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Surface waves in non-homogeneous fibre-reinforced anisotropic elastic media with voids

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ABSTRACT

This paper investigates the propagation of surface waves in a non homogeneous fibre-reinforced elastic media with voids. The general surface wave speed is derived to study the effects of voids on surface waves in non homogeneous fibre-reinforced elastic solid and discussed its particular cases for Stoneley, Love and Rayleigh waves. The results obtained in this investigation are more general in the sense that some earlier published results are obtained from our result as special cases. Also by neglecting non homogeneity and the reinforced elastic parameters, the results reduce to well known isotropic medium.

Key words: Surface waves, non-homogeneous, fiber reinforced, anisotropic, voids.

INTRODUCTION

There are many types of surface waves [1-6] but we only focus on Stoneley, Love and Rayleigh waves. In earthquake the movement is due to the surface waves. These are also used for detecting cracks and other defects in materials. Lord Rayleigh [3] was the first to observe such kind of waves in 1885. That's why we called it Rayleigh waves. Sengupta and Nath [7] investigated surface waves in fibre-reinforced anisotropic elastic media but their decomposition of displacement vector was not correct that's why some errors are found in their investigations [8].

The idea of continuous self-reinforcement at every point of an elastic solid was introduced by Belfield [9]. The superiority of fibre-reinforced composite materials over other structural materials attracted many authors to study different type of problems in this field. Fibre-reinforced composite structures are used due to their low weight and high strength. Two important components namely concrete and steel of a reinforced medium are bound together as a single unit so that there can be no relative displacement between them i.e. they act together as a single anisotropic unit. The artificial structures on the surface of the earth are excited during an earthquake, which give rise to violent vibrations in some cases [10, 11]. Engineers and architects are in search of such reinforced elastic materials for the structures that resist the oscillatory vibration. The propagation of waves depends upon the ground vibration and the physical properties of the structure material. Kakar et al. [12-16] discussed surface wave propagation in non homogeneous media.

In classical theory of elasticity, the voids is an important generalization. Nunziato and Cowin [17] and Cowin and Nunziato [18] discusse the theory in elastic media with voids. Puri and Cowin [19] studied the effects of voids on plane waves in linear elastic media and it is evident that pure shear waves remain unaffected by the presence of pores. Chandrasekharaiah [20] and [21] discussed the effects of voids on propagation of surface and plane waves respectively.

Aim of this paper is to investigate the propagation of surface waves in a non homogeneous fibre-reinforced elastic media with voids. The general surface wave speed is derived to study the effect of voids on surface waves. Particular

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cases for Stonely, Love and Rayleigh waves are discussed. The results obtained in this investigation are more general in the sense that some earlier published results are obtained from our result as special cases. For homogeneous medium our results are well agreement to fibre-reinforced materials. It is also observed that the corresponding classical results follow from this analysis, in homogeneous media, by neglecting reinforced parameters. Results for homogeneous media can be deduced from this investigation.

2. Formulation of the Problem:

Medium is consisting of two non-homogeneous anisotropic fibre-reinforced semi-infinite elastic solid media M_1 and M_2 with different elastic and reinforcement parameters. The non-homogeneity of the material is depending on the space variable. It is assumed that non-homogeneity grows or decays slowly. Its rate of growth or decay is proportional to its value at that point i.e.

$\frac{d\lambda}{dx_2} \propto \lambda;$ where λ is an elastic parameter. This implies

$$\frac{d\lambda}{dx_2} = m\lambda,$$

where m is a constant, which is positive for non-homogeneity growth and negative for decay.
Above equation implies

$$\lambda = \lambda_o e^{m x_2}$$

For m =0, $\lambda = \lambda_o$, Thus for $m = 0$, the medium is homogeneous.
The two media are perfectly welded in contact at a plane interface. Let us take orthogonal Cartesian axes $ox_1x_2x_3$ with the origin at o . ox_2 is pointing vertically upwards into the medium M ($x_2 \geq 0$). Each of the media M ($x_2 \geq 0$) and M_1 ($x_2 \leq 0$) separated at x_2 =0. Both media are rotating about an axis.

It is assumed that the waves travel in the positive direction of the x_1-axis and at any instant, all particles have equal displacements in any direction parallel to ox_3. In view of that assumptions, the propagation of waves will be independent of x_3 .
In the presence of voids, the general equuation for a fibre-reinforced linearly elastic anisotropic media w.r.t. a direction $\bar{a} \equiv (a_1, a_2, a_3)$ is as under [7, 11] .

$$\tau_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu_T\varepsilon_{ij} + \alpha(a_k a_m \varepsilon_{km}\delta_{ij} + \varepsilon_{kk}a_i a_j) + 2(\mu_L - \mu_T)(a_i a_k \varepsilon_{kj} + a_j a_k \varepsilon_{ki})$$
$$+ \beta(a_k a_m \varepsilon_{km} a_i a_j) + \beta\delta_{ij}\phi,$$

Where, τ_{ij} are components of stress and strain tensor is $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ -and λ, μ_T are elastic parameters. α, β and $(\mu_L - \mu_T)$ are reinforced anisotropic elastic parameters, u_i are the displacement vectors components.
In the absence of body forces, the field equations may be taken as follows:

$$\tau_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu_T\varepsilon_{ij} + \alpha(a_k a_m \varepsilon_{km}\delta_{ij} + \varepsilon_{kk}a_i a_j) + 2(\mu_L - \mu_T)(a_i a_k \varepsilon_{kj} + a_j a_k \varepsilon_{ki})$$
$$+ \beta(a_k a_m \varepsilon_{km} a_i a_j) + \xi\delta_{ij}\phi,$$

$$\tau_{ij,j} = \rho\ddot{u}_i$$

$$\alpha\phi_{,ii} - \omega_0\phi - \varpi\dot{\phi} - \xi u_{i,i} = \rho\kappa\ddot{\phi}$$

In these equations, ϕ is the so-called volume fraction field. $\alpha, \beta, \omega_0, \varpi$ and κ are new material constants characterizing the presence of voids. Where ε_{ijk} is the Levi-Civita tensor, τ_{ij} are components of stress, ρ is the mass density and u_i is the displacement vector. Comma followed by index shows partial derivative with respect to coordinate. Also Einstein summation convention over repeated indexes is used.

The propagation equations of small elastic disturbances are as follows.

In component form, the equation of motion in the presence of voids becomes

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$$\left.\begin{aligned} \tau_{11,1}+\tau_{12,2}+\xi\phi_{,1}&=\rho\ddot{u}_1, \\ \tau_{21,1}+\tau_{22,2}+\xi\phi_{,2}&=\rho\ddot{u}_2, \\ \tau_{31,1}+\tau_{32,2}&=\rho\ddot{u}_3. \end{aligned}\right\} \tag{2.1}$$

$$\alpha\left(\phi_{,11}+\phi_{,22}\right)-\omega_0\phi-\varpi\dot{\phi}-\xi\left(u_{1,1}+u_{2,2}\right)=\rho\kappa\ddot{\phi}$$

In the present problem we consider exponentially decaying non-homogeneous material. Hence density, elastic module and elastic parameters may be taken in the following form.

$$\rho=\rho_o\, e^{-mx_2}$$

$$\lambda=\lambda_0\, e^{-mx_2} \qquad \mu_T=\mu_{T_0}e^{-mx_2}$$

$$\alpha=\alpha_0 e^{-mx_2} \qquad \mu_L=\mu_{L0}e^{-mx_2}$$

$$\beta=\beta_0 e^{-mx_2} \qquad \mu_T=\mu_{T_0}e^{-mx_2}$$

We choose the fibre direction as $\bar{a}$ = (1, 0, 0).

Now using the above equations and taking all derivatives w.r.t. x_3 zero. The equation (2.1a) becomes

$$(\lambda+2\alpha+4\mu_L-2\mu_T+\beta)u_{1,11}+(\alpha+\lambda+\mu_L)u_{2,21}+\mu_L u_{1,22}-m\mu_T\left(u_{1,2}+u_{2,1}\right)+\xi\phi_{,1}=\rho\ddot{u}_1, \tag{2.2a}$$

similarly equations (2.1b) and (2.1c) takes the following form

$$(\alpha+\lambda+\mu_L)u_{1,12}+\mu_T u_{2,11}+(\lambda+2\mu_T)u_{2,22}-m\left(\lambda+\alpha\right)u_{1,1}-m\left(\lambda+2\mu_T\right)u_{2,2}+\xi\phi_{,2}=\rho\ddot{u}_2, \tag{2.2b}$$

$$\mu_L u_{3,11}+\mu_T u_{3,22}-m\mu_T u_{3,2}=\rho\ddot{u}_3, \tag{2.2c}$$

$$\alpha\left(\phi_{,11}+\phi_{,22}\right)-\omega_0\phi-\varpi\dot{\phi}-\xi\left(u_{1,1}+u_{2,2}\right)=\rho\kappa\ddot{\phi}. \tag{2.2d}$$

Similarly, we can get similar relations in M_1 with ρ $\alpha,\lambda,$ μ_T and β are replaced by $\rho',\ \alpha',\lambda',\mu_T'$ and β'.

3. Solution of the Problem

We seek harmonic solutions in the form

$$u_1,u_2,u_3=\left(\hat{u}_1(x_2),\hat{u}_2(x_2),\hat{u}_3(x_2)\right)\exp\left[i\omega(x_1-ct)\right],$$

Thus equations (2.2a, b) of motion becomes

$$\left[\mu_L D^2-m\mu_T D+\omega^2\left\{\rho c^2-(\lambda+2\alpha+4\mu_L-2\mu_T+\beta)\right\}\right]\hat{u}_1+ i\omega\left[(\alpha+\lambda+\mu_L)D-m\mu_T\right]\hat{u}_2+i\omega\xi\hat{\phi}=0 \tag{3.1a}$$

$$\left[(\lambda+2\mu_T)D^2-m(\lambda+2\mu_T)D+\omega^2\left\{\rho c^2-\mu_L\right\}\right]\hat{u}_2 +i\omega\left[(\alpha+\lambda+\mu_L)D-m(\alpha+\lambda)\right]\hat{u}_1+\xi D\hat{\phi}=0. \tag{3.1b}$$

$$\mu_L u_{3,11}+\mu_T u_{3,22}-m\mu_T u_{3,2}=\rho\ddot{u}_3, \tag{3.1c}$$

$$\left\{\alpha\left(D^2-\omega^2\right)-\omega_0+i\omega c\varpi+\omega^2c^2\rho\kappa\right\}\hat{\phi}-\xi\left(i\omega\hat{u}_1+D\hat{u}_2\right)=0 \tag{3.1d}$$

Where, $D=\dfrac{d}{dx_2}$

Similarly we can get similar relations in M_1 with ρ α μ_T and β are replaced by $\rho',\ \alpha',\ \mu_T'$ and β'.

Thus coupled equations (3.1a,b,d) becomes

$$(\hbar_1 D^2-m\hbar_5 D-\omega^2\hbar_3+\omega^2\rho c^2)\hat{u}_1+i\omega(\hbar_2 D-m\hbar_5)\hat{u}_2+i\omega\xi\hat{\phi}=0\ ,$$

$$(\hbar_4 D^2 - m\hbar_4 D - \omega^2 \hbar_1 + \omega^2 \rho c^2)\hat{u}_2 + i\omega(\hbar_2 D - m(\hbar_2 - \hbar_1))\hat{u}_1 + \xi D\hat{\phi} = 0 \ ,$$

$$\left\{\alpha\left(D^2 - \omega^2\right) - \omega_0 + i\omega c\varpi + \omega^2 c^2 \rho\kappa\right\}\hat{\phi} - \xi\left(i\omega\hat{u}_1 + D\hat{u}_2\right) = 0 \ ,$$

and uncoupled equation (3.1c) becomes

$$\left\{\hbar_5 D^2 - m\hbar_5 D - \omega^2(\hbar_1 - \rho c^2)\right\}\hat{u}_3 = 0$$

where

$$\hbar_1 = \mu_L, \ \hbar_2 = (\alpha + \lambda + \mu),$$

$$\hbar_3 = (\lambda + 2\alpha + 4\mu - 2\mu + \beta) \ ,$$

$$\hbar_4 = (\lambda + 2\mu) \ \ and \ \ \hbar_5 = \mu_T$$

The uncoupled equation has the following solution,

$$u_3 = \left(Ee^{-\eta_1 \omega x_2} + E_1 e^{-\eta_2 \omega x_2}\right)e^{i\omega(x_1 - ct)},$$

where η_1 and η_2 are roots of the equation $\hbar_5\eta^2 - m\hbar_5\eta - \omega^2(\hbar_1 - \rho c^2) = 0$.

$$\eta_{1,2} = \frac{1}{2}\left(m \pm \sqrt{m^2 + \frac{4\left(\hbar_1 - \rho c^2\right)}{\hbar_5}}\right)$$

For positive real root η_1, it is necessary that $0 < 4\rho c^2 < \hbar_5 m^2 + 4\hbar_1$ and in the homogeneous medium $0 < \rho c^2 < \hbar_1$ otherwise transverse component does not exist. For boundedness

$$u_3 = Ee^{-\eta_1 \omega x_2} \ exp\left\{i\omega(x_1 - ct)\right\},$$

Above set of coupled equations can be written as

$$\left.\begin{aligned}
&(\hbar_1 D^2 - m\hbar_5 D - A_1)\hat{u}_1 + i\omega(\hbar_2 D - m\hbar_5)\hat{u}_2 + i\omega\xi\hat{\phi} = 0\\
&(\hbar_4 D^2 - m\hbar_4 D - A_2)\hat{u}_2 + i\omega(\hbar_2 D - m(\hbar_2 - \hbar_1))\hat{u}_1 + \xi D\hat{\phi} = 0\\
&(D^2 - A_3)\hat{\phi} - \xi(i\omega\hat{u}_1 + D\hat{u}_2) = 0
\end{aligned}\right\} \tag{3.2}$$

where

$$A_1 = \omega^2 \hbar_3 - \omega^2 \rho c^2$$

$$A_2 = \omega^2 \hbar_1 - \omega^2 \rho c^2$$

$$A_3 = \omega^2 + \frac{\omega_0 - i\omega c\varpi - \omega^2 c^2 \rho\kappa}{\alpha}$$

From above set of equations, we have

$$\begin{vmatrix}
(\hbar_1 D^2 - m\hbar_5 D - A_1) & i\omega(\hbar_2 D - m\hbar_5) & i\omega\xi\\
i\omega(\hbar_2 D - m(\hbar_2 - \hbar_1)) & (\hbar_4 D^2 - m\hbar_4 D - A_2) & \xi D\\
-i\omega\xi & -\xi D & (D^2 - A_3)
\end{vmatrix}(\hat{u}_1, \hat{u}_2, \hat{\phi}) = 0$$

This implies

$$(D^6 + C_1 D^5 - C_2 D^4 - C_3 D^3 + C_4 D^2 + C_5 D - C_6)(\hat{u}_1, \hat{u}_2, \hat{\phi}) = 0$$

where

$$C_1 = \frac{m\hbar_4}{\hbar_1 \hbar_4}(\hbar_2 - \hbar_1)$$

$$C_2 = \frac{1}{\hbar_1\hbar_4}\left(\hbar_4 A_1 + \hbar_1(A_2 + \hbar_4 A_3 - \xi^2) - \omega^2\hbar_2^2 - m\hbar_4\hbar_5\right)$$

$$C_3 = \frac{m}{\hbar_1\hbar_4}\left\{\hbar_2\left(\xi^2 - A_2 - \hbar_4 A_3\right) - \hbar_4 A_1 - \hbar_2\left(\hbar_1 - \hbar_2\right) + \omega^2\hbar_2^2 - \hbar_1\hbar_4 A_3\right\}$$

$$C_4 = \frac{1}{\hbar_1\hbar_4}\left\{\begin{array}{l}\left(A_1A_2 + \hbar_4 A_1 A_3 + A_2A_3 - \hbar_2\omega^2 A_3\right) - \xi^2(A_1 + 2\omega^2\hbar_2 + \hbar_4\omega^2) \\ -\omega^2\left(m\hbar_5\right)\left(m\hbar_1 - m\hbar_2\right)\end{array}\right\}$$

$$C_5 = \frac{m}{\hbar_1\hbar_4}\left(\hbar_1 + \hbar_2\right)\hbar_4$$

$$C_6 = \frac{1}{\hbar_1\hbar_4}\left(A_1A_2A_3 - \omega^2 A_2\xi^2 - m^2\hbar_5\left(\hbar_2 - \hbar_1\right)\right).$$

For homogeneous medium, m = 0, this implies $C_1 = C_3 = C_5 = 0$ and C_2 , C_4 and C_6 must be positive for real positive roots. If there are also no voids then the above equation is easy to solve.

Let α_i , i=1,2,…6 be six positive real roots, then solution by normal mode method has the following form

$$\hat{u}_1 = \sum_{n=1}^{6} M_n\, e^{-\alpha_n x_2}, \tag{3.3a}$$

$$\hat{u}_2 = \sum_{n=1}^{6} M_{1n}\, e^{-\alpha_n x_2}, \tag{3.3b}$$

$$\hat{\phi} = \sum_{n=1}^{6} M_{2n}\, e^{-\alpha_n x_2}, \tag{3.3c}$$

where M_n , M_{1n} and M_{2n} , are some parameters depending on c and ω. By using Eqs. (3.2a-c) into Eqs. (3.2), we get the following relations,

$$M_{1n} = H_{1n}M_n$$

$$M_{2n} = H_{2n}M_n$$

where

$$H_{1n} = \frac{i\omega\left(A_2 + (\hbar_2 - \hbar_4)\alpha_n^2\right) + m(\hbar_2 - \alpha_n\hbar_4)}{\hbar_1\alpha_n^3 + (\hbar_2\omega^2 - A_1)\alpha_n + m\hbar_2\alpha_n^2 + m\omega(\hbar_2 - \hbar_1)},$$

$$H_{2n} = \frac{\alpha_n^2 - A_3}{\xi(\alpha_n H_{1n} - i\omega)} \qquad n = 1, 2, 3, 4, 5, 6.$$

Hence we obtain the expressions of the displacement components function and stresses as follows

$$u_1 = \sum_{n=1}^{6} M_n\, e^{-\alpha_n x_2}\, exp\left\{i\omega(x_1 - ct)\right\},$$

$$u_2 = \sum_{n=1}^{6} H_{1n}M_n\, e^{-\alpha_n x_2}\, exp\left\{i\omega(x_1 - ct)\right\},$$

$$u_3 = Ee^{-\eta_1\omega x_2}\, exp\left\{i\omega(x_1 - ct)\right\},$$

$$\phi = \sum_{n=1}^{6} H_{2n}M_n\, e^{-\alpha_n x_2}\, exp\left\{i\omega(x_1 - ct)\right\}.$$

Also it is found that

$$\tau_{12} = \sum_{n=1}^{6} \hbar_1\left(-\alpha_n + i\omega H_{1n}\right)M_n e^{-\alpha_n x_2}\, exp\left\{i\omega(x_1 - ct)\right\}$$

$$\tau_{22}=\sum_{n=1}^{6}\{i\omega(\hbar_2-\hbar_1)-\hbar_4\alpha_n H_{1n}+\xi H_{2n}\}M_n e^{-\alpha_n x_2}\, exp\{i\omega(x_1-ct)\}$$

$$\tau_{23}=-\eta\omega E\hbar_5\, e^{-\eta_1\omega x_2}\, exp\{i\omega(x_1-ct)\}.$$

Similar expressions can be obtained for second mediun and present them with dashes as follows

$$u_1'=\sum_{n=1}^{6}M_n'\, e^{-\alpha_n' x_2}\, exp\{i\omega(x_1-ct)\},$$

$$u_2'=\sum_{n=1}^{6}H_{1n}'M_n'\, e^{-\alpha_n' x_2}\, exp\{i\omega(x_1-ct)\},$$

$$u_3'=Fe^{-\eta_1'\omega x_2}\, exp\{i\omega(x_1-ct)\},$$

$$\phi'=\sum_{n=1}^{6}H_{2n}'M_n'\, e^{-\alpha_n' x_2}\, exp\{i\omega(x_1-ct)\}.$$

Also it is found that

$$\tau_{12}'=\sum_{n=1}^{6}\hbar_1'\left(-\alpha_n'+i\omega H_{1n}'\right)M_n' e^{-\alpha_n' x_2}\, exp\{i\omega(x_1-ct)\},$$

$$\tau_{22}'=\sum_{n=1}^{6}\{i\omega(\hbar_2'-\hbar_1')-\hbar_4'\alpha_n' H_{1n}'+\xi' H_{2n}'\}M_n' e^{-\alpha_n' x_2}\, exp\{i\omega(x_1-ct)\},$$

$$\tau_{23}'=-\eta'\omega F\hbar_5 e^{-\eta'\omega x_2}\, exp\{i\,\omega(x_1-ct)\}.$$

In order to determine the secular equations, we have the following boundary conditions.

4. Boundary conditions

I. The displacement components and their rate of change w.r.t. x_2 , between the mediums are continuous, i.e. $u_1=u_1',\ u_2=u_2',\ u_3=u_3',\ \phi=\phi'$, $u_{1,2}=u_{1,2}',\ u_{2,2}=u_{2,2}',\ u_{3,2}=u_{3,2}'$ and $\phi_{,2}=\phi_{,2}'$ on $x_2=0$, for all x_1 and *t*.

II. Stress and their deritive w.r.t x_2 are continuous , i.e. $\tau_{12}=\tau_{12}',\ \tau_{22}=\tau_{22}',\ \tau_{23}=\tau_{23}'$, $\tau_{12,2}=\tau_{12,2}',\ \tau_{22,2}=\tau_{22,2}',\ \tau_{23,2}=\tau_{23,2}'$ also $\tau_{11,2}=\tau_{11,2}'$, $\tau_{13,2}=\tau_{13,2}'$ and $\tau_{33,2}=\tau_{33,2}'$ on $x_2=0$, for all x_1 and *t*

Boundary conditions implies the following equations

$$\left.\begin{aligned}&\sum_{n=1}^{6}M_n=\sum_{n=1}^{6}M_n'\\&\sum_{n=1}^{6}H_{1n}M_n=\sum_{n=1}^{6}H_{1n}'M_n'\\&E=F\\&\sum_{n=1}^{6}H_{2n}M_n=\sum_{n=1}^{6}H_{2n}'M_n'\end{aligned}\right)$$

$$\sum_{n=1}^{6}\alpha_n M_n=\sum_{n=1}^{6}\alpha_n' M_n'$$

$$\sum_{n=1}^{6}\alpha_n H_{1n}M_n=\sum_{n=1}^{6}\alpha_n' H_{1n}'M_n'$$

$$\eta_1 E=\eta_1' F$$

$$\sum_{n=1}^{6}\alpha_n H_{2n}M_n=\sum_{n=1}^{6}\alpha_n' H_{2n}'M_n'$$

$$\sum_{n=1}^{6} \hbar_1 \left(-\alpha_n + i\omega H_{1n}\right) M_n = \sum_{n=1}^{6} \hbar_1' \left(-\alpha_n' + i\omega H_{1n}'\right) M_n' ,$$

$$\sum_{n=1}^{6} \{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 \alpha_n H_{1n} + \xi H_{2n} \} M_n = \sum_{n=1}^{6} \{ i\omega(\hbar_2' - \hbar_1') - \hbar_4' \alpha_n' H_{1n}' + \xi' H_{2n}' \} M_n' ,$$

$$\hbar_5 \eta_1 E = \hbar_5' \eta_1' F ,$$

$$\sum_{n=1}^{6} \hbar_1 \alpha_n \left(-\alpha_n + i\omega H_{1n}\right) M_n = \sum_{n=1}^{6} \hbar_1' \alpha_n' \left(-\alpha_n' + i\omega H_{1n}'\right) M_n' ,$$

$$\sum_{n=1}^{6} \alpha_n \{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 \alpha_n H_{1n} + \xi H_{2n} \} M_n = \sum_{n=1}^{6} \alpha_n' \{ i\omega(\hbar_2' - \hbar_1') - \hbar_4' \alpha_n' H_{1n}' + \xi' H_{2n}' \} M_n' ,$$

$$\hbar_5 \eta_1^2 E = \hbar_5' \eta_1'^2 F ,$$

$$\sum_{n=1}^{6} \alpha_n \{ \left(i\omega\hbar_3 - \alpha_n (\hbar_2 - \hbar_1) H_{1n} \right) \} M_n = \sum_{n=1}^{6} \alpha_n' \{ \left(i\omega\hbar_3' - \alpha_n' (\hbar_2' - \hbar_1') H_{1n}' \right) \} M_n' ,$$

$$\sum_{n=1}^{6} \alpha_n \{ i\omega(\hbar_2 - \hbar_1) - (\hbar_4 - 2\hbar_5) \alpha_n H_{1n} \} M_n = \sum_{n=1}^{6} \alpha_n' \{ i\omega(\hbar_2' - \hbar_1') - (\hbar_4' - 2\hbar_5') \alpha_n' H_{1n}' \} M_n' ,$$

$$\xi \alpha_n H_{2n} M_n = \xi' \alpha_n' H_{2n}' M_n' ,$$

From above set of equations, the four equations containing *E* and *F* implies that *E* = *F* = *0*. From remaining twelve equations, for non-trivial solution we have

$$\det\left(a_{ij}\right) = 0, \quad i = j = 1, 2, \ldots, 12. \tag{4.1}$$

Where

$a_{1p} = 1, \quad a_{1p+6} = -1, \quad ; a_{2p} = H_{1p}; \quad a_{2p+6} = -H_{1p}';$

$a_{3p} = H_{2p}; \; a_{3p+6} = -H_{2p}'$

$a_{4p} = \alpha_p \; ; \quad a_{4p+6} = -\alpha_p',$

$a_{5p} = \alpha_p H_{1p} \; ; \; a_{5p+6} = -\alpha_p' H_{1p}',$

$a_{6p} = \alpha_p H_{2p} \; ; \; a_{6p+6} = -\alpha_p' H_{1p}',$

$p = 1, 2, \ldots 6;$

$a_{7p} = \hbar_1 \left(-\alpha_p + i\omega H_{1p}\right), \; p = 1, 2, \ldots, 6;$

$a_{7q} = -\hbar_1' \left(-\alpha_q' + i\omega H_{1q}'\right), \; q = 7, 6, \ldots, 12.$

$a_{8p} = \left\{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 \alpha_p H_{1p} + \xi \alpha_n H_{2n} \right\}, \; p = 1, 2, \ldots, 6;$

$a_{8q} = -\left\{ i\omega(\hbar_2' - \hbar_1') - \hbar_4' \alpha_q' H_{1q}' + \xi \alpha_n H_{2n} \right\}, \; q = 7, 6, \ldots, 12.$

$a_{9p} = \alpha_p \{ \left(i\omega\hbar_3 - \alpha_n (\hbar_2 - \hbar_1) H_{1p} \right) \}, \;\; p = 1, 2, \ldots, 6;$

$a_{9q} = \alpha_q' \{ \left(i\omega\hbar_3' - \alpha_q' (\hbar_2' - \hbar_1') H_{1q}' \right) \}, \; q = 7, 8, \ldots, 12.$

$a_{10p} = \alpha_p \left\{ i\omega(\hbar_2 - \hbar_1) - (\hbar_4 - 2\hbar_5) \alpha_p H_{1p} \right\}, \quad p = 1, 2, \ldots, 6.$

$a_{10q} = \alpha_p' \left\{ i\omega(\hbar_2' - \hbar_1') - (\hbar_4' - 2\hbar_5') \alpha_p' H_{1p}' \right\}, \; q = 7, 8, \ldots, 12.$

$a_{12p} = \xi \alpha_p H_{2p} \; , \;\; p = 1, 2, \ldots, 6.$

$a_{12q} = -\xi \alpha_q H_{2q} \, , \;\; q = 1, 2, \ldots, 6.$

5. Particular cases

5.1 Stoneley waves

Equation (4.1) is the secular equation for Stonely waves [4].

5.2:- Love waves

To investigate the rotational effects on Love waves in a fibre reinforced viscoelastic media of higher order, we replace medium M_1 by an infinitely extended horizontal plate of finite thickness d and bounded by two horizontal plane surfaces $x_2 = 0$ and $x_2 = d$. Medium M is semi infinite as in the general case.

The boundary conditions of Love wave are as follows

The displacement component u_3 and τ_{12} between the mediums are continuous, i.e.

$u_3 = u_3'$ and $\tau_{23} = \tau_{23}'$ on $x_2 = 0$

$\tau_{23}' = 0$ on $x_2 = d$, for all x_1 and *t*,

where

$$u_3 = E e^{-\eta_1 \omega x_2} e^{i\omega(x_1 - ct)},$$

$$u_3' = E' e^{\eta_1' \omega x_2} e^{i\omega(x_1 - ct)} + F' e^{-\eta_1' \omega x_2} e^{i\omega(x_1 - ct)},$$

This implies

$$E - E' - F' = 0,$$

$$\mu_T \eta_1 E + \mu_T' \eta_1' E' - \mu_T' \eta_1' F' = 0,$$

$$e^{\omega \eta_1 d} E' - e^{-\omega \eta_1 d} F' = 0.$$

For non trivial solution implies

$$\begin{vmatrix} 1 & -1 & -1 \\ \hbar_5 \eta_1 & \hbar_5' \eta_1' & -\hbar_5' \eta_1' \\ 0 & e^{\omega \eta_1' d} & -e^{-\omega \eta_1' d} \end{vmatrix} = 0,$$

This gives the wave velocity of Love waves propagating in a fiber-reinforced medium. It is interesting to note that voids and non-homogeneity does not affect the velocity of Love waves.

5.3 Rayleigh waves

Rayleigh wave is a special case of the above general surface wave. In this case we consider a model where the medium M_2 is replaced by vacuum. Since the boundary $x_2 = 0$ is adjacent to vacuum. It is free from surface traction. So the stress boundary condition in this case may be expressed as

$\tau_{12} = 0$, $\tau_{22} = 0$, $\tau_{12,2} = 0$, $\tau_{22,2} = 0$, also $\tau_{11,2} = 0$ and $\tau_{33,2} = 0$ on $x_2 = 0$, for all x_1 and *t*

Thus above set of equations reduces to

$$\sum_{n=1}^{6} \hbar_1 (-\alpha_n + i\omega H_{1n}) M_n = 0,$$

$$\sum_{n=1}^{6} \{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 \alpha_n H_{1n} \} M_n = 0,$$

$$\sum_{n=1}^{6} \hbar_1 \alpha_n (-\alpha_n + i\omega H_{1n}) M_n = 0,$$

$$\sum_{n=1}^{6} \alpha_n \{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 \alpha_n H_{1n} - \xi \alpha_n H_{2n} \} M_n = 0,$$

$$\sum_{n=1}^{6} \alpha_n \{ (i\omega \hbar_3 - \alpha_n (\hbar_2 - \hbar_1) H_{1n}) - \xi \alpha_n H_{2n} \} M_n = 0,$$

$$\sum_{n=1}^{6} \alpha_n \{ i\omega(\hbar_2 - \hbar_1) - (\hbar_4 - 2\hbar_5) \alpha_n H_{1n} - \xi \alpha_n H_{2n} \} M_n = 0,$$

For non trival solution

$$\det(E_{mn}) = 0, \quad n=1,2,..,6 \qquad (5.1)$$

where

$$E_{1n} = \hbar_1\left(-\alpha_n + i\omega H_{1n}\right),$$

$$E_{2n} = \left\{i\omega(\hbar_2 - \hbar_1) - \hbar_4\alpha_n H_{1n}\right\}$$

$$E_{3n} = \hbar_1\alpha_n\left(-\alpha_n + i\omega H_{1n}\right)$$

$$E_{4n} = \alpha_n\left\{i\omega(\hbar_2 - \hbar_1) - \hbar_4\alpha_n H_{1n} - \xi\alpha_n H_{2n}\right\}$$

$$E_{5n} = \alpha_n\{\left(i\omega\hbar_3 - \alpha_n(\hbar_2 - \hbar_1)H_{1n} - \xi\alpha_n H_{2n}\right)\}$$

$$E_{6n} = \alpha_n\left\{i\omega(\hbar_2 - \hbar_1) - (\hbar_4 - 2\hbar_5)\alpha_n H_{1n} - \xi\alpha_n H_{2n}\right\}$$

Equation (5.1) is the secular equation for Rayleigh wave.

DISCUSSION AND CONCLUSION

Very few researchers did work in that field because of complicated nature of the governing equations of the fibre-reinforced anisotropic with voids. The method used in this study provides a quite successful in dealing with such problems. This method gives exact solutions in the fibre-reinforced anisotropic elastic media without any assumption. Special cases only for Stonely, Love and Rayleigh waves were considered.

It is observed that in the case of homogeneous and without voids only one mode propagate in the medium but in the case of non-homogeneous without voids two mode exist. In the case of homogeneous with voids, three modes exist. In non-homogeneous with voids maximum six modes can propagate and it is depend upon the nature of material.

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