseteq B_1\subseteq B_2$ (vi) $B_3\subseteq B_2\subseteq B_1$.

For (i) and (ii), we have $(B_1\Gamma)^2 B_1 = B_1\Gamma B_1\Gamma B_1\subseteq B_1\Gamma B_2\Gamma B_3\subseteq B$, implies $B_1\subseteq B$, as B is Γ-idempotent. Similarly for other possibilities we have $B_2\subseteq B$ or $B_3\subseteq B$.

Conversely, suppose that each bi-ternary Γ-ideal of T is prime, so is semiprime, by Theorem 3.5. Thus by Theorem 4.5, each bi-ternary Γ-ideal of T is Γ-idempotent.

Theorem 9

If the set of bi-ternary Γ-ideals of a ternary Γ-semiring T is totally ordered, then the concepts of primeness and strongly primeness coincide.

Proof

Let *B* be a prime bi-ternary Γ-ideal of *T*. To show that *B* is a strongly prime bi-ternary Γ-ideal of T, let B_1, B_2, B_3 be any bi-ternary Γ-ideals of *T* such that

$B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2\subseteq B$.

As the set of bi-ternary Γ-ideals of ternary Γ-semiring *T* is totally ordered, then for B_1, B_2, B_3 we have the following six possibilities:

(i) $B_1\subseteq B_2\subseteq B_3$ (ii) $B_1\subseteq B_3\subseteq B_2$
(iii) $B_2\subseteq B_3\subseteq B_1$ (iv) $B_2\subseteq B_1\subseteq B_3$
(v) $B_3\subseteq B_1\subseteq B_2$ (vi) $B_3\subseteq B_2\subseteq B_1$.

For (i) and (ii), we have $(B_1\Gamma)^2 B_1 = ((B_1\Gamma)^2 B_1)\cap((B_1\Gamma)^2 B_1)\cap((B_1\Gamma)^2 B_1)$ $\subseteq B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2\subseteq B$.

Implies $B_1\subseteq B$, as *B* is a prime bi-ternary Γ-ideal of *T*. Similarly for other possibilities we have $B_2\subseteq B$ or $B_3\subseteq B$. This shows that *B* is a strongly prime bi-ternary Γ-ideal of *T*. Thus every prime bi-ternary Γ-ideal of T is a strongly prime bi-ternary Γ-ideal of T. Now let *B* be a strongly prime bi-ternary Γ-ideal of T. To show that *B* is a prime bi-ternary Γ-ideal of T, let B_1, B_2, B_3 be any bi-ternary Γ-ideals of T such that $B_1\Gamma B_2\Gamma B_3\subseteq B$.Implies $B_1\Gamma B_2\Gamma B_3 \cap B_2\Gamma B_3\Gamma B_1 \cap B_3\Gamma B_1\Gamma B_2\subseteq B$. Implies either $B_1\subseteq B$ or $B_2\subseteq B$ or $B_3\subseteq B$, as *B* is a strongly prime bi-ternary Γ-ideal of T. This shows that *B* is a prime bi-ternary Γ-ideal of T. Thus every strongly prime bi-ternary Γ-ideal of T is a prime bi-ternary Γ-ideal of T.

Theorem 10

For a ternary Γ-semi ring T, the following assertions are equivalent:

(1) The set of bi-ternary Γ-ideals of T is totally ordered by set inclusion.

(2) Each bi-ternary Γ-ideal of T is strongly irreducible.

(3) Each bi-ternary Γ-ideal of T is irreducible.

Proof

(1) ⇒(2) Let the set of bi-ternary Γ-ideals of Tis totally ordered by set inclusion. To show that each bi-ternary Γ-ideal of T is strongly irreducible, let *B* be an arbitrary bi-ternary Γ-ideal of T and B_1, B_2, B_3 be any bi-ternary Γ-ideals of T such that $B_1\cap B_2\cap B_3\subseteq B$.Since the set of bi-ideals of T is totally ordered by set inclusion, then

$B_1 \cap B_2 \cap B_3 = B_1$ or B_2 or B_3. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. So B is strongly irreducible. Hence each bi-ternary Γ-ideal of T is strongly irreducible.

(2) ⇒ (3) Let each bi-ternary Γ-ideal of T is strongly irreducible. To show that each bi-ternary Γ-ideal of T is irreducible, let B be an arbitrary bi-ternary Γ-ideal of T and B_1, B_2, B_3 be any bi-ternary Γ-ideals of T such that $B_1 \cap B_2 \cap B_3 = B$. This implies $B \subseteq B_1$, $B \subseteq B_2$ and $B \subseteq B_3$. On the other hand, by hypothesis we have $B_1 \subseteq B$ or $B_2 \subseteq B$ or $B_3 \subseteq B$. Hence either$B_1 = B$ or $B_2 = B$ or $B_3 = B$. Thus B is an irreducible bi-ternary Γ-ideal of T. Hence each bi-ternary Γ-ideal of T is irreducible.

(3) ⇒ (1) Let each bi-ternary Γ-ideal of Tis irreducible. To show that the set of bi-ternary Γ-ideals of T is totally ordered by set inclusion, let B_1, B_2 be any two bi-ternary Γ-ideals of T. Then by Theorem 22, $B_1 \cap B_2$ is also a bi-ternary Γ-ideal of Tand so is irreducible bi-ternary Γ-ideal of T. Since $B_1 \cap B_2 \cap T = B_1 \cap B_2$, implies $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$ or $T = B_1 \cap B_2$, implies either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $B_1 = B_2 = T$. Hence the set of bi-ternary Γ-ideals of T is totally ordered by set inclusion.

Conclusion

In this paper mainly we start the study of prime Bi-ternary Γ-ideals, irreducible bi-ternary Γ-ideals in ternary Γ-semi rings. We characterize them and results in this paper may apply to many algebraic structures for further research.

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REFERENCES

1. M. Amyari and M. S. Moslehian, Approximate homomorphisms of ternary semigroups, Letters Math. Physics 77 (2006) 1–9.
2. N. Bazunova, A. Borowiec and R. Kerner, Universal differential calculus on ternary algebras, Letters Math. Physics 67 (2004) 195–206.
3. R. Carlsson, Cohomology of associative triple systems, Proc. Amer. Math. Soc. 60(1976) 1–7.
4. T. K. Dutta and S. Kar, on prime ideals and prime radical of ternary semirings, *Bull. Cal. Math. Soc.* 97(5) (2005) 445–454.
5. T. K. Dutta and S. Kar, Onsemiprime ideals and irreducible ideals of ternary semirings, *Bull. Cal. Math. Soc.* 97(5) (2005) 467–476.
6. R. Kerner, The cubic chess coard, class, Quantum Grav. 14 (1997) A203–A225.
7. R. Kerner, Ternary algebraic structures and their applications in physics, Univ. P&M. Curie preprint, Paris (2000) Arxiv Math-Ph/0011023.
8. S. Kar, on quasi-ideals and bi-ideals in ternary semirings, *Int. J. Math. Math.Sci*. 2005, no. 18, 3015–3023.
9. M. Kapranov, I. M. Gelfand and A. Zelevinskii, Discriminants, Resultants and Multidimensional Determinants, Birkhauser, Berlin, 1994.
10. R. Lawrence, Algebras and Triangle relations, in Topological Methods in Field Theory, (J. Mickelson and O.

Pekonetti, eds.), *World Sci*., Singapore, 1992, pp. 429–447.

11. W. G. Lister, Ternary rings, Trans. Amer. Math. Soc. 154 (1971) 37–55.
12. D. Madhuduhana Rao, A. Anjaneyulu and A. Gangadhara Rao, Prime Γ-radicals in Γ-semi groups, *International eJournal of Mathematics and Engineering* 138 (2011) 1250-1259.
13. M. Shabir and M. Bano, Prime bi-ideals in ternary semigroups, Quasigroups Related Systems 16 (2008) 239–256.
14. N. P. Sokolov, Introduction to the theory of Multidimensional Matrices, Naukova, Dumka, Kiev, 1972.
15. Sajani Lavanya, MadhusudhanaRao and Syam Julius Rajendra., on Quasi-Ternary Γ-Ideals and Bi-Ternary Γ-Ideals in Ternary **Γ**-Semirings-*International Journal of Mathematics and Statistics Invention*, Volume 6, Issue 3, September 2015, pp 05-14.
16. G. Srinivasa Rao and D. Madhusudhana Rao, Prime Radicals and Completely Prime Radicals in Ternary Semirings-*International Journal of Basic Science and Applied Computing (IJBSAC)*, Volume-1 Issue-4, February 2015, pp 1-5.
17. Syam Julius Rajendra. V, Dr. Madhusudhana Rao. D and Sajani Lavanya. M-On Pure PO-Ternary Γ-Ideals in Ordered Ternary Γ-Semirings, *IOSR Journal of Mathematics (IOSR-JM)*, Volume 11, Issue 5 Ver. IV (Sep. - Oct. 2015), PP 05-13.
18. Y. Sarala, A. Anjaneyulu and D. Madhusudhana Rao, Prime radicals in ternary semi groups, *International Organization Scientific Research Journal of Mathematics (IOSR-JM)* ISSN: 2278-5728 Volume 4, Issue 5 (Jan.-Feb. 2013), pp 43-53.