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Advances in Applied Science Research, 2013, 4(2):163-168

ISSN: 0976-8610
CODEN (USA): AASRFC

The successive over relaxation method in multi-layer grid refinement scheme

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ABSTRACT

The successive over-relaxation (SOR) method has been widely used as an iterative method to solve large sparse linear system. When solving a partial differential equation over a rectangular domain with Dirichlet boundary conditions, the multi-layer grid refinement method can be used to generate the linear system, with higher efficiency than uniform grid theme. In this paper, we will study the SOR method in the multi-layer grid refinement scheme. A heuristic estimation for the optimal parameter of the SOR method is given and numerical experiments are carried out to verify the estimation in this scheme.

Key words: partial differential equation, iterative methods, SOR method, multi-layer grid refinement method

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INTRODUCTION

The multi-layer grid refinement method [**5**] is used to solve a partial differential equation of the form

$$A(x,y)u_{xx} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y) \qquad (1.1)$$

where *A*, *C*, *D*, *E*, *F* are functions of *x* and *y*, with the Dirichlet boundary conditions on a rectangular region. A numerical solution of the partial differential equation is based on the finite difference method, which involves a five-point scheme. It discretizes the PDE into a set of difference equations so that a linear system can be generated and solved. Normally, the uniform grid scheme is applied because of its ease in use, but due to the large number of grids, the coefficient matrix *A* could be extremely large most of the time. Subsequently may the computation time be substantial. However, in many cases only a small subdomain in the region is of great interest. It is not necessary to put very small grids on the entire region. The multi-layer grid refinement reduces the size of the coefficient matrix *A* by using much fewer grids in the whole region than the normal uniform grid scheme. We place fine grids in the interested domain and coarse grids in other domains in the region, with a special consideration [**5**] of obtaining the partial derivatives of the inner boundary points. Figure 1.1 illustrates a possible grid pattern of the multi-layer grid refinement method. As a result, the size of the coefficient matrix of the linear system and the computational time are substantially reduced without sacrificing the accuracy of the solutions.

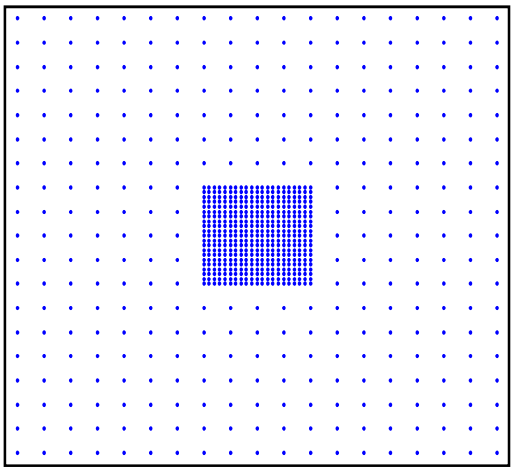

Figure 1.1. Grid pattern for the two-layer scheme

2. The SOR method

Numerical iterative methods [**3, 6, 7, 8, 9**] have been used to solve for solutions to a large sparse linear system

$$Au = b \tag{2.1}$$

where *A* is a given matrix and *b* is a given vector. There are some basic iterative methods e.g. the Jacobi method[**2**], the Gauss-Seidel(GS) method[**8**], and many others. In this paper we focus on the use of the successive over-relaxation (SOR) method [**8, 10**] to solve the linear system.

We let *A* be written as

$$A = D - C_L - C_U. \tag{2.2}$$

The matrix *D* is a diagonal matrix with the same diagonal elements as *A*; C_L and C_U are strictly lower and strictly upper triangular matrices of *A*, respectively. With an introduction of parameter ω acting on the Gauss-Seidel method, the iterative method becomes a robust stand-alone method. It is called the SOR method.

The iteration of the SOR method is given by

$$u^{(n+1)} = (D - \omega C_L)^{-1}[(1-\omega)D + \omega C_U]u^{(n)} + (D - \omega C_L)^{-1}\omega b, \tag{2.3}$$

where the parameter ω is the over-relaxation factor and the iteration matrix G is defined as

$$G = (D - \omega C_L)^{-1}[(1-\omega)D + \omega C_U]. \tag{2.4}$$

We note that if the value of ω is equal to 1, then the SOR method and the GS method are identical. However, with an optimal choice of ω, the rate of convergence of the SOR method can be increased significantly. The rate of convergence of an iterative method is defined by

$$\mathrm{R(G)} = -\log\big(\rho(\mathrm{G})\big). \tag{2.5}$$

The analytical value of the optimal value of ω can be found for certain linear systems, see [**8**], the Model Problem 1 described in section 3 is one example. If the linear system is generated by the central finite difference scheme, then the optimal ω is proven to be

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\rho(B)^2}} \tag{2.6}$$

where $\rho(B)$ is the spectral radius of the Jacobi iteration matrix $B = I - D^{-1}A$. With such choice of the parameter, the rate of convergence of the SOR method can be increased by several orders of magnitude. However, in general, the optimal value of ω is not easy to obtain. We note that a general procedure [**10**] for finding the optimal value of ω may be applied but not efficient in the multi-layer scheme. Therefore in this paper we introduce a heuristic estimation formula for such scheme.

3. Numerical Experiments

In this research, we perform numerous experiments using the multi-layer grid refinement method on the following two model problems.

3.1.1. *Model Problem 1 (MP1).*

$$u_{xx} + u_{yy} = 0 \tag{3.1}$$

over the region Ω= [0, 1] × [0, 1]. The boundary conditions are given by

$$u(0,y) = \cos y \quad , \quad u(1,y) = e^{-1}\cos y, \quad u(x,0) = e^{-x} \quad , \quad u(x,1) = e^{-x}\cos 1. \tag{3.2}$$

The exact solution is $u = e^{-x}\cos y$.

3.1.2. *Model Problem 2 (MP2).*

$$u_{xx} + u_{yy} + D(x,y)u_x + E(x,y)u_y + u = G(x,y) \tag{3.3}$$

where $D(x,y) = \sin x \sin y$, $E(x,y) = \cos x \cos y$ and $G(x,y) = -\sin x \cos y$,

over the region Ω = [0, 1] × [0, 1]. The boundary conditions are given by

$$u(0,y) = 0\,, u(1,y) = \sin 1 \cos y,\ u(x,0) = \sin x\,, u(x,1) = \sin x \cos 1. \tag{3.4}$$

The exact solution is $u = \sin x \cos y$.

In this paper, we use the stopping criteria $\dfrac{\left\| u^{(i)} - \bar{u} \right\|_2}{\left\| \bar{u} \right\|_2} \le \varepsilon$ where $u^{(i)}$ is the approximate solutions at the i^{th} iteration of the iterative method and ε is a preset small tolerance.

As described earlier, the SOR method involves an over-relaxation parameter ω. The rate of convergence of an iterative method is determined by the spectral radius of the iterative matrix. The smaller the spectral radius is, the faster the method converges. For the SOR method, its rate of convergence is very sensitive to the choice of ω. If the uniform grids are placed over a rectangular region, the spectral radius of the iterative matrix of the SOR method is

$$\rho(G_{SOR}) = \begin{cases} \left[\dfrac{\omega\rho(B) + \sqrt{\omega^2\rho^2(B) - 4(\rho(B)-1)}}{2} \right]^2, & if \quad 0 < \omega \le \omega_{opt} \\ \omega - 1 \quad , & if \quad \omega_{opt} \le \omega < 2. \end{cases} \tag{3.5}$$

The spectral radius has an absolute minimum value at the optimal value of ω_{opt} , see Figure 3.1.

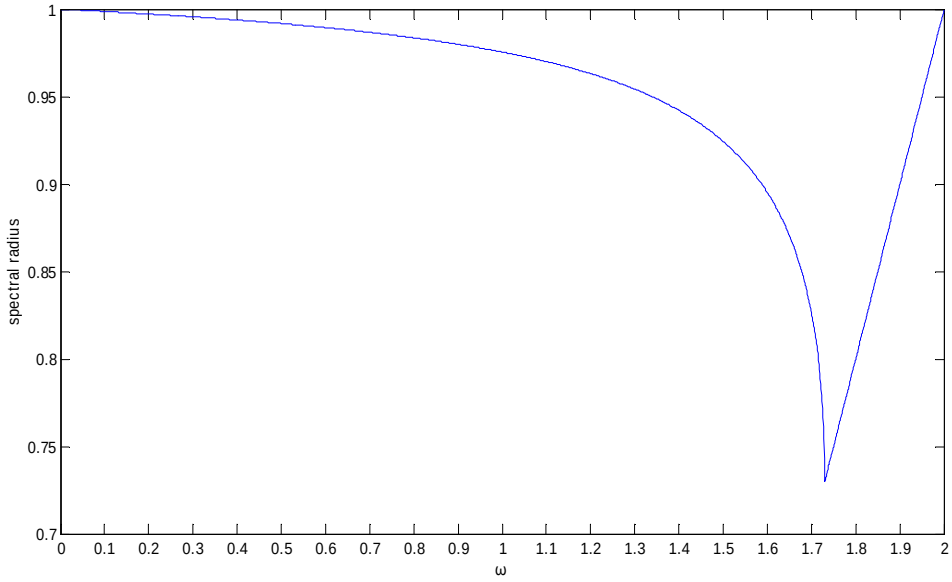

Figure 3.1. Spectral radius vs. ω

The above figure shows the spectral radius versus ω for MP1 with uniform grid size h = 1/20 over the entire domain. The optimal value of ω can be analytically computed by (2.6) in which $\rho(B) = \cos(\pi h)$. The spectral radius curve is very steep around the optimal value of ω. It means that if the value of ω is a little bit off the optimal value, the rate of convergence decreases significantly. For example, when ω = 1.7, the spectral radius of the SOR method is 0.8262 and the rate of convergence is 0.0829. The number of iteration required in this situation to obtain an accuracy of 10^{-6} is 72. When ω = 1.730, the spectral radius is 0.73, and the rate of convergence is 0.137. The number of iteration required to an accuracy of 10^{-6} is 44. We can see that the number of iteration reduced by 40%.

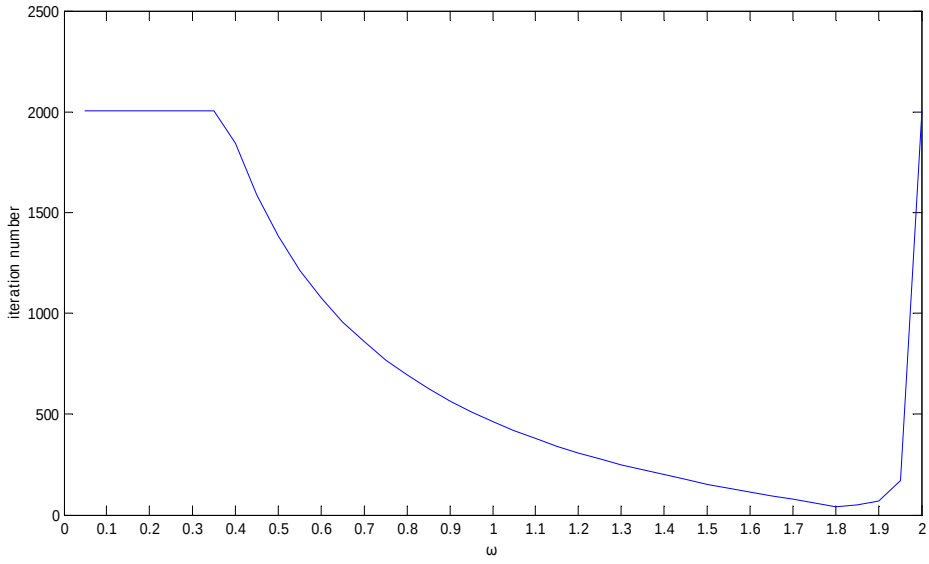

Figure 3.2. Iteration number vs. ω (*H* = 1/20, *h* = 1/40)

For multi-layer grid refinement method the analytical optimal value of ω for the SOR method is not known, a linear search is used to find the optimal value. Numerical experiments are conducted, see figures below, to illustrate similar graphs as in Fig. 3.1 that indicates the importance of getting the optimal value of ω. We set the value of ω from 0 to 2 with a step size 0.05 to obtain the numbers of iteration required. In the two-layer grid refinement scheme of MP1 and MP2, the interested region is placed at [0.4, 0.6]×[0.4, 0.6] with fine grids. Figures 3.2 and 3.3 present the iteration numbers vs. ω for MP1 with different sizes of H and h, where H is the mesh size of the coarse grid and h is the mesh size of the fine grid. The maximum number of iteration has been set to be 2000. Therefore, in the figures, the flat segments at 2000 level indicate that the number of iterations are larger than 2000. It clearly displays the importance of the value of ω.

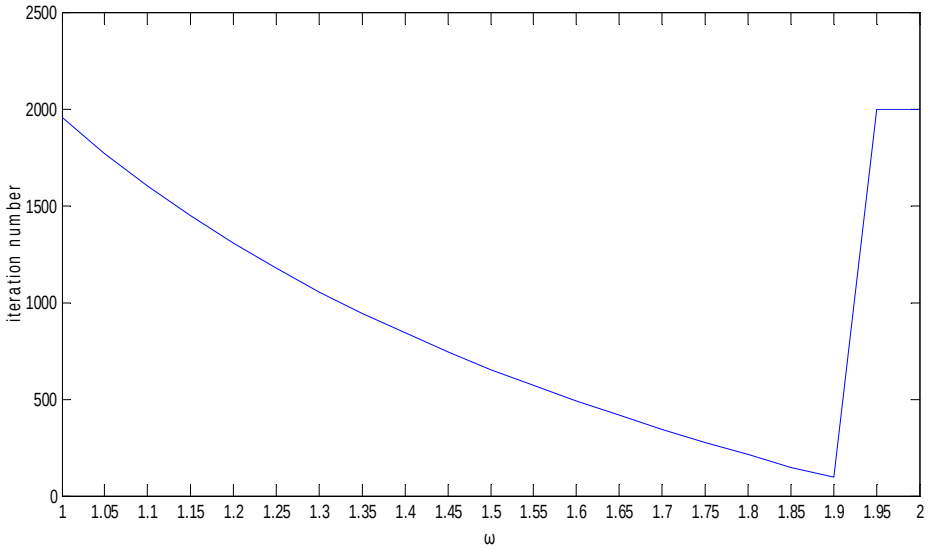

Figure 3.3. Iteration number vs. ω (*H* = 1/40, *h* = 1/80)

For simplicity, we consider two different grid sizes. Since the discretization has been changed, the formula for spectral radius of *B* can no longer be the same as in one uniform grid. Let *H* and *h* be the grid sizes for the coarse grid domain and the fine grid domain respectively, and let *R* be the ratio of the gird sizes *H/h*. As we mentioned in the beginning of this paper, the second layer is placed in the center of the first layer in our research. A heuristic estimation of *ρ(B)* for two different grid sizes is given by

$$\rho(B) = \frac{1}{R\sqrt{R}}\cos(\pi H) + (1 - \frac{1}{R\sqrt{R}})\cos(\pi h) \qquad (3.6)$$

It is logical to estimate that the spectral radius is a linear combination of the spectral radii by the two different discretization. In this paper, we attempt to estimate the optimal value of ω for the SOR method used in multi-layer refinement grid scheme. We propose that the formula (2.6) is still valid in this scheme.

Below are the tables that illustrate the values of *ρ*(*B*) by actual computation via MATLAB, *ρ*(*B*)_*M*, and by the estimation from (3.6), *ρ*(*B*)_*F*. The optimal values of ω are also displayed in three different cases: actual values, *Opt.ω* via linear search, values using (2.6) with the actual *ρ*(*B*)_*M*, *ω*_*M* and values using (2.6) with the estimation of *ρ*(*B*)_*F* , *ω*_*F*. We note that the linear search is conducted with a step size 0.005 over the interval (0, 2).

Table 3.1. Spectral radius and the optimal omega for MP1

H	*h*	*ρ*(*B*)_*M*	*ω*_*M*	Opt.ω	*ω*_*F*	*ρ*(*B*)_*F*
1/10	1/20	0.9724	1.6218	1.6550	1.6346	0.9747
	1/40	0.9914	1.7682	1.7950	1.7662	0.9912
	1/80	0.9978	1.8747	1.8900	1.8586	0.9971
	1/160	0.9994	1.9353	1.9450	1.9144	0.9990
1/20	1/40	0.9927	1.7843	1.8000	1.7984	0.9937
	1/80	0.9976	1.871	1.8850	1.8757	0.9978
	1/160	0.9994	1.9323	1.9400	1.9279	0.9993
1/40	1/80	0.9981	1.8838	1.8900	1.8930	0.9984
	1/160	0.9994	1.9308	1.9400	1.9330	0.9994

Table 3.2. Spectral radius and the optimal omega for MP2

H	*h*	*ρ*(*B*)_*M*	*ω*_*M*	Opt.ω	*ω*_*F*	*ρ*(*B*)_*F*

1/10	1/20	0.9735	1.6279	1.6550	1.6346	0.9747
	1/40	0.9917	1.7720	1.7950	1.7662	0.9912
	1/80	0.9978	1.8768	1.8900	1.8586	0.9971
	1/160	0.9995	1.9364	1.9450	1.9144	0.9990
1/20	1/40	0.9930	1.7882	1.8050	1.7984	0.9937
	1/80	0.9977	1.8732	1.8900	1.8757	0.9978
	1/160	0.9994	1.9335	1.9450	1.9279	0.9993
1/40	1/80	0.9982	1.8860	1.8950	1.8930	0.9984
	1/160	0.9994	1.9320	1.9400	1.9330	0.9994

The tables show very good estimates obtained for various combinations of mesh sizes *H* and *h*. These tables also gave a positive evidence for assuming that (2.6) is still an excellent formula to obtain the optimal value of ω.

For the efficiency in terms of number of iterations required, numerical experiments are also performed using estimated value from our proposed formula. Tables 3.3 and 3.4 below show the iterative numbers when computing MP1 and MP2 in SOR method, using the actual optimal omega "Opt.ω"from linear search and "*ω_F*" by the heuristic *ρ(B)_F* for the over-relaxation parameter, respectively. The ones with *ω_F* yield to very comparable efficiency to those of the actual optimal omegas.

Table 3.3. Iterations numbers in SOR method with different omegas for MP1

H	*h*	*Matrix Size*	*Opt.ω*	*Iteration numbers of opt* ω	ω_F	*Iteration numbers of* ω_*F*
1/10	1/20	97	1.6520	16	1.6346	18
1/10	1/40	153	1.7920	28	1.7662	36
1/20	1/40	417	1.8000	42	1.7984	42
1/40	1/80	1729	1.8900	77	1.8930	79
1/40	1/160	5457	1.9400	140	1.9330	146

Table 3.4. Iterations numbers in SOR method with different omegas for MP2

H	*h*	*Matrix Size*	*Opt.ω*	*Iteration numbers of opt* ω	ω_F	*Iteration numbers of* ω_*F*
1/10	1/20	97	1.6520	19	1.6346	21
1/10	1/40	153	1.7920	31	1.7662	40
1/20	1/40	417	1.8000	35	1.7984	35
1/40	1/80	1729	1.8900	86	1.8930	86
1/40	1/160	5457	1.9400	172	1.9330	186

We need to point out that *ρ*(*B*) will be different from the above estimations when the interested subdomain is located in the positions other than the center of the region. However, it is very reasonable and logical to put the interested region on the center stage. Thus we focus our research on the case where the second layer is placed in the center.

CONCLUSION

In a multi-layer grid refinement environment, the optimal value of the parameter for the robust standalone SOR method is not known. It has been shown earlier in the paper that the optimal value is critical to make the SOR method efficient. This paper introduced an estimation formula so that it could produce excellent estimation in this environment. Numerical studies had been carried out to confirm the results.

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