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Stability analysis of a prey-predator model with a reserved area

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ABSTRACT

The study presented here generalizes a model given by B. Dubey in which he proposed and analyzed a non-linear mathematical model to study the dynamics of fishery resource having two zones. In this paper, we have considered a prey-predator fishery model with prey dispersal in a two-patch environment, one is assumed to be a free fishing zone and the other is a reserved zone, where fishing and other extractive activities are prohibited. The local and global stability analysis has been carried out. Biological equilibria of the system along with the conditions of their existence are obtained. Criteria for the coexistence of predator-prey are obtained. Numerical simulation has also been performed in support of analysis.

Keywords: Prey, Predator, Reserved area, Stability.

INTRODUCTION

Population dynamics refers to changes in the sizes of populations of organisms through time, and predator-prey interactions may play an important role in explaining the population dynamics of many species. The co-existence of interacting biological species has been of great interest in the past few decades like Anderson and Lee [1], Chaudhuri and Johnson [3],Ganguly and Chaudhuri[9], Krishna et.al [13]and Pradhan and Chaudhuri [16]. The combined harvesting of two competing species was studied in detail by Chaudhuri [4]. Collings[5] studied the nonlinear behavior of predator-prey model with refuge protecting a constant proportion of prey and wit temperature dependent parameters chosen appropriately for a mite interaction on fruit species. Krivan [14] proposed a mathematical model and investigated the effects of optimal anti predator behavior of prey in predator-prey system.Dubey et.al.[7]propose and analyses a mathematical model to study the dynamics of a fishery resource system in an aquatic environment that consists with of free fishing zone and a reserve zone.Kar and Chaudhuri [10], investigated a dynamic reaction model in the case of a prey-predator type fishery system, where only the prey species is subjected to harvesting, taking taxation as a control instrument. Dubey et.al.[8]proposed and analyzed a mathematical model to study the dynamics of one prey, two predators system with ratio dependent predators growth rate. Kar [12], in their paper, offer some mathematical analysis of the dynamics of a two prey, one predator system in the presence of a time delay.

Kar and Matsuda [11] investigated a prey-predator model with Holling type of predation and harvesting of mature predator species.Braza[2] analyzed a ‘two predator, one prey model in which one predator interferes significantly with other. A generalized predator-prey system with exploited terms and the existence of eight positive periodic solutions was studied by Zhanga and Tianb [19]. A Lotka-Volterra predator-prey system with a single delay was used by Yan and Zhang [18] in their investigation.Singh et.al. [17] proposed a generalized mathematical model to study the depletion of resources by two kinds of populations, one is weaker and others stronger.A model of prey-predator with a generalized transmission function for unsaturated zone has been analyzed by Mehta et al. [7].We consider the two cases: one when the predator is wholly dependent on the prey and other when the predator is partially dependent on the prey in the unreserved zone.we study the coexistence and stability behavior of predator-prey system in the habitat [6].

MATHEMATICAL MODEL

Let us consider the prey-predator model, wherex(t) be the density of prey species in unreserved zone, y(t) the density of prey species in reserved zone and z(t) the density of the predator species at any time $t \geq 0$.Let σ_1 be the migration rate coefficient of prey species from unreserved to reserved zone and σ_2 the migration rate coefficient of prey species from reserved to unreserved zone. It is assumed that the prey species in both zones are growing logistically. We assume that the prey grows logistically in both zones with carrying capacity K and L, intrinsic growth rate coefficients r and s of prey species in unreserved and reserved zones respectively; β_1 is the depletion rate coefficient of the prey species due to the predator, and β_0 is the natural death rate coefficient of the predator species. Q(z) represents the growth rate of predator.

Using the symbols, notations and basic assumptions of Dubey [6], the dynamics of system may be governed by the following system of ordinary differential equations:

$$\begin{aligned} \frac{dx}{dt} &= rx(1-\frac{x}{Ax+K}) - \sigma_1 x + \sigma_2 y - \beta_1 xz, \\ \frac{dy}{dt} &= sy(1-\frac{y}{L}) + \sigma_1 x - \sigma_2 y, \\ \frac{dz}{dt} &= Q(z) - \beta_0 z, \\ & x(0) \geq 0, y(0) \geq 0, z(0) \geq 0. \end{aligned} \tag{1}$$

In model (1) $r, s, \sigma_1, \sigma_2, \beta_1, \beta_2, \beta_0$ and A are assumed to be positive constants and consider the predatoris wholly dependent on the prey species, i.e. $Q(z) = \beta_2 xz$. (2)

Existence of Equilibria:

It can be seen that model (1) ,when Q(z) satisfies (2) , has only three nonnegative equilibrium , namely $E_0(0,0,0)$, $E_1(\hat{x},\hat{y},0)$ and $\overline{E}(\overline{x},\overline{y},\overline{z})$. The equilibrium E_0 exists obviously and we shall show the existence of E_1 and $\overline{E}$ as follows:

3.1 Existence of E_1 $(\hat{x},\hat{y},0)$

Here $\hat{x}$ *and* $\hat{y}$ are positive solution of the following algebraic equations:

$$rx\left(1-\frac{x}{Ax+K}\right) - \sigma_1 x + \sigma_2 y = 0, \tag{3a}$$

$$sy\left(1-\frac{y}{L}\right) + \sigma_1 x - \sigma_2 y = 0. \tag{3b}$$

From equation (3a) we have

$$y = \frac{1}{\sigma_2}\left[\frac{rx^2}{Ax+K} - (r-\sigma_1)x\right] \tag{4}$$

Putting the value of y from equation (4) into equation (3b),

$$ax^3 + bx^2 + cx + d = 0 \quad (5)$$

where

$$a = \left[\frac{s}{L\sigma_2^{\,2}K^2}\left\{ r^2 - 2r(r-\sigma_1)A + (r-\sigma_1)^2 A^2 \right\} \right]$$

$$b = \left[\frac{-2rs(r-\sigma_1)}{KL\sigma_2^{\,2}} + \frac{2(r-\sigma_1)^2 A}{KL\sigma_2^{\,2}} - \frac{\sigma_1 A^2}{K^2} - \left(\frac{S-\sigma_2}{\sigma_2 K^2}\right)\{rA - (r-\sigma_1)A^2\} \right]$$

$$c = \left[\frac{s}{L\sigma_2^{\,2}}(r-\sigma_1)^2 - \frac{(s-\sigma_2)r}{\sigma_2 K} + \frac{2(s-\sigma_2)(r-\sigma_1)A}{\sigma_2 K} - \frac{2\sigma_1 A}{K} \right]$$

$$d = \left[\frac{(s-\sigma_2)(r-\sigma_1)}{\sigma_2} - \sigma_1 \right]$$

It may be noted that equation (5) has a unique positive solution x = x* if the following inequalities hold:

$$\frac{(s-\sigma_2)r}{K} - \frac{2(s-\sigma_2)(r-\sigma_1)A}{\sigma_2 K} + \frac{2\sigma_1 A}{K} \le \frac{s(r-\sigma_1)^2}{L\sigma_2} \quad (6a)$$

$$(r-\sigma_1)(s-\sigma_2) \le \sigma_1\sigma_2. \quad (6b)$$

From the model system(1) we note that if there is no migration of the prey species from reserved to unreserved zone (i.e. $\sigma_2 = 0$) and r- $\sigma_1 < 0$, then $\frac{dx}{dt} < 0$. Similarly if there is no migration from of the prey species from unreserved to reserved zone (i.e. $\sigma_1 = 0$)and s - $\sigma_2 < 0$,then $\frac{dy}{dt} < 0$. Hence it is natural to assume that

$$r > \sigma_1 \;\; and \;\; s > \sigma_2. \quad (6c)$$

Knowing the value of $\hat{x}$,the value of $\hat{y}$ can be computed from equation (5),It may also be noted that for $\hat{y}$ to be positive, we must have

$$\hat{x} > \frac{K}{r}(r-\sigma_1). \quad (7)$$

3.2 Existence of $\bar{E}(\bar{x}, \bar{y}, \bar{z})$

Here $\bar{x}, \bar{y}, \bar{z}$ arethe positive solutions of the following algebraic equations:

$$rx\left(1 - \frac{x}{Ax+K}\right) - \sigma_1 x + \sigma_2 y - \beta_1 xz = 0,$$

$$sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y = 0.$$

$$\beta_2 xz - \beta_0 z = 0.$$

Solving the above equations, we get,

$$\bar{x} = \frac{\beta_0}{\beta_2}, \tag{8a}$$

$$\bar{y} = \frac{1}{2s\beta_2}\left[(s-\sigma_2)L\beta_2 + \sqrt{\{(s-\sigma_2)L\beta_2\} + 4s\sigma_1\beta_0 L\beta_2}\right] \tag{8b}$$

$$\bar{z} = \frac{\beta_2}{\beta_0\beta_1}\left[\sigma_2\bar{y} + (r-\sigma_1)\frac{\beta_0}{\beta_2} - \frac{r\beta_0^{\ 2}}{\beta_2(A\beta_0+\beta_2 K)}\right] \tag{8c}$$

For z to be positive, we must have

$$\left[\sigma_2\bar{y} + (r-\sigma_1)\frac{\beta_0}{\beta_2} > \frac{r\beta_0^{\ 2}}{\beta_2(A\beta_0+\beta_2 K)}\right] \tag{9}$$

.

Equation (9) gives a threshold value of the carrying capacityof the free access zone for the survival of predators. In the following lemma,we show that all solutions of model (1) are nonnegative and bounded.

Lemma .1

The set $\Omega = \{(x,y,z)\in R_3^{\ +} : 0 < w = x+y+z \le \frac{\mu}{\eta}\}$

for all solutions initiating in the interior of the positive octant.

Where η is a constant such that

$0 < \eta < \beta_{0,}$

$\mu = \frac{K}{4r}(r+\eta)^2 + \frac{L}{4s}(s+\eta^2),$ $\qquad \beta_1 \ge \beta_{2.}$

Proof .Let

$\omega(t) = x(t) + y(t) + z(t)$ *and* $\eta > 0$

be a constant. Then

$$\frac{dw}{dt} + \eta w = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dx} + \eta(x+y+z)$$

$$= (r+\eta)x - \frac{rx^2}{Ax+K} + (s+\eta)y - \frac{sy^2}{L} - (\beta_1-\beta_2)xz - (\beta_0-\eta)z \tag{10}$$

Since β_1 is the depletion rate coefficient of prey due to its intake by the predator and β_2 is the growth rate coefficient of predator due to its interaction with their prey, and hence it is natural to assume that $\beta_1 \ge \beta_2$.

Now choose η such that $0 < \beta_0$.

Then equation (10) can be written as

$$\frac{dw}{dt} + \eta w = (r+\eta)x - \frac{rx^2}{Ax+K} + (s+\eta)y - \frac{sy^2}{L}$$

$$\frac{dw}{dt}+\eta w=(r+\eta)x-\frac{rx^2}{Ax+K}+(s+\eta)y-\frac{sy^2}{L}$$

$$\frac{K}{4r}(r+\eta)^2-\frac{r}{K}\left[\frac{x}{1+\frac{Ax}{K}}-\frac{K}{2r}(r+\eta)\right]^2$$

$$+\frac{L}{4s}(s+\eta)^2-\frac{s}{L}\left[y-\frac{L}{2s}(s+\eta)\right]^2$$

$$\frac{dw}{dt}+\eta w\le\frac{K}{4r}(r+\eta)^2+\frac{L}{4s}(s+\eta)^2=\mu\,\text{(say)}$$

By using the ordinary differential equations rule, we obtain

$$0 < w\,\{x\,(t), y\,(t), z(t)\} \le \frac{\mu}{\eta}(1-e^{-\eta t})+\{x(0),y(0),z(0)\}e^{-\eta t}$$

Taking limit when $t \to \infty$, we have, $0 < w\,(t) \le \frac{\mu}{\eta}$, proving the lemma.

3.3 Stability Analysis

The Variation matrix of the system (1) is

$$J=\begin{pmatrix} r-r(\frac{2xk+x^2A}{(Ax+k)^2})-\sigma_1-\beta_1 z & \sigma_2 & -\beta_1 x \\ \sigma_1 & s-\frac{2sy}{L}-\sigma_2 & 0 \\ \beta_2 z & 0 & \beta_2 x-\beta_0 \end{pmatrix} \quad (11)$$

The characteristic equation of the Variation matrix (11) at $E_0(0,0,0)$ is

$$\lambda^3-(r-\sigma_1+r\sigma_2-\beta_0)\lambda^2+(rs-\sigma_1 s-r\sigma_2)(1-\beta_0)\lambda+(rs-\sigma_1 s-r\sigma_2)\beta_0=0. \quad (12)$$

The characteristic equation of the Variation matrix (11) at $E_1(\hat{x},\hat{y},0)$ is

$$\left[\lambda^2+\left\{r(\frac{2\hat{x}k+\hat{x}^2A}{(A\hat{x}+k)^2})-(r-\sigma_1)-(s-\sigma_2)\right\}\lambda+\left\{(r-r(\frac{2\hat{x}k+\hat{x}^2A}{(A\hat{x}+k)^2})-\sigma_1)(s-\frac{2s\hat{y}}{L}-\sigma_2)-\sigma_1\sigma_2\right\}\right]$$

$$[\beta_2\hat{x}-\beta_0-\lambda]=0. \quad (13)$$

By computing the variationmatrices correspondingto each equilibrium, we note the following:

1. E_0 is a saddle point with stable manifold locally in the z-direction from the equation (12).
2. If $\beta_2 \hat{x} > \beta_0$ then E_1 is a saddle point with stable manifold locally in the xy-plane and with unstable manifold locally in the z- direction from the equation (13).
3. If $\beta_2 \hat{x} < \beta_0$ then E_1 is locally asymptotically stable (13).

Theorem 1. The model system (1) under the assumption (2) cannot have any periodic solution in the interior of the quadrant of the xy- plane.

Proof . Let H(x,y) = $\frac{1}{xy}$.Clearly H(x,y) is positive in the interior of the positive quadrant of thexy-plane.

$$h_1(x,y) = rx(1-\frac{x}{Ax+K}) - \sigma_1 x + \sigma_2 y,$$

$$h_2(x,y) = sy\,(1-\frac{y}{L}) + \sigma_1 x - \sigma_2 y.$$

Then

$$\Delta(x,y) = \frac{\delta}{\delta x}(h_1 H) + \frac{\delta}{\delta y}(h_2 H)$$

$$\Delta(x,y) = -\frac{1}{y}\left[\frac{rK}{(Ax+K)^2} + \frac{\sigma_2 y}{x^2}\right] - \frac{1}{x}\left[\frac{s}{L} + \frac{\sigma_1 x}{y^2}\right] < 0. \qquad (14)$$

From the above equation, we note that Δ (x,y) does not change sign and is not identically zero in the interior of the positive quadrant of the xy- plane.In the following theorem, we show that $\overline{E}$ is locally asymptotically stable.

Theorem 2. The interior equilibrium $\overline{E}$ is locally asymptotically stable.

Proof. In order to prove this theorem, we first linearize model (1) by taking the following

transformation.

x = $\bar{x} + X$, $y = \bar{y} + Y$, $z = \bar{z} + Z$

Now we considered the following positive definite function:

$$V(t) = \frac{1}{2}X^2 + \frac{1}{2}c_1 Y^2 + \frac{1}{2}c_2 Z^2 \qquad (15)$$

Where c_1 and c_2 are positive constant to bechosen suitably.

Now differentiating V with respect to time t along the linear version of model (1) we get

$$\frac{dV}{dt} = -\left(\frac{r\bar{x}}{A\bar{x}+K} + \frac{\sigma_2\bar{y}}{\bar{x}}\right)X^2 - c_1\left(\frac{s\bar{y}}{L} + \frac{\sigma_1\bar{x}}{\bar{y}}\right)Y^2 + XY\left(\sigma_2 + c_1\sigma_1\right) + XZ\left(c_2\beta_2\bar{z} - \beta_1\bar{x}\right)$$

Choosing $c_2 = \frac{\beta_1\bar{x}}{\beta_2\bar{z}}$ we note that V is negative definite if

$$\left(\sigma_2 + c_1\sigma_1\right)^2 < 4c_1\left(\frac{s\bar{y}}{L} + \frac{\sigma_1\bar{x}}{\bar{y}}\right)\left(\frac{r\bar{x}}{A\bar{x}+K} + \frac{\sigma_2\bar{y}}{\bar{x}}\right)$$

It can be written as

$$\left(\sigma_2 - c_1\sigma_1\right)^2 + 4c_1\sigma_1\sigma_2 < 4c_1\left(\frac{s\bar{y}}{L} + \frac{\sigma_1\bar{x}}{\bar{y}}\right)\left(\frac{r\bar{x}}{A\bar{x}+K} + \frac{\sigma_2\bar{y}}{\bar{x}}\right)$$

If we choose $c_1 = \dfrac{\sigma_2}{\sigma_1}$,then above condition is satisfied and show that V is Liapunov function.

Theorem 3.The interior equilibrium $\bar{E}$ is globally asymptotically stable with respect to all solutions.

Proof. Consider the following positive definite function about $\bar{E}$,

$$W(t) = \left(x - \bar{x} - \bar{x}\ln\frac{x}{\bar{x}} \right) + c_1\left(y - \bar{y} - \bar{y}\ln\frac{y}{\bar{y}} \right) + c_2\left(z - \bar{z} - \bar{z}\ln\frac{z}{\bar{z}} \right). \tag{16}$$

Differentiating W with respect to time t along the solutions of model (1), we get

$$\frac{dW}{dt} = -\frac{r}{Ax+K}(x-\bar{x})^2 - \frac{c_1 s}{L}(y-\bar{y})^2 + (x-\bar{x})(z-\bar{z})(c_2\beta_2 - \beta_1)$$
$$+\sigma_2(x-\bar{x})\left(\frac{\bar{x}y - x\bar{y}}{x\bar{x}}\right) + c_1\sigma_1(y-\bar{y})\left(\frac{x\bar{y} - \bar{x}y}{y\bar{y}}\right).$$

Choosing $c_1 = \dfrac{\bar{y}\sigma_2}{\bar{x}\sigma_1}$ and $c_2 = \dfrac{\beta_1}{\beta_2}$, $\dfrac{dW}{dt}$ can further be written as

$$\frac{dW}{dt} = -\frac{r}{Ax+K}(x-\bar{x})^2 - \frac{\bar{y}\sigma_2 s}{x\sigma_1 L}(y-\bar{y})^2 - \frac{\sigma_2}{x\bar{x}y}(\bar{x}y - x\bar{y})^2.$$

Which is negative definite. Hence W is a Liapunov function with respect to $\bar{E}$ whose domain contains the region of attraction Ω, proving the theorem.

4.Numerical Simulation :
For simulation let us take

$r = 3.3, s = 2.002, K = 29, L = 48, \sigma_1=2.8, \sigma_2 = 2.001, A =0.001, \beta_0 = 3, \beta_1 = 2, \beta_2 = 1$ (17)

For the above values of the parameters for the model (1) given and we get an equilibrium point
$\hat{x} = 30.5149, \quad \hat{y} = 45.2730$ (18)

The following graphical presentation shows the stability of the above equilibrium point.

The result (18) of numerical simulation are displayed graphically. In Figure (1.1) prey (x) and prey (y) population are plotted against time (where predator (z) is absent), from this graph it can be said that initial value of the population tend to their corresponding value of equilibrium point E_1 and hence coexist in the form of stable steady state, assuming the local stability of E_1.

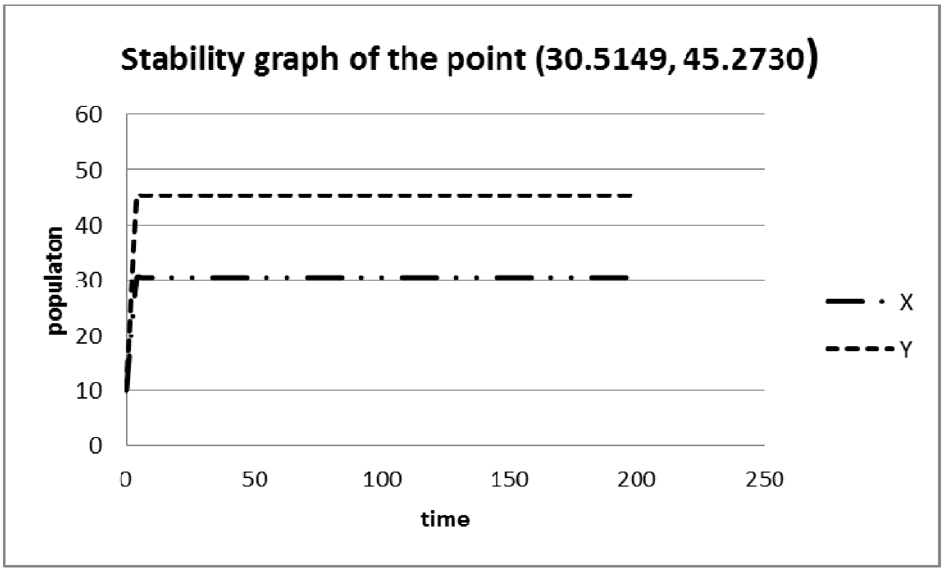

Figure-1.1

i) Figure (1.1) correspond to model (1) when the predator is wholly dependent on the prey. The predator is zero equilibrium level (z = 0), the total density of the prey species at equilibrium level is 85.7879 (30.5149+45.2730).

When predator is wholly dependent on the prey, we observe that the positive equilibrium $\overline{E}(\overline{x}, \overline{y}, \overline{z})$ exists and it is given by values of

$$\overline{x} = 3, \quad \overline{y} = 14.203, \quad \overline{z} = 5.35 \tag{19}$$

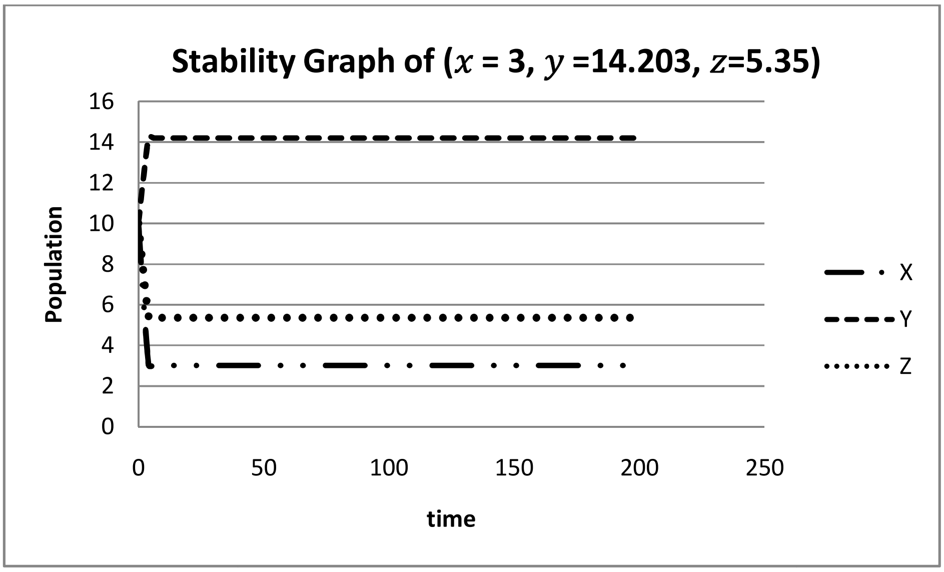

Figure-1.2

The result (19) of numerical simulation are displayed graphically. In Figure (1.2) prey (x), prey (y) and predator (z) population are plotted against time, from this graph it can be said that initial value the population showing stable behavior tend to their corresponding value of equilibrium point $\overline{E}$ and hence coexist in the form of stable steady state.

ii) Figure (1.2) correspond to model (1) when the predator is wholly dependent on the prey. Then density of the predator is 5.35while the total density of the prey has decreased from 85.7879 to 17.203.

CONCLUSION

In this paper, we have analyzed a prey-predator fishery model with prey dispersal in a two-patch environment, one is assumed to be a free fishing zone and the other is a reserved zone where fishing and other extractive activities are prohibited. we have discussed the local and global stability of the system. It has been observed that the asymptotic stability of the controlled system is proved using the Liapunov function. Finally, extensive numerical examples and simulation are introduced

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