Vibration of damped simply supported orthotropic rectangular plates resting on elastic winkler foundation, subjected to moving loads

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ABTSRACT

In this work the fourth order differential equation governing the vibration of damped orthotropic rectangular plate resting on Winkler foundation was reduced to second order coupled differential equation by separating the variables. The coupled differential equation was solved using numerical methods .The classical edge condition used as an illustrative example is the simple support conditions. The effects of some physical phenomena were investigated and the results obtained are discussed and graphically presented.

Keywords: Winkler foundation, Orthotropic Rectangular Plates, Viscous Damping, and Dynamic Moving Loads.
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INTRODUCTION

Vibration of rectangular plates is an interesting subject because of its wide applications in structural Engineering and transport engineering. Structures like railway bridges, highway bridges, cranes, road pavements etc, can actually be modeled as rectangular plates. The applicability of this subject in the study of these structures we can hardly do without, has accelerated research on the study and understanding of rectangular plates. As a way of review we quickly discuss some early works on the vibration of rectangular plates and some interesting results obtained. In [6] the differential equation relating to the breaking of railway bridges was developed and well discussed. Some other research works focused more on vibration of solids and structures under moving loads, such work includes [2], who found that the theoretical considerations are applicable in calculations relating to dynamic stresses in railway and highway bridges, suspension bridges, rails, sleepers, cranes etc. In mechanics moving loads are defined as loads that vary in both time and space. In [3] the dynamic response of plates on elastic foundation to distributed moving loads was investigated and it was reported that the natural frequency of rectangular plates traversed by moving concentrated forces is greater than that of plates subjected to moving concentrated masses and that the presence of foundation modulus reduces the deflection of the plate.

In most of the works the type of plates considered are isotropic rectangular plates which are uniform in all direction. In application not all plates are isotropic, another important type of plate is the orthotropic rectangular plates which found applications in the modeling of the dynamic response of rigid concrete pavements. In [1], orthotropic rectangular plate was used to model the dynamic of rigid roadway pavement under dynamic load, the method used was the modified Bolotin Method. The dynamic moving traffic load is expressed as a concentrated load of harmonically varying magnitude, moving straight along the plate with a constant velocity, It was found that this dynamic load approach may lead to more economic solutions as compared to those obtained from the conventional static load approach.

Viscous damping is the dissipation of energy and the consequence reduction or decay of motion. To understanding the control and mechanical response of vibrating structures, viscous damping should be properly understood. Most of the early works neglected damping, but recently, interesting studies and results are now emerging on the effects of viscous damping on the vibration of rectangular plates on elastic foundation. Some of such works include [4], it was found that the deflection profile of the plate depends on the magnitude of the damping coefficient. In [5], the

effect of viscous damping on the dynamic response of isotropic rectangular plates on Pasternak foundations are studied and it was shown that viscous damping in the presence of Pasternak foundation actually reduces the buildup of amplitude, thereby reducing the possibility of resonance. In this work we pay attention to the vibration or the dynamic behavior of damped Orthotropic rectangular plate resting on Winkler foundation, subjected to dynamic loads.

MATERIALS AND METHODS

Methods of Solution

The governing equation of the problem is given as:

$$\left[\alpha_1 \frac{\partial^4 w}{\partial x^4} + 2\alpha_2 \frac{\partial^4 w}{\partial x^2 y^2} + \alpha_3 \frac{\partial^4 w}{\partial y^4}\right] + M\frac{\partial^2 w}{\partial t^2} + 2M\gamma\frac{\partial w}{\partial t} + Kw = p(x,y,t) \tag{1}$$

where:

w = w(x,y,t) is the deflection of the plate.

t = time in seconds.

γ =viscous damping coefficient.

α_1= flexural rigidity in the x direction.

α_2 = effective torsional rigidity.

α_3= flexural rigidity in the y direction.

E= Young's modulus

M= mass density per unit area

H= thickness of plate

P(x,y,t)= the applied load, which taken to be;

$$P(x,y,t)= \frac{1}{r}\left(mg - m\frac{d^2 w}{dt^2}\right)\left[H\left(x - vt + \frac{r}{2}\right) - H\left(x - vt - \frac{r}{2}\right)\right]\partial(y - y_1) \tag{2}$$

r= length of the load.

H(x)= Heaviside step function

$\partial(x)$= dirac delta function

g= acceleration due to gravity.

v= velocity.

K= foundation stiffness.

The above governing partial differential equation (1) was developed under the following assumptions:

- The small strain in the system is still governed by Hook's law.

-The plate is resting on elastic foundation.

-The load is taken to be a distributed time load.

-There is no deformation in the middle of the plate, i.e the plate remains the same before and after bending.

The governing equation of the problem is solved using separation of variable in series form. We assume the following;

Let,

$$W(x,y,t)= \sum_{n=1}^{N}\sum_{m=1}^{M} A_{mn}(t)W_n(x)W_m(y) \tag{3}$$

Where: n=1,2,3,…,N and m=1,2,3,…M. M and N are fixed positive integer.

A(t) is a function of time.

If we substitute (3) into (1), we have that;

$$\Big[\propto_1 \sum_{n=1}^{N}\sum_{m=1}^{M} A_{mn}(t)w_n^{\ iv}(x)w_m(y) +$$

$$2 \propto_2 \sum_{n=1}^{N}\sum_{m=1}^{M} A_{mn}(t)w_n^{\ ii}(x)w_m^{\ ii}(y) +$$

$$\propto_3 \sum_{n=1}^{N}\sum_{m=1}^{M} A_{mn}(t)w_n(x)w_m^{\ iv}(y)\Big] + M\sum_{n=1}^{N}\sum_{m=1}^{M}\ddot{A}_{mn}(t)$$

$$w_n(x)$$

$$w_m(y) +$$

$$2M\gamma\sum_{n=1}^{N}\sum_{m=1}^{M}\dot{A}_{mn}(t)w_n(x)w_m(y) + K\sum_{n=1}^{N}\sum_{m=1}^{M} A_{mn}(t)\,w_n(x)w_m(y) =$$

$$p(x,y,t) \tag{4}$$

On further simplifications, we have that;

$$\sum_{n=1}^{N}\sum_{m=1}^{M}[A_{mn}(t)\{\propto_1 w_n^{\ iv}(x)w_m(y) + 2\propto_2 w_n^{\ ii}(x)w_m^{\ ii}(y) +$$

$$\propto_3 w_n(x)w_m^{\ iv}(y)\} +$$

$$M\ddot{A}_{mn}(t)w_{n(x)}w_m(y) + 2M\gamma\dot{A}_{mn}(t)w_n(x)w_m(y) + KA_{mn}(t)w_n(x)w_m(y)] =$$

$$\frac{1}{r}\Big(mg - m\frac{d^2w}{dt^2}\Big)\Big[H\Big(x-vt+\frac{r}{2}\Big) - H\Big(x-vt-\frac{r}{2}\Big)\Big]\partial(y-y_1) \tag{5}$$

When we substitute the p(x,y,t), from (2).

The equation governing the undamped free vibration of an orthotropic rectangular plate is as follows;

$$\Big[\propto_1 \frac{\partial^4 w}{\partial x^4} + 2\propto_2 \frac{\partial^4 w}{\partial x^2 y^2} + \propto_3 \frac{\partial^4 w}{\partial y^4}\Big] + Kw + \omega^2 Mw = 0 \tag{6}$$

By substituting (3) into (6), and taking $\mu_{mn} = -\omega^2 M$,

We have that

$$\propto_1 w_n^{\ iv}(x)w_m(y) + 2\propto_2 w_n^{\ ii}(x)w_m^{\ ii}(y) + \propto_3 w_n(x)w_m^{\ iv}(y) +$$

$$Kw_n(x)w_m(y) = \mu_{mn}w_n(x)w_m(y) \tag{7}$$

On substituting (7) into (5), and putting (3) in the RHS of (5), we have the simplified equation governing the vibration problem of damped orthotropic rectangular plate resting on Winkler foundation, subjected to dynamic loading.

$$\sum_{n=1}^{N}\sum_{m=1}^{M}[\mu_{mn}A_{mn}(t)Mw_n(x)w_m(y) + M\ddot{A}_{mn}(t)w_{n(x)}w_m(y) + 2$$

$$M\gamma\dot{A}_{mn}(t)w_n(x)w_m(y)] =$$

$$\frac{1}{r}(mg - m\sum_{n=1}^{N}\sum_{m=1}^{M}\{\ddot{A}_{mn}(t)w_n(x)w_m(y) + 2v\dot{A}_{mn}(t)w_n^{\ i}(x)w_m(y) +$$

$$v^2 A_{mn}(t)w_n^{\ ii}(x)w_m(y)\}) \Big[H\Big(x-vt+\frac{r}{2}\Big) - H\Big(x-vt-\frac{r}{2}\Big)\Big]\partial(y-y_1) \tag{8}$$

Multiply both sides of (8) by $w_i(x)w_j(y)$ and we integrate along the edges of the rectangular plate of dimension (axb). And further apply the Orthogonality of $w_n(x)\ and\ w_m(y)$ with the following relation between dirac delta function and the Heaviside unit function.

$$\int_{x_1}^{x_2} \partial(x-x_o)\,dx = H(x-x_o).$$

We obtain;

$$
\ddot{A}_{mn}(t) + 2\gamma \dot{A}_{mn}(t) + \mu_{mn} A_{mn}(t)
$$

$$
= \frac{1}{\varphi r M}\{ mg w_j(y_1) \int_{vt-\frac{r}{2}}^{vt+\frac{r}{2}} w_i(x)\,dx
$$

$$
- m \sum_{n=1}^{N} \sum_{m=1}^{M} [\ddot{A}_{mn}(t) w_m(y_1) w_n(y_1) \int_{vt-\frac{r}{2}}^{vt+\frac{r}{2}} w_n(x) w_i(x)\,dx
$$

$$
+ 2v \dot{A}_{mn}(t) w_m(y_1) w_j(y_1) \int_{vt-\frac{r}{2}}^{vt+\frac{r}{2}} w_n^{\ i}(x) w_i(x)\,dx
$$

$$
+ v^2 A_{mn}(t) w_m(y_1) w_j(y_1) \int_{vt-\frac{r}{2}}^{vt+\frac{r}{2}} w_n^{\ ii} w_i(x)\,dx] \}
$$

(9)

Where φ is an arbitrary constant.

Equation (9) is the generalized Ordinary coupled differential equation to be solved for some specific boundary conditions.

Simply Supported Rectangular plate (as an illustrative example)
We have different kinds of classical edge supports, but for the sake of this study we limit our consideration to simply supported edges alone.

The boundary condition for simply supported rectangular plates is given as;

$$
w(0, y, t) = w(a, y, t) = w_{xx}(0, y, t) = w_{xx}(a, y, t) = 0
$$
$$
w(x, 0, t) = w(x, b, t) = w_{yy}(x, 0, t) = w_{yy}(x,, t) = 0
$$

With the initial condition

$$
w(x, y, 0) = w_t(x, y, 0) = 0
$$
(10)

The Normalized deflection curve for simply supported boundary condition for rectangular plate has been obtained in [3] to be

$$
w_n(x) w_m(y) = \frac{2}{\sqrt{ab}} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}
$$
Where n = 1,2,3,…, and m = 1,2,3,…,   (11)

To obtain the eigen values, we substitute (11) into (7) to have;

$$
\mu_{mn} = \left[\propto_1 \frac{n^4 \pi^4}{a^4} + 2 \propto_2 \frac{n^2 m^2 \pi^4}{a^2 b^2} + \propto_3 \frac{m^4 \pi^4}{b^4} \right] + K
$$
(12)

The exact governing equation for simply supported damped orthotropic rectangular plate resting on Winkler foundation can be obtained by putting (11) into (9).

To have

$$\ddot{A}_{mn}(t) + 2\gamma \dot{A}_{mn}(t) + \mu_{mn} A_{mn}(t)$$
$$= \frac{1}{\varphi r M}\left[\frac{4mga}{\pi i \sqrt{ab}}\sin\frac{i\pi vt}{a}\sin\frac{i\pi y_1}{b}\right.$$
$$- m\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\frac{4}{ab}\sin\frac{m\pi y_1}{b}\sin\frac{i\pi y_1}{b}\ddot{A}_{mn}(t)\left(\frac{a}{\pi d_1}\cos\frac{d_1\pi rt}{a}\sin\frac{d_1\pi r}{2a}\right.\right.$$
$$- \frac{a}{\pi p}\cos\frac{p\pi vt}{a}\sin\frac{p\pi r}{2a}\right)$$
$$+ \frac{8n\pi v}{a^2 b}\dot{A}_{mn}(t)\sin\frac{m\pi y_1}{b}\sin\frac{\pi y_1}{b}\left(\frac{a}{p\pi}\sin\frac{p\pi vt}{a}\sin\frac{p\pi r}{2a}\right.$$
$$- \frac{a}{d_1\pi}\sin\frac{d_1\pi vt}{a}\sin\frac{d_1\pi r}{2a}$$
$$- \frac{4\pi^2 n^2}{a^3 b}v^2 A_{mn}(t)\sin\frac{m\pi y_1}{b}\sin\frac{j\pi y_1}{b}\left(\frac{a}{\pi d_1}\cos\frac{d_1\pi vt}{a}\sin\frac{d_1\pi r}{2a}-\right.$$
$$\left.\left.\frac{a}{\pi p}\cos\frac{p\pi vt}{a}\sin\frac{p\pi r}{2a}\right)\right] \tag{13}$$

Where i ≠ n

$$\ddot{A}_{mn}(t) + 2\gamma \dot{A}_{mn}(t) + \mu_{mn} A_{mn}(t) = \frac{1}{\varphi r M}\left[\frac{4mga}{\pi n\sqrt{ab}}\sin\frac{i\pi y_1}{b}\sin\frac{ir\pi}{2a}\sin\frac{i\pi vt}{a} -\right.$$
$$m\sum_{n=1}^{N}\sum_{m=1}^{M}\left\{\frac{4}{ab}\sin\frac{m\pi y_1}{b}\sin\frac{i\pi y_1}{b}\ddot{A}_{mn}(t)\left(\frac{r}{2}-\frac{a}{2n\pi}\sin\frac{n\pi r}{a}\cos\frac{2nvt\pi}{a}\right) +\right.$$
$$\frac{8n\pi v}{a^2 b}\dot{A}_{mn}(t)\sin\frac{m\pi y_1}{b}\sin\frac{j\pi y_1}{b}\left(\frac{a}{2n\pi}\sin\frac{2nvt\pi}{a}\sin\frac{n\pi r}{a}\right) -$$
$$\frac{4\pi^2 n^2 v^2}{a^3 b}A_{mn}(t)\sin\frac{m\pi y_1}{b}\sin\frac{j\pi y_1}{b}\left(\frac{r}{2} -\right.$$
$$\left.\left.\frac{a}{2n\pi}\sin\frac{n\pi r}{a}\cos\frac{2nvt\pi}{a}\right)\right\}\right] \tag{14}$$

RESULTS AND DISCUSSION

Equations (13) and (14) are Second order coupled differential equations. Equation (13) is for when i ≠ n, and equation (14) is for when n=i. The coupled differential equations are solved using the finite difference method i.e. the central difference method. The resulting Tridiagonal matrix is of the form;

$$[2 - hP(t_i)]T_{i-1} + [-4 + 2h^2 Q(t_i)]T_i + [2 + hP(t_i)]T_{i+1} = 2h^2 R(t_i)$$

Where h is chosen appropriately, and $i = 1, 2, 3, \ldots n$

The resulting tridiagonal matrices were solved using MATLAB. The following values are assumed for the corresponding variables: $a = 10, b = 5,$ $v =$12m/s, 24m/s and 36m/s, K=0, 20,80. The values used for the flexural rigidity in the x-direction (α_1), the effective torsonal rigidity (α_2) and the flexural rigidity in the y-direction (α_3), is that of Veneer, given as 0.297, 0.21 and 0.69, respectively. The values of the damping ratios (γ) are assumed to be 0,100,150, respectively.

In figure 1, we see that the maximum deflection is much higher when damping ratio (γ)=0, and as the damping ratio is increased, the deflection is reduced and the vibration also stabilizes with time.

In figure 2, observed that when the foundation modulus is reduced to zero, the mid-plate deflection increased and when the foundation modulus K, is increased, the maximum deflection is reduced.

In figure 3, various velocities are considered; we see that at high velocity, the maximum deflection is attained at a shorter time.
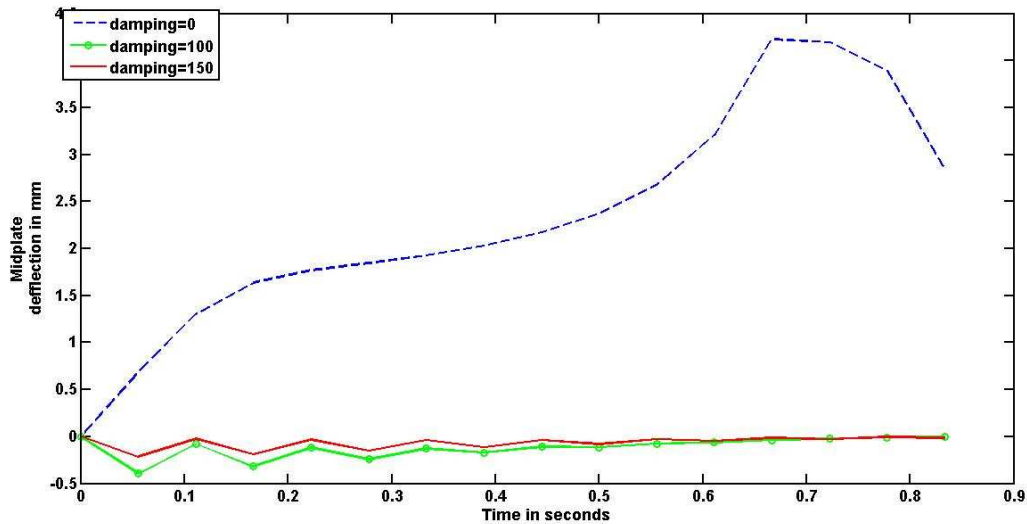

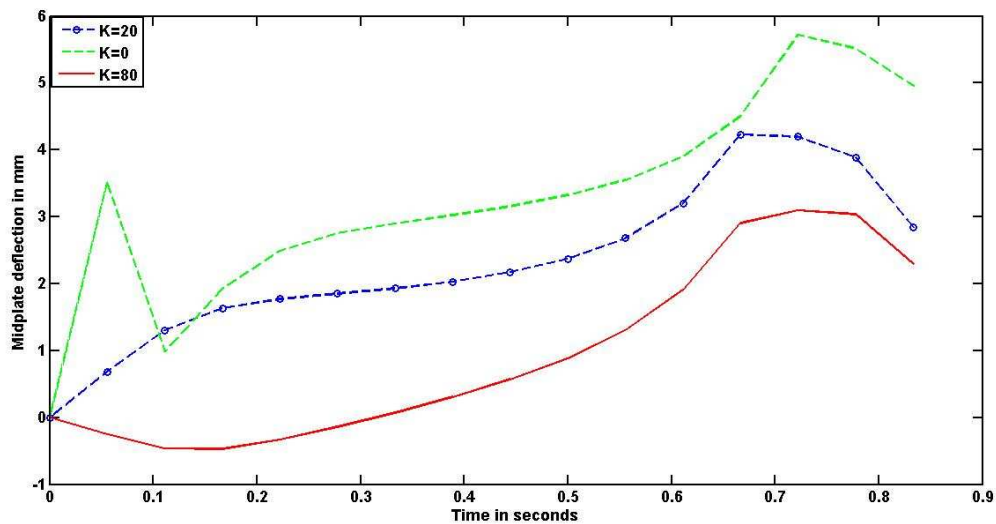**Figure 1: Deflection per time at various damping ratios**

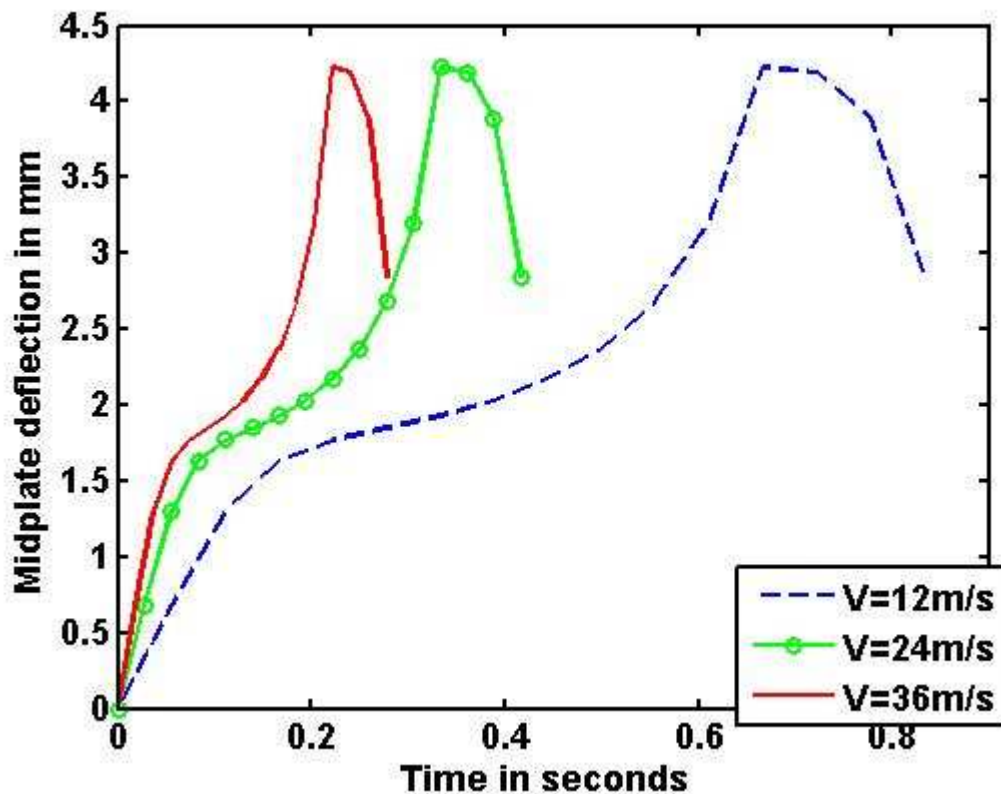**Figure 2: Mid-plate deflection per time at various values of K**

Figure 3: Deflection per time at various velocities

CONCLUSION

Damping plays a very significant role in the vibration of solid structures, as it has been shown that deflection profile depends greatly on the damping ratio. The deflection profile also proves to be more stable in the presence of foundation coupled with viscous damping. These results further show that very high speed can be detrimental to solid structures, especially Highway bridges and other structures subjected to dynamic loads.

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