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Stability analysis of some epidemic models with vertical transmission and different incidences

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ABSTRACT

Korobeinikov and Wake [9] introduced a family of Lyapunov functions for three-compartmental epidemiological models which appear to be useful for more sophisticated models. In this paper we have reinvestigated the models of Korobeinikov and Wake [9] with different incidences. The basic reproduction number $\Re_0$ is identified and local stability of the equilibrium states is discussed. The Global stability of the equilibrium states is proved by constructing a Lyapunov function. Some numerical simulations are given to illustrate the analytical results.

Keywords: Infectious disease, Epidemiological model, Equilibrium, Stability, Lyapunov function.
AMS Subject Classification: 92D30.

INTRODUCTION

Population sizes of individuals are affected not only by ecological interactions such as competition, predation, and parasitism, but also by the effects of infectious diseases [6, 7]. Infectious diseases are said to be of SIS type if individuals have no immunity after an infection, so that susceptible move to the infective class when infected and then back to the susceptible class after recovery. If there is temporary immunity in a recovered class after an infection, then the disease is of SIRS type. An SIR model is a special case of an SIRS model in which the immunity is permanent, so that recovered individuals never lose their immunity.The incidence in an epidemiological model is the rate at which susceptible become infectious. The form of the incidence rate that is used in the classical Kermack-Mckendrick model [8] is the simple mass action λSI where S and I denote the number of susceptible and infectious, respectively, λ is called the infection coefficient. The standard incidence is $\frac{\lambda SI}{N}$ where N is the total population size and λ is called the daily contact rate. Another kind of incidence is the saturation incidence $\frac{\lambda SI}{(c+S)}$ where c is a constant. When the number of susceptible S is large compared to c that incidence is approximately I. This kind of incidence was proposed by Anderson and May [1]. Many researchers Esteva and Matias [5], Hethcote and Levin [7], Liu et al. [11] have proposed transmission laws in which the nonlinearities are more than quadratic. Several different incidence rates have been proposed by researchers in epidemic models. Different models for mutually intersecting species are also studied [12, 14]. A model of prey-predator with a generalized transmission function for unsaturated zone has been analyzed by Mehta et. al. [13]. A model for Hepatitis C with saturated chronic infection rate has been studied by Ujjainkar et. al. [15]. One of the basic and important research subjects in mathematical epidemiology is the global stability of the equilibrium states of the epidemic models. Generally, an epidemic model admits two types of equilibrium states. The first one is the disease-free equilibrium state E_0, whose global stability means biologically

that the disease always dies out. The second one is the endemic equilibrium state E^*. Epidemiologically, if E^* is globally asymptotically stable, the disease will persist at the endemic equilibrium level if it is initially present. Korobeinikov and Wake [9] introduced a family of Lyapunov functions for three-compartmental epidemiological models and global stability of the endemic equilibrium states is proved. In this paper we have reanalyzed the epidemic models of Korobeinikov and Wake [9] with vertical transmission and different incidences.

2. SIS model with vertical transmission and simple mass action incidence.

Let S be the number of susceptible and I be the number of infective individuals with size N = S + I. Following Korobeinikov and Wake [9], the proposed model is

$$\begin{aligned} \frac{dS}{dt} &= \gamma N - \lambda SI - p\gamma I + \delta I - \sigma S \\ \frac{dI}{dt} &= \lambda SI - (\delta + \sigma + \varepsilon - p\gamma)I \end{aligned} \tag{2.1}$$

The simple mass action incidence λSI is used instead of the standard incidence $\frac{\beta SI}{N}$. The parameters in this and other models in this paper are:

$\gamma =$ Natural birth rate constant
$\sigma =$ Natural death rate constant
$\delta =$ Recovery rate constant
$\alpha =$ Loss of immunity rate constant
$p =$ Constant
ε = Disease related death rate constant.

Equilibria of model (2.1) can be obtained by equating right hand side to zero. This provides two equilibria: an infection-free equilibrium $E_0 = (S_0, I_0)$, with

$$S_0 = \frac{\gamma N}{\sigma}, \qquad I_0 = 0$$

and an endemic equilibrium $E^* = (S^*, I^*)$, with

$$S^* = \frac{\gamma N}{\sigma \Re_0}, I^* = \frac{\gamma}{(\sigma + \varepsilon)}\left(1 - \frac{1}{\Re_0}\right)N.$$

The parameter $\Re_0 = \frac{\gamma \lambda N}{\sigma(\delta + \sigma + \varepsilon - p\gamma)}$ is called the basic reproduction number. The condition $\Re_0 > 1$ ensures existence of the positive endemic equilibrium state E^*. In order to get an idea of stability, the variation matrix of the system (2.1) is

$$J = \begin{bmatrix} -(\lambda I + \sigma) & -(\lambda S + p\gamma - \delta) \\ \lambda I & \lambda S - (\delta + \sigma + \varepsilon - p\gamma) \end{bmatrix}.$$

Theorem 2.1: If $\Re_0 < 1$, the infection-free equilibrium state E_0 is locally stable; if $\Re_0 = 1$, E_0 is stable and if $\Re_0 > 1$, E_0 is unstable.

Proof: For the equilibrium point E_0, the variation matrix is

$$J_{E_0} = \begin{bmatrix} -\sigma & -\left(\frac{\lambda \gamma N}{\sigma} + p\gamma - \delta\right) \\ 0 & \frac{\lambda \gamma N}{\sigma} - (\delta + \sigma + \varepsilon - p\gamma) \end{bmatrix}.$$

The corresponding eigen values are

$$\lambda_1 = -\sigma, \quad \lambda_2 = \frac{\lambda\gamma N}{\sigma} - (\delta + \sigma + \varepsilon - p\gamma).$$

For stability λ_1 and λ_2 should be negative so that we obtain $\frac{\lambda\gamma N}{\sigma} < (\delta + \sigma + \varepsilon - p\gamma)$ i.e. $\Re_0 < 1$. Thus, if $\Re_0 < 1$, the infection-free equilibrium state E_0 is locally stable; if $\Re_0 = 1$, E_0 is stable; and if $\Re_0 > 1$, E_0 is unstable.

Global properties of the system (2.1) are given by the following Theorem.

Theorem 2.2: The infection-free equilibrium state E_0 of the system (2.1) is globally stable. **Proof:** Consider a Lyapunov function $L = I$ then the Lyapunov derivative $\dot{L} = \dot{I} = 0$ only if I=0. Thus, E_0 is globally asymptotically stable.

Theorem 2.3: The endemic equilibrium state E^* of the system (2.1) is globally stable.

Proof: Consider a Lyapunov function

$$U(S,I) = S^*\left(\frac{S}{S^*} - \ln\frac{S}{S^*}\right) + \frac{(\sigma+\varepsilon)}{(\delta+\sigma+\varepsilon-p\gamma)} I^*\left(\frac{I}{I^*} - \ln\frac{I}{I^*}\right) \tag{2.2}$$

which is defined and continuous for all S, I > 0, can be applied to the system (2.1).It is easy to see that the endemic equilibrium state $E^* = (S^*, I^*)$ is the only extremum and the global minimum of the function $U(S,I)$ in $\boldsymbol{R}_+^2$. From system (2.1), it follows that

$$\lambda S^* I^* = \gamma N + (\delta - p\gamma) I^* - \sigma S^* = (\delta + \sigma + \varepsilon - p\gamma) I^* \tag{2.3}$$

In the case of system (2.1), using (2.3), the derivative of the function $U(S,I)$ satisfies

$$\dot{U}(S,I) = \frac{\partial U}{\partial S}\dot{S} + \frac{\partial U}{\partial I}\dot{I}$$

$$= \left(1 - \frac{S^*}{S}\right)[\gamma N - \lambda SI + (\delta - p\gamma)I - \sigma S]$$

$$+ \frac{(\sigma+\varepsilon)}{(\delta+\sigma+\varepsilon-p\gamma)}\left(1 - \frac{I^*}{I}\right)[\lambda SI - (\delta+\sigma+\varepsilon-p\gamma)I]$$

$$= \gamma N - \lambda SI + (\delta - p\gamma)I - \sigma S$$

$$- \gamma N\frac{S^*}{S} + \lambda S^* I - (\delta - p\gamma)I\frac{S^*}{S} + \sigma S^*$$

$$+ \frac{(\sigma+\varepsilon)}{(\delta+\sigma+\varepsilon-p\gamma)}(\lambda SI - \lambda SI^*) - (\sigma+\varepsilon)(I - I^*)$$

$$= [\gamma(N - pI) + \delta I]\left(2 - \frac{S}{S^*} - \frac{S^*}{S}\right)$$

$$= -[\gamma(N - pI) + \delta I]\frac{S}{S^*}\left(1 - \frac{S^*}{S}\right)^2$$

That is $\dot{U}(S,I) \le 0$ for all $S, I \ge 0$ ensured by $(N - pI) \ge 0$. Since $\dot{U}(S,I) = 0$ holds only for $S = S^*$ and the endemic equilibrium state E^* is the only invariant set of the system on the line $S = S^*$, by the asymptotic stability theorem [2] the equilibrium state E^* is globally asymptotically stable.

3. SIRS model with vertical transmission and simple mass action incidence.

In this section we consider the following SIRS model with vertical transmission and simple mass action incidence. In the SIRS model the total population is N = S + I + R, where R is the removed individuals. Using the same parameters as the analogous SIS model in section 2, the differential equations are

$$\begin{aligned} \frac{dS}{dt} &= \gamma(N - pI) - \lambda SI - \sigma S + \alpha R \\ \frac{dI}{dt} &= \lambda SI + p\gamma I - (\sigma + \delta)I \\ \frac{dR}{dt} &= \delta I - \alpha R - [\gamma N - \sigma(S + I)] \end{aligned} \tag{3.1}$$

We assume that the population size N is constant, so we do not need an equation for the removed class R. Thus, our model (3.1) becomes

$$\begin{aligned} \frac{dS}{dt} &= (\gamma + \alpha)N - \lambda SI - (\alpha + p\gamma)I - (\alpha + \sigma)S \\ \frac{dI}{dt} &= \lambda SI - (\sigma + \delta - p\gamma)I \end{aligned} \tag{3.2}$$

The equilibria of system (3.2) are: an infection-free equilibrium $E_0 = (S_0, I_0)$, with

$$S_0 = \left(\frac{\alpha + \gamma}{\alpha + \sigma}\right)N, \quad I_0 = 0$$

and an endemic equilibrium $E^* = (S^*, I^*)$, with

$$S^* = \frac{(\alpha + \gamma)N}{(\alpha + \sigma)\Re_0}, \; I^* = \frac{(\alpha + \gamma)}{(\alpha + \sigma + \delta)}\left(1 - \frac{1}{\Re_0}\right)N.$$

The parameter $\Re_0 = \dfrac{(\alpha + \gamma)\lambda N}{(\alpha + \sigma)(\sigma + \delta - p\gamma)}$ is called the basic reproduction number. The condition $\Re_0 > 1$ ensures existence of the positive endemic equilibrium state E^*.The variation matrix of the system (3.2) is given by

$$J = \begin{bmatrix} -(\lambda I + \alpha + \sigma) & -(\lambda S + \alpha + p\gamma) \\ \lambda I & \lambda S - (\sigma + \delta - p\gamma) \end{bmatrix}.$$

Theorem 3.1: If $\Re_0 < 1$, the infection-free equilibrium state E_0 is locally stable; if $\Re_0 = 1$, E_0 is stable and if $\Re_0 > 1$, E_0 is unstable.

Proof: For the equilibrium point E_0, the variation matrix is

$$J_{E_0} = \begin{bmatrix} -(\alpha+\sigma) & -\left[\lambda\left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)N+\alpha+p\gamma\right] \\ 0 & \lambda\left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)N-(\sigma+\delta-p\gamma) \end{bmatrix}.$$

The corresponding eigen values are

$$\lambda_1 = -(\alpha+\sigma), \quad \lambda_2 = \lambda\left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)N-(\sigma+\delta-p\gamma).$$

For stability λ_1 and λ_2 should be negative so that we obtain $\lambda\left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)N < (\sigma+\delta-p\gamma)$ i.e. $\Re_0 < 1$.Thus, if $\Re_0 < 1$, the infection-free equilibrium state E_0 is locally stable; if $\Re_0 = 1$, E_0 is stable; and if $\Re_0 > 1$, E_0 is unstable.

Global properties of the system (3.2) are given by the following Theorem.

Theorem 3.2: The infection-free equilibrium state E_0 of the system (3.2) is globally stable. **Proof:** Consider a Lyapunov function $L = I$ then the Lyapunov derivative $\dot{L} = \dot{I} = 0$ only if I=0. Thus, E_0 is globally asymptotically stable.

Theorem 3.3: The endemic equilibrium state E^* of the system (3.2) is globally stable.

Proof: After a small alteration, the Lyapunov function

$$V(S,I) = S^*\left(\frac{S}{S^*}-\ln\frac{S}{S^*}\right)+\frac{\sigma}{(\sigma+\delta-p\gamma)}I^*\left(\frac{I}{I^*}-\ln\frac{I}{I^*}\right) \tag{3.3}$$

which is defined and continuous for all S, I > 0.It is easy to see that the endemic equilibrium state $E^* = (S^*, I^*)$ is the only extremum and the global minimum of the function $V(S,I)$ in $\boldsymbol{R}_+^2$. From system (3.2), it follows that

$$\lambda S^* I^* = (\gamma+\alpha)N-(\alpha+p\gamma)I^*-(\alpha+\sigma)S^* = (\sigma+\delta-p\gamma)I^* \tag{3.4}$$

In the case of system (3.2), using (3.4), the derivative of the function $V(S,I)$ satisfies

$$\dot{V}(S,I) = \frac{\partial V}{\partial S}\dot{S}+\frac{\partial V}{\partial I}\dot{I}$$

$$= \left(1-\frac{S^*}{S}\right)[(\gamma+\alpha)N-\lambda SI-(\alpha+p\gamma)I-(\alpha+\sigma)S]$$

$$+\frac{\sigma}{(\sigma+\delta-p\gamma)}\left(1-\frac{I^*}{I}\right)[\lambda SI-(\sigma+\delta-p\gamma)I]$$

$$= (\gamma+\alpha)N-\lambda SI-(\alpha+p\gamma)I-(\alpha+\sigma)S$$

$$-(\gamma+\alpha)N\frac{S^*}{S}+\lambda S^*I+(\alpha+p\gamma)I\frac{S^*}{S}+(\alpha+\sigma)S^*$$

$$+\frac{\sigma}{(\sigma+\delta-p\gamma)}(\lambda SI-\lambda SI^*)-\sigma(I-I^*)$$

$$=[(\gamma+\alpha)N-(\alpha+p\gamma)I]\left(2-\frac{S}{S^*}-\frac{S^*}{S}\right)$$

$$=-[\gamma(N-pI)+\alpha(N-I)]\frac{S}{S^*}\left(1-\frac{S^*}{S}\right)^2$$

That is $\dot{V}(S,I)\le 0$ for all $S,I\ge 0$ ensured by $[\gamma(N-pI)+\alpha(N-I)]\ge 0$. Since $\dot{V}(S,I)=0$ holds only for $S=S^*$ and the endemic equilibrium state E^* is the only invariant set of the system on the line $S=S^*$, by the asymptotic stability theorem [2] the equilibrium state E^* is globally asymptotically stable.

4. SIRS model with vertical transmission and non-linear incidence.

Here we consider the SIRS model with vertical transmission and the non- linear incidence $\frac{\beta SI}{1+aI}$. Using the parameters defined previously, β as the infection coefficient and a as the constant, the differential equations are

$$\begin{aligned}\frac{dS}{dt}&=\gamma(N-pI)-\frac{\beta SI}{1+aI}-\sigma S+\alpha R\\ \frac{dI}{dt}&=\frac{\beta SI}{1+aI}+p\gamma I-(\sigma+\delta)I\\ \frac{dR}{dt}&=\delta I-\alpha R-[\gamma N-\sigma(S+I)]\end{aligned}\qquad(4.1)$$

We do not need an equation for the removed class R, since the population size N is constant. Thus, our model (4.1) becomes

$$\begin{aligned}\frac{dS}{dt}&=(\gamma+\alpha)N-\frac{\beta SI}{1+aI}-(\alpha+p\gamma)I-(\alpha+\sigma)S\\ \frac{dI}{dt}&=\frac{\beta SI}{1+aI}-(\sigma+\delta-p\gamma)I\end{aligned}\qquad(4.2)$$

The system of equations (4.2) possesses two equilibrium points: an infection-free equilibrium $E_0=(S_0,I_0)$, with

$$S_0=\left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)N,\qquad I_0=0$$

and an endemic equilibrium $E^*=(S^*,I^*)$, with

$$S^*=\frac{(\alpha+\gamma)(1+aI^*)N}{(\alpha+\sigma)\Re_0},\ I^*=\frac{(\alpha+\gamma)\left(1-\frac{1}{\Re_0}\right)N}{\left((\alpha+\sigma+\delta)+\frac{(\alpha+\gamma)aN}{\Re_0}\right)}.$$

The parameter $\Re_0=\frac{(\alpha+\gamma)\beta N}{(\alpha+\sigma)(\sigma+\delta-p\gamma)}$ is called the basic reproduction number. The condition $\Re_0>1$ ensures existence of the positive endemic equilibrium state E^*.

The variation matrix of the system (4.2) is

$$J=\begin{bmatrix} -\left(\frac{\beta I}{1+aI}+\alpha+\sigma\right) & -\left(\frac{\beta S}{(1+aI)^2}+\alpha+p\gamma\right) \\ \frac{\beta I}{1+aI} & \frac{\beta S}{(1+aI)^2}-(\sigma+\delta-p\gamma) \end{bmatrix}.$$

Theorem 4.1: If $\Re_0 < 1$, the infection-free equilibrium state E_0 is locally stable; if $\Re_0 = 1$, E_0 is stable and if $\Re_0 > 1$, E_0 is unstable.

Proof: For the equilibrium point E_0, the variation matrix becomes

$$J_{E_0}=\begin{bmatrix} -(\alpha+\sigma) & -\left[\beta\left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)N+\alpha+p\gamma\right] \\ 0 & \beta\left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)N-(\sigma+\delta-p\gamma) \end{bmatrix}.$$

The eigen values of J_{E_0} are

$$\lambda_1=-(\alpha+\sigma),\ \lambda_2=\beta\left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)N-(\sigma+\delta-p\gamma).$$

For stability λ_1 and λ_2 should be negative so that we obtain $\beta\left(\frac{\alpha+\gamma}{\alpha+\sigma}\right)N<(\sigma+\delta-p\gamma)$ i.e. $\Re_0 < 1$.Thus, if $\Re_0 < 1$, the infection-free equilibrium state E_0 is locally stable; if $\Re_0 = 1$, E_0 is stable; and if $\Re_0 > 1$, E_0 is unstable.

Global properties of the system (4.2) are given by the following Theorem.

Theorem4.2: The infection-free equilibrium state E_0 of the system (4.2) is globally stable. **Proof:** Consider a Lyapunov function $L=I$ then the Lyapunov derivative $\dot{L}=\dot{I}=0$ only if I=0. Thus, E_0 is globally asymptotically stable.

Theorem 4.3: The endemic equilibrium state E^* of the system (4.2) is globally stable.

Proof: Consider a Lyapunov function

$$W(S,I)=(S-S^*\ln S)+\frac{\sigma}{(\sigma+\delta-p\gamma)}(I-I^*\ln I) \tag{4.3}$$

which is defined and continuous for all S, I > 0. It is easy to see that the endemic equilibrium state $E^*=(S^*,I^*)$ is the only extremum and the global minimum of the function $W(S,I)$ in R_+^2. From system (4.2), it follows that

$$\frac{\beta S^*I^*}{1+aI^*}=(\gamma+\alpha)N-(\alpha+p\gamma)I^*-(\alpha+\sigma)S^*=(\sigma+\delta-p\gamma)I^* \tag{4.4}$$

In the case of system (4.2), using (4.4), the derivative of the function $W(S,I)$ satisfies

$$\dot{W}(S,I)=\frac{\partial W}{\partial S}\dot{S}+\frac{\partial W}{\partial I}\dot{I}$$

$$=\left(1-\frac{S^*}{S}\right)[(\gamma+\alpha)N-\frac{\beta SI}{1+aI}-(\alpha+p\gamma)I-(\alpha+\sigma)S]$$

$$+\frac{\sigma}{(\sigma+\delta-p\gamma)}\left(1-\frac{I^*}{I}\right)\left(\frac{\beta SI}{1+aI}-(\sigma+\delta-p\gamma)I\right)$$

$$=(\gamma+\alpha)N-\frac{\beta SI}{1+aI}-(\alpha+p\gamma)I-(\alpha+\sigma)S$$

$$-(\gamma+\alpha)N\frac{S^*}{S}+\frac{\beta S^*I}{1+aI}+(\alpha+p\gamma)I\frac{S^*}{S}+(\alpha+\sigma)S^*$$

$$+\frac{\sigma}{(\sigma+\delta-p\gamma)}\left(\frac{\beta SI}{1+aI}-\frac{\beta SI^*}{1+aI}\right)-\sigma(I-I^*)$$

$$=[(\gamma+\alpha)N-(\alpha+p\gamma)I]\left[1-\frac{S^*}{S}+\left(1-\frac{S}{S^*}\right)\left(\frac{1+aI^*}{1+aI}\right)\right]$$

$$=[(\gamma+\alpha)N-(\alpha+p\gamma)I]\left[1-\frac{S^*}{S}+\left(1-\frac{S}{S^*}\right)\frac{S^*}{S}\right]$$

$$=0$$

That is $\dot{W}(S,I)=0$ for all $S,I\geq 0$ .Hence, by the asymptotic stability theorem [2] the equilibrium state E^* is globally asymptotically stable.

5. Numerical Example and Concluding remarks

We have investigated numerically the model given in section 4. If we choose the parameters for model (4.2) as follows: $\gamma=1,\beta=0.6,\ N=2,\ \sigma=0.8,\ \alpha=0.1,\ \delta=0.2,\ p=0.1$, a=1 then we get unique positive equilibrium point (S = 2.020408163, I = 0.346938776). Here the basic reproduction number $\Re_0$ = 1.6296 > 1. For the above choice of parameters we see that the components S and I approach to their steady state values as time goes to infinity, the disease becomes endemic (see fig 1).

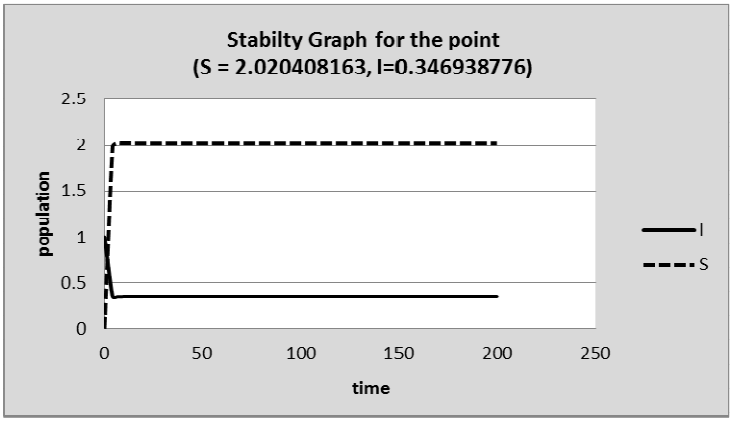

Figure 1

On increasing the value of 'a', the susceptible population increases and infective population decreases significantly. The details of the numerical results are given in Annexure followed by the references. If immunity is permanent, then the *SIRS* model reduces to the *SIR* model.

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Annexure

γ	β	N	σ	α	δ	p	a	S*	I*
1	0.6	2	0.8	0.1	0.2	0.1	1	2.020408163	0.346938776
1	0.6	2	0.8	0.1	0.2	0.1	1.5	2.112	0.272
1	0.6	2	0.8	0.1	0.2	0.1	2	2.171052632	0.223684211
1	0.6	2	0.8	0.1	0.2	0.1	2.5	2.212290503	0.189944134
1	0.6	2	0.8	0.1	0.2	0.1	3	2.242718447	0.165048544
1	0.6	2	0.8	0.1	0.2	0.1	3.5	2.266094421	0.145922747
1	0.6	2	0.8	0.1	0.2	0.1	4	2.284615385	0.130769231
1	0.6	2	0.8	0.1	0.2	0.1	4.5	2.299651568	0.118466899
1	0.6	2	0.8	0.1	0.2	0.1	5	2.312101911	0.108280255