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Advances in Applied Science Research, 2014, 5(3):195-203

Advances in Applied Science Research

Pelagia Research Library

ISSN: 0976-8610
CODEN (USA): AASRFC

Stability analysis of a system of coupled harmonic oscillators

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ABSTRACT

We present here a critical study of coupled harmonic oscillators. These are essentially two pendulums for which the oscillating masses are connected by a spring of stiffness k. We derive the equation of motion of the masses using the Euler-Lagrange equations. Exact analytical solutions of the equations are obtained using Laplace transforms enabling us to give a graphical profile as well as the phase portraits. Stability analysis of the nonlinear system is investigated by the direct method.

Key words: Harmonic oscillators, Euler-Lagrange Equations, Laplace Transforms, Stability.

INTRODUCTION

The dynamics of coupled oscillators are significant in mechanics, electronics and biological systems. An early use of mathematical modeling in biological systems was Van der Pol's use in 1928 of a driven Van der Pol oscillator[1] to explain some normal, and pathological, rhythms of the heart.

In this work, we focus on the dynamics of a system of coupled pendulums oscillating in phase space. The harmonic oscillator and as well the system it will model normally has a single degree of freedom. However, most complicated systems have more degrees of freedom, for instance two or more masses and one spring (each being attached to fixed points and each other). In this case, the behaviour of the variables influences that of the other and this leads in turn to a coupling of the oscillation.

1.1 Anatomy of an oscillator

Here we give a formal mathematical definition of an oscillator. It is essentially a dynamical system that produces periodic behaviour. For example, in $\mathbb{R}^d$ we have the model;

$$\dot{x}_1 = f_1\left(x_1,\dots,x_d\right), \qquad \dot{x}_d = f_d\left(x_1,\dots,x_d\right)$$

with a periodic orbit

$$P(t) = \left(p_1(t),\dots,p_d(t)\right)$$

with period $T > 0$, i.e.

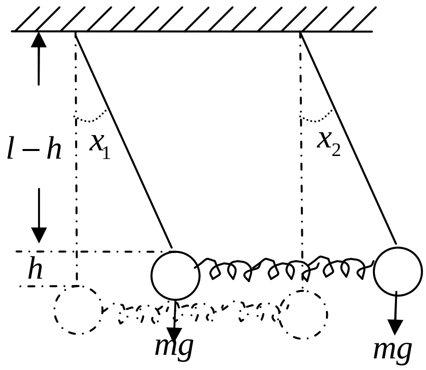
$$P(t+T) = P(t)$$

such that T is the smallest possible choice of periodicity of all components.

1.2 Derivation of the Equations of Motion

Consider a system of coupled pendulums of masses shown in the figure below. We will assume for simplicity that the masses *m* and lengths *l* on each pendulums are equal. The equation of motion of the coupled system can be derived by the application of energy methods. In particular we employ the Lagrangian method [3] using the generalized angular coordinates x_1 and x_2. The Lagrangian of the system is given by $L = T - V$ where;

T = Total kinetic energy of the system
V = Total potential energy of the system



From the above figure, we have;

$$\frac{l-h}{l} = \cos x_1 \quad \Rightarrow \quad h = l - l\cos x_1$$

The potential energy of the first mass is;

$$V_1 = mgh = mgl\left(1-\cos x_1\right)$$

Similarly for the second mass;

$$V_2 = mgh = mgl\left(1-\cos x_2\right)$$

The potential energy of the spring is given by;

$$V_3 = \frac{1}{2}k\left(lx_2 - lx_1\right)^2 = \frac{1}{2}kl^2\left(x_2 - x_1\right)^2$$

Kinetic energy of the masses are respectively; $T_1 = \dfrac{1}{2}mv_1^2$ and $T_2 = \dfrac{1}{2}mv_2^2$;

where $v_1 = \dfrac{d}{dt}\left(lx_1\right) = l\dot{x}_1$ and $v_2 = \dfrac{d}{dt}\left(lx_2\right) = l\dot{x}_2$; $\dot{x}_i = \dfrac{dx_i}{dt}, \quad i = 1,2.$

The Lagrangian then becomes;

$$L = T_1 + T_2 - \left(V_1 + V_2 + V_3\right)$$

i.e.

$$L = \left(\tfrac{1}{2}ml^2\dot{x}_1^2 + \tfrac{1}{2}ml^2\dot{x}_2^2\right) - \left[mgl\left(1-\cos x_1\right) + mgl\left(1-\cos x_2\right) + \tfrac{1}{2}kl^2\left(x_2 - x_1\right)^2\right]$$

The Euler-Lagrange equations are given by;

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0, \qquad i = 1,2.$$

We have;

$$\frac{\partial L}{\partial x_1} = mgl\sin x_1 + kl^2\left(x_2 - x_1\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{d}{dt}\left(\tfrac{1}{2}ml^2\dot{x}_1^2\right) = ml^2\ddot{x}_1$$

Hence the equation of motion of the first mass is;

$$-mgl\sin x_1 + kl^2\left(x_2 - x_1\right) - ml^2\ddot{x}_1 = 0 \tag{1}$$

$$m\ddot{x}_1 = -\frac{mg}{l}\sin x_1 + k\left(x_2 - x_1\right) \tag{2}$$

Similarly for the second mass we get;

$$\frac{\partial L}{\partial x_2} = -mgl\sin x_2 + kl^2\left(x_2 - x_1\right)$$

and;

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{d}{dt}\left(\tfrac{1}{2}ml^2\dot{x}_2^2\right) = ml^2\ddot{x}_2$$

Hence the equation of motion for the second mass is;

$$m\ddot{x}_2 = -\frac{mg}{l}\sin x_2 + k\left(x_1 - x_2\right)$$

Assuming small oscillations i.e, $\sin x_1 \approx x_1$, $\sin x_2 \approx x_2$, equations (1) and (2) can be rewritten as

$$m\ddot{x}_1 = -\frac{mg}{l}x_1 + k\left(x_2 - x_1\right) \tag{3}$$

$$m\ddot{x}_2 = -\frac{mg}{l}x_2 + k\left(x_1 - x_2\right) \tag{4}$$

With the initial conditions $x_1 = x_2 = 0,\ \dot{x}_1 = v,\ \dot{x}_2 = 0$ at $t = 0$.

Let $L\left[x_i(t)\right] = F_i(s),\ i = 1,2$. Then show that the Laplace transforms[1] of our differential equations (i) and (ii) are;

$$m\left(s^2F_1 - v\right) = -\frac{mg}{l}F_1 + k\left(F_2 - F_1\right) \tag{5}$$

$$ms^2F_2 = -\frac{mg}{l}F_2 + k\left(F_1 - F_2\right) \tag{6}$$

Solve (5) and (6) simultaneously to get

$$F_1(s) = \frac{v\left(s^2 + \frac{g}{l} + \frac{k}{m}\right)}{\left(s^2 + \frac{g}{l} + \frac{2k}{m}\right)\left(s^2 + \frac{g}{l}\right)} = \frac{v}{2}\left(\frac{1}{s^2 + \frac{g}{l} + \frac{2k}{m}} + \frac{1}{s^2 + \frac{g}{l}}\right)$$

$$F_2(s)=\frac{\left(\frac{2vk}{m}\right)}{\left(s^2+\frac{g}{l}+\frac{2k}{m}\right)\left(s^2+\frac{g}{l}\right)}=\frac{v}{2}\left(\frac{1}{s^2+\frac{g}{l}}-\frac{1}{s^2+\frac{g}{l}+\frac{2k}{m}}\right)$$

The inverse transforms are respectively;

$$x_1(t)=\frac{v}{2}\left(\frac{\sin\left(\sqrt{\frac{g}{l}+\frac{2k}{m}}\right)t}{\sqrt{\frac{g}{l}+\frac{2k}{m}}}+\frac{\sin\left(\sqrt{\frac{g}{l}}\right)t}{\sqrt{\frac{g}{l}}}\right),\quad x_2(t)=\frac{v}{2}\left(\frac{\sin\left(\sqrt{\frac{g}{l}}\right)t}{\sqrt{\frac{g}{l}}}-\frac{\sin\left(\sqrt{\frac{g}{l}+\frac{2k}{m}}\right)t}{\sqrt{\frac{g}{l}+\frac{2k}{m}}}\right)$$

The derivatives of the above solutions are respectively;

$$y_1(t)=\frac{v}{2}\left(\cos\left(\sqrt{\frac{g}{l}+\frac{2k}{m}}\right)t+\cos\left(\sqrt{\frac{g}{l}}\right)t\right),\quad y_2(t)=\frac{v}{2}\left(\cos\left(\sqrt{\frac{g}{l}}\right)t-\cos\left(\sqrt{\frac{g}{l}+\frac{2k}{m}}\right)t\right)$$

2.0 Conditions for stability, continuous time systems

In this section we establish stability conditions for continuous time dynamical systems[2].

Given a system $\dot{\boldsymbol{x}}=A\boldsymbol{x}$, $\boldsymbol{x}\in\mathbb{R}^n$, $A=\left(a_{ij}\right)_{n\times n}$; $\boldsymbol{x}\left(0\right)=\boldsymbol{x}_0$. Formally the solution is given by;

$$\boldsymbol{x}\left(t\right)=e^{tA}\boldsymbol{x}_0$$

e^{tA} being the so-called *exponential matrix*. We attempt to diagonalize the matrix A by a coordinate transformation:

$$\left.\begin{aligned}\tilde{\boldsymbol{x}}&=A\boldsymbol{x}\\ \boldsymbol{x}&=S\boldsymbol{z}\\ \tilde{\boldsymbol{x}}&=S\tilde{\boldsymbol{z}}\end{aligned}\right\}\quad\Rightarrow\quad\tilde{\boldsymbol{z}}=S^{-1}AS\boldsymbol{z}$$

It turns out that $S^{-1}AS$ is a diagonal matrix which we label D, i.e.

$$D=\begin{pmatrix}\lambda_1 & & \bigcirc\\ & \ddots & \\ \bigcirc & & \lambda_n\end{pmatrix}$$

This is most of the time possible, but not always. Suppose temporarily that it is possible, i.e. that A is diagonalizable. Then λ_i are the eigenvalues of A and the i-th column of S the eigenvector for λ_i.

We can represent the exponential matrix as;

$$e^{tA}=I+tA+\frac{t^2}{2!}A^2+\frac{t^3}{3!}A^3+\cdots$$

and thus

$$S^{-1}e^{tA}S=S^{-1}S+tS^{-1}AS+\frac{t^2}{2!}S^{-1}ASS^{-1}AS+\cdots$$

$$=e^{tD}$$

$$= \begin{pmatrix} e^{t\lambda_1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & e^{t\lambda_n} \end{pmatrix}$$

Hence, the solutions become;

$$\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}_0 = Se^{tD}S^{-1}\boldsymbol{x}_0$$

and in transformed coordinates this becomes

$$\boldsymbol{z}(t) = e^{tD}\boldsymbol{z}_0 = \begin{pmatrix} e^{t\lambda_1} z_{01} \\ \vdots \\ e^{t\lambda_n} z_{0n} \end{pmatrix}$$

This leads directly to the following proposition.

2.1 Proposition

If A is diagonalizable

- The system is asymptotically stable if all eigenvalues λ_i satisfy Re $\lambda_i < 0$
- The system is stable if all eigenvalues λ_i satisfy Re $\lambda_i \le 0$.
- The system is strongly unstable if Re $\lambda_i > 0$ for at least one eigenvalue λ_i.

2.2 Stability Analysis of the coupled pendulums

For the purpose of stability analysis we must vectorize the coupled system of differential equations;

$$\ddot{x}_1 = -\frac{g}{l}x_1 + \frac{k}{m}(x_2 - x_1) \tag{7}$$

$$\ddot{x}_2 = -\frac{g}{l}x_2 + \frac{k}{m}(x_1 - x_2) \tag{8}$$

Let $u_1 = x_1$ and $u_2 = x_2$; set $\dot{u}_1 = u_3 \Rightarrow \ddot{u}_1 = \dot{u}_3$

Similarly we set $\dot{u}_2 = u_4 \Rightarrow \ddot{u}_2 = \dot{u}_4$.

The equations become;

$$\dot{u}_1 = u_3$$

$$\dot{u}_2 = u_4$$

$$\dot{u}_3 = -\frac{g}{l}u_1 + \frac{k}{m}(u_2 - u_1)$$

$$\dot{u}_4 = -\frac{g}{l}u_2 + \frac{k}{m}(u_1 - u_2)$$

Or in matrix form, we get;

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(g/l + k/m) & k/m & 0 & 0 \\ k/m & -(g/l + k/m) & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

i.e. $\dot{\boldsymbol{u}} = A\boldsymbol{u}$, where;

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(g/l + k/m) & k/m & 0 & 0 \\ k/m & -(g/l + k/m) & 0 & 0 \end{pmatrix}$$

The eigenvalues of the matrix *A* are given by;

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \left| \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(g/l + k/m) & k/m & 0 & 0 \\ k/m & -(g/l + k/m) & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0$$

For computational convenience set;

$$\alpha = -(g/l + k/m), \;\; \beta = k/m$$

We then have

$$\left| \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ \alpha & \beta & -\lambda & 0 \\ \beta & \alpha & 0 & -\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \quad \lambda^4 - 2\alpha\, \lambda^2 + \left(\alpha^2 - \beta^2\right) = 0 \qquad (9)$$

where $\alpha^2 - \beta^2 = (g/l + k/m)^2 - (k/m)^2 = (g/l + 2k/m)\, g/l$

In equation (9) let $v = \lambda^2$, hence

$$v^2 - 2\alpha v + \left(\alpha^2 - \beta^2\right) = 0$$

with roots

$$v = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4\left(\alpha^2 - \beta^2\right)}}{2} = \alpha \pm \beta$$

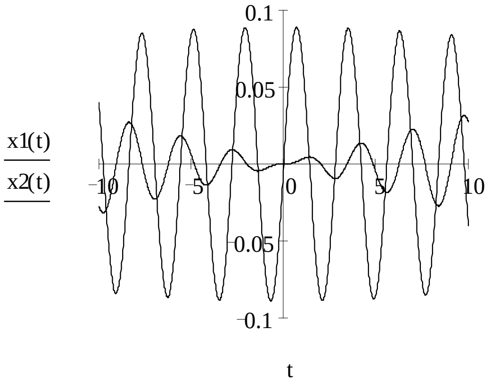
Hence the two roots of equation (9) are given by; $\lambda_1^{\pm} = \pm\sqrt{g/l + 2k/m}, \;\; \lambda_2^{\pm} = \pm\sqrt{g/l}$.

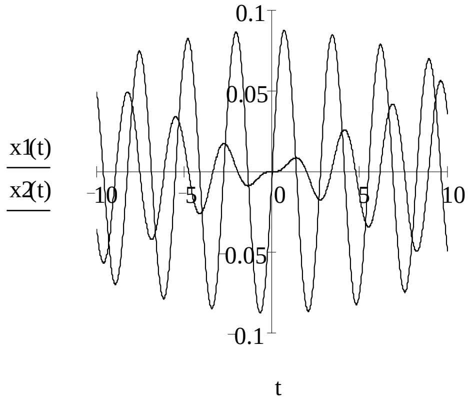
We thus conclude using proposition (2.1) that the coupled system is asymptotically stable for the roots λ_1^-, λ_2^- , and strongly unstable for the roots λ_1^+, λ_2^+ .

3.0 Simulations

g := 9.8 l := 2 m := 3 v := 0.2

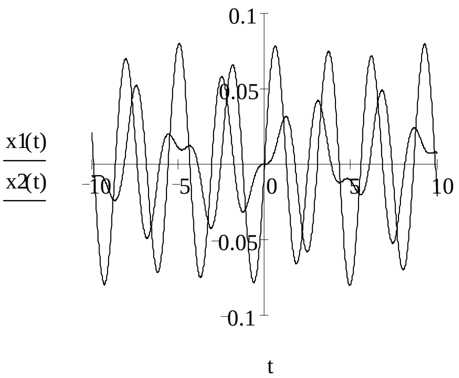
k = 0.5 k = 1

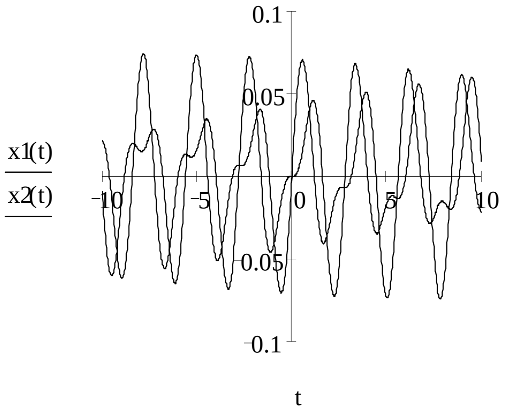




k = 5

k = 10

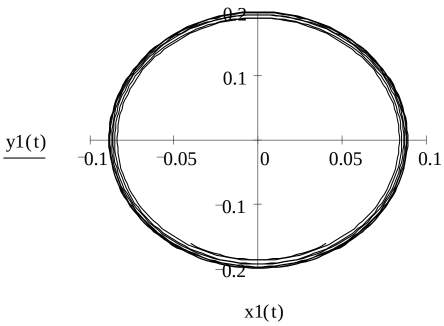


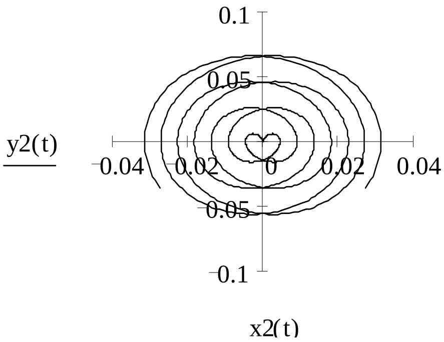


3.0 Phase Portraits

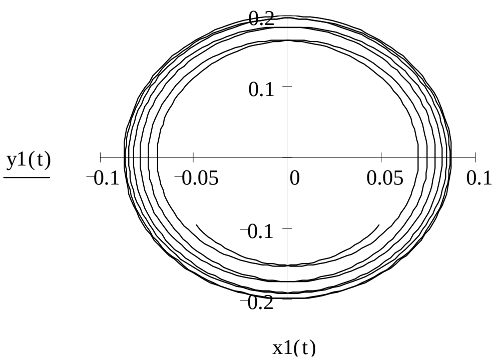
g := 9.8 l := 2 m := 3 v := 0.2

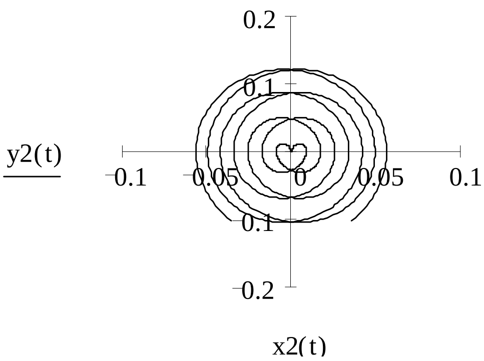
k = 0.5



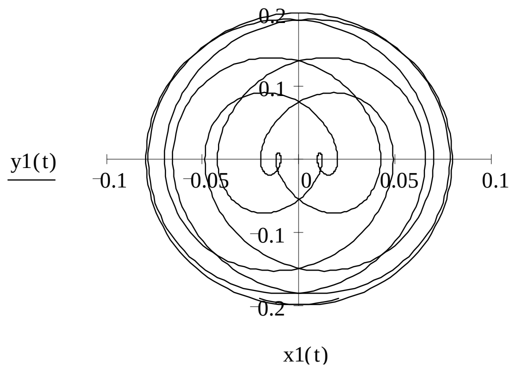


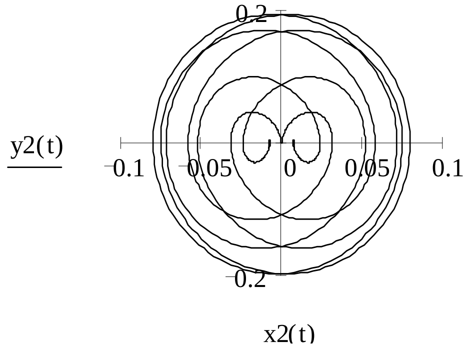
k = 1



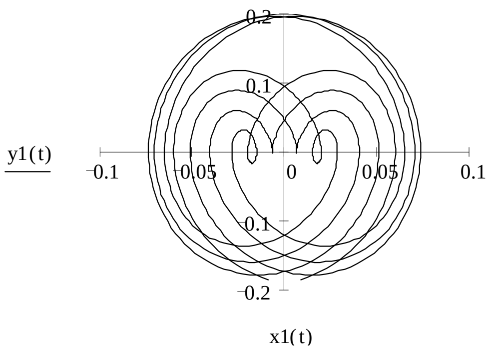


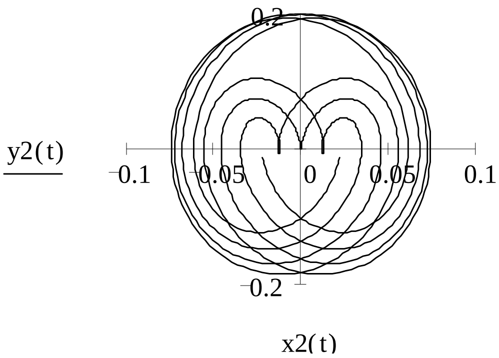
k = 5





k=10





RESULTS AND DISCUSSION

We recall that a *phase portrait* is a geometric representation of the trajectories of a dynamical system in the phase plane. Each set of initial conditions is represented by a different curve, or point. Phase portraits are an invaluable tool in studying dynamical systems. The plot of typical trajectories in the state space reveals information such as whether an attractor, a repeller or limit cycle is present for the chosen parameter value [4,5].

We observe that for the given system above, there is not only one circle-shaped trajectory, but a number of parallel running lines. The trajectory is circling one fixed point, while symmetrically, after some random time, circling another. Furthermore the trajectories depicted are closed curves indicating that for the dynamical system energy is conserved.

Moreover, if we follow two very close segments of the trajectory, we will see that they run into different regions of the phase space after some time. Assuming we have an infinite long trajectory (also called attractor), then we can imagine, that this trajectory fills a 2-dimensional plane. This is not the case however; a measurement of the dimension of this trajectory would reveal a fractal dimension between two and three. These observed properties are typical for chaotic systems: Although small perturbations of such a system cause exponential divergence of its state, after some time the system will come back to a state that is arbitrary close to a former state and pass through a similar evolution.

CONCLUSION

In this work we have studied a mathematical model of a coupled system of harmonic oscillators. The equation of motion were derived employing generalized coordinates and the Euler-Lagrange equations. Exact analytical solutions of the equations were obtained using Laplace transforms, furthermore the trajectories as well as the phase portraits were depicted. Stability analysis of the nonlinear system was investigated by the direct method and it was observed that the coupled system is asymptotically stable for the strictly negative roots λ_1^-, λ_2^-, and strongly unstable for the strictly positive roots λ_1^+, λ_2^+.

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