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Some remarks on the stability and boundedness of solutions of certain non-autonomous T-period differential equations of second order

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ABSTRACT

This paper extends some known results on the stability and boundedness of solutions of certain T- period of non-autonomous differential equations of second order.

Keywords: Differential Equation; Second Order; Stability; Boundedness; Lyapunov Function; T- Period.

INTRODUCTION

The equation studied in this paper is of the form

$$x''(t) + g'(u) + b(t)\, h(u) = P(t; x, x') \qquad 1.1$$

where Q, g, b, h, and P are continuous which depend only on the arguments display. The symbol prime indicate differentiation with respect to the independent variable 't' and all solutions considered here are assumed to be real. The derivatives of 1.1 exist and are continuous. Moreover the stability and the Boundedness of the equation 1.1 will be assumed.

Ever since Lyapunov proposed his famous direct (or second) method on the stability of motion, numerous method have been proposed in the relevant literature to derive suitable Lyapunov function and hereby, in particular, many papers and books have been devoted to the study of stability and boundedness of solutions of certain second, third, fourth, fifth and sixth order nonlinear differential equations.

(See for example, Anderson [2]; Antosiewict [3]; Bihari [4]; Burton [5]; Hatvani [6-8]; Iggidr [9]; Mirosanu and Viadimirescu [10]; Napoles [12]). So far, the most efficient tool for the study of the stability and boundedness of solutions of a given non-autonomous differential equations of second order is provided by Lyapunov theory. This theory is based on the use of positive definite functions that are non-increasing along the solutions of differential equation under consideration.

Atkinson [1] remarks that "the autonomous case where a = b = 1, or constant independent of t ϵ f. This study shall adopt Burton [5] who developed a theory whereby all these properties (boundedness, stability and periodicity) been investigated differently and used one major theorem the Lyapunov second method.

1.2 DEFINITIONS

Let $\dot{X} = f(t, x)$ (1.2)

Where $\dot{X}\ \epsilon\ \mathrm{R}^n$ be a system of n coupled order equations, we shall give the following definition for completeness.

DEFINITIONS 1.2.1
A Lyapunov function V defined as $v: 1\ X\ R^n \to R$ is said to be Complete for $X\ \epsilon\ R^n$

(i) $V(t,X)\ \geq 0$ (ii) $V(t,X) = 0$, if and only if $X = 0$ and (iii) $\dot{V}(t,X) \leq -C|X|$, where C is any positive constant and $|X|$ given by $|X| = \sum_{i=1}^{n}(x_i^2)\frac{1}{2}$ such that $|X| \to \infty\ as\ X\ \to \infty$

DEFINITION 1.2.2
A Lyapunov function V defined as $V : 1\ \times\ R^n \times R$ is said to be Incomplete if for $X\ \epsilon\ \mathrm{R}^n$ (i) and (ii) of the above definition is satisfied and in addition (iii) $\dot{V}(t,X)/2 \cdot 3 \leq -\mathrm{C}|X|_{(T)}$ where C is any positive constant and $|X|_{(T)}$ given by$|X|_{(T)}\ = \sum_{i=1}^{n}(x_i^2)\frac{1}{2}$ such that $|X|_{(T)}\ \to \infty$ as $\mathrm{X} \to \infty$.

GENERALIZED THEOREMS (BURTON [5]).
Considered the general differential equation
$\dot{\mathrm{X}} = \mathrm{f}\ (\mathrm{t}, \mathrm{x})$ 1.3

Equation 1.3 can be written as a linear equation
$\dot{X} = \mathrm{A}\ (\mathrm{x}) + \mathrm{P}\ (\mathrm{t})$ 1.4

Equation 1.4 can be written as homogeneous systems as $\dot{\mathrm{X}} = \mathrm{A}\ (\mathrm{t})\ \mathrm{x}$ 1.5

Where $\mathrm{A}\ (\mathrm{t})$ is an *n* x *n* matrix of unknown coefficient
P; $R\ \to\ R^n$ is a continuous function.

The following scheme will be employed because of the use of Lyapunov functions.

(i) If $\mathrm{f}\ (\mathrm{t}, 0) = 0$ and if there exist a function
$\mathrm{V}: (0, \infty)$ x $R^n \to R$, that;
$w_1(|X|) \leq \mathrm{V}(\mathrm{t}, \mathrm{X})\ \leq -w_2(|X|)$ and
$\dot{\mathrm{V}}(\mathrm{t}, \mathrm{X})\ /\ 2 \cdot 2 \leq -w_3(|X|)$

Where $wi(i = 1, 2, 3)$ are strictly increasing continuous function defined as $wi(0, \infty) \to (0, \infty)$ with $w(s) > 0$ and $w(0) = 0$ as wedges. Then the solutions of equation 1.2 is uniformly as asymptotically stable.

(ii) If there exist function $\mathrm{V}: (0, \infty)$ x $R^n \to R$, such that,
$w_1(|X|) \leq \mathrm{V}(\mathrm{t}, \mathrm{X})\ \leq w_2(|X|)$ and
$\dot{\mathrm{V}}(t, X) \leq -w_3(|X|) + M(M > 0)$,

then the solutions of equation 1.2 are ultimately bounded and uniformly ultimately bounded

(iii) If the solution of Equation 1.3 and 1.4 are unique, the Equation 1.3 has a periodic solution.

We shall state without proof,

THEOREM OF BURTON [5].
Theorem A [5]: If f is Lip Schitz in X and periodic in t with period T and if the solutions are uniformly bounded and uniformly for any given bound (say) B, then equation 1.4 has a T – periodic solution.

Theorem B [5]: Assume the following conditions hold.
(i) $\mathrm{f}(\mathrm{t}, +\mathrm{T}, \mathrm{X}) = \mathrm{f}(\mathrm{t}, \mathrm{X})$ for all t and some $\mathrm{T} > 0$;
(ii) all solutions of equation 1.3 are bounded;
(iii) each solution of equation 1.3 is equi-asymptotically stable;
(iv) the zero solution of the homogeneous system corresponding to equation 1.3 is uniformly asymptotically stable.

Then equation 1.3 has a globally stable non-autonomous solution.

2.0 STATEMENT OF RESULTS

The following results will be basic to the proofs of lemma 2.2 and 2.3.

Theorem 2.2: Let g and h be continuous and also periodic with period w together, and the following conditions holds;

(i). $H_o = \frac{h(x)-h(o)}{x} \leq \alpha \in 1_0,\ x \neq 0\ and\ h(0) = 0$

(ii). $G_o = \frac{g(y)-g(o)}{Y} \leq \beta,\ y \neq 0\ and\ g(0) = 0$

(iii) $\mathrm{a(t,),b(t)}$ continuous with $0 < 90 < a \leq a(t) \leq \mathrm{a_1}, 0 < bo < b(\mathrm{t}) \leq \mathrm{b1}$ and

(iv) $|P(t: x, y)| \leq M\ \ (M = constant)$.

Then Equation 1.1 has a globally stable and boundedness of periodic solution with ω, as the period.

3.0 SOME PRELIMINARIES

We shall use the function $\mathrm{V(t,x,y)}$ defined below to prove the main theorem of this paper.

Let $\left\{2\,\mathrm{V(t,x,y)} = \frac{\delta}{\alpha \propto \beta}\,\mathrm{H(t)}(\propto \mathrm{ab} + \beta^2)x^2 + \frac{1}{9}\mathrm{y}^2 + 2\beta\right\}$ 3.1

Where $H(t)$ is defined = exP $\left(-\int_o^t \mathrm{a(s)ds}\right)$ where a, b, $\alpha, \beta, \delta > 0, for\ all\ x, t \in H(t)$.

Lemma 3.3 assume theorem 2.1 holds, there exist positive constants $Mi\ (\mathrm{a}, b, \alpha, \beta, \delta), i = 1{,}2$ such that

$Mi\ (x^2 + y^2) \leq V(t;\ x, y) \leq K_2(x^2 + y^2)$ 3.2

Proof: From equation 3.1, it is clear that $V(t{:}\,0,0) \equiv 0$.

Equation 3.1 also gives;

$2V(t;\ x, y) = \frac{\sigma}{\mathrm{a}\alpha\,\beta} H(t)\left\{\propto \mathrm{ab}x^2 + \beta^2(x + \frac{1}{\beta}\mathrm{y})^2 + \frac{1-\mathrm{a}\beta^2}{\mathrm{a}}\mathrm{y}^2\right\}$ 3.3

$2V(t;\ x, y) = \frac{\sigma}{\mathrm{a}\alpha\,\beta} H(t)\left\{\alpha\mathrm{ab}x^2 + \frac{1-\beta^2}{\mathrm{a}}\mathrm{y}^2\right\}$ 3.4

$\geq Mi(x^2 + y^2)$ 3.5

Where $Mi\frac{\sigma}{a\alpha\,\beta} \cdot Min\ \left\{\alpha\mathrm{ab}, \frac{1-\alpha\beta^2}{\mathrm{a}}\right\}$

Therefore $2V(t;\ x, y) \geq Mi(x^2 + y^2)$.

Using in equality on equation 3.1, $x\,y\ \leq \frac{1}{2}(x^2 + y^2)$

Gives, $2V(t;\ x, y) \geq \frac{\sigma}{\mathrm{a}\alpha\,\beta} H(t)\left\{(\alpha\mathrm{ab}x^2)\ x^2 + \frac{1}{\mathrm{a}}\mathrm{y}^2 + \beta(x^2 + y^2)\right\}$ 3.6

implies that, $2V \geq M_2\ (x^2 + y^2)$ 3.7

where $M_2 = \frac{\sigma}{\mathrm{a}\alpha\,\beta} \cdot Max\left\{(\alpha\mathrm{ab} + \beta(\beta + 1), (\frac{1-\mathrm{a}\beta}{\mathrm{a}})\right\}\cdot$

From equation 3.5 and 3.7, we have;

$M_1\ (x^2 + y^2) \leq V(t;\ x, y) \leq M_1\ (x^2 + y^2)$ 3.8

This proves the Lemma.

Lemma 3.2 Assump Theorem 2.1 holds, there exist positive constants, $M_j = M_j(a, b, \alpha, \delta)$, where $(j = 3, 4)$ such that for any solution (x, y) $(1 \cdot 1)$.

$$\dot{V}\Big|(1\cdot 1) \equiv \frac{d}{dt}V\Big|(1\cdot 1)(t; x, y) \le -M_3(x^2+y^2) + M_4(|x|+|y|)|p(t; x, y)| \qquad 3.9$$

Proof; from equation 1.1, we have;

$$\dot{V}|_{1.1} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\dot{x} + \frac{\partial v}{\partial y}\dot{y} = -H(t)R(x,y) + H(t)\frac{\partial v}{\partial x}y + \frac{\partial v}{\partial y}(-ag(y) - bh(u) + p(t)$$

Where $R(x,y) = \left\{(\alpha ab + \beta^2)x^2 + \frac{1}{a}y^2 + 2\beta xy\right\}$

$$\forall t, x, y \le \frac{\delta}{\alpha\beta}H(t)\{R(x,y) + M_2(x^2+y^2)\}$$

$$M_b(|x|+|y|)P(t; x, y)\} \qquad 3.10$$

Where $M_b = Max\ (b\alpha, \alpha\beta)$ and M_2 is defined in the equation 3.8 from the definition of H(t), we have equation 3.10 reduces to

$$\dot{V}(t;\ x, y) \le -M_3(x^2+y^2) + M_b(|x|+|y|)P(t;\ x, y) \qquad 3.11$$

With $M_3 = 2M_{2.}$

Equation 3.11 can be simplified to give

$$\forall(t, x, y) \le M_3(x^2+y^2) + M_4(x^2+y^2)½\ P(t; x, y) \qquad 3.12$$

With $M_4 = \sqrt{2M}$

This completes the proof of the Lemma.

4.0 PROOF OF THE MAIN RESULTS

Proof of Theorem 2.1 from the two proved Lemma given above, it had been established that the function $\vee(t;\ x, y)$ is a Lyapunov function for the system $\dot{x}$ = y

$\dot{y} = a(t)g(y) - b(t)\ h(x) + P(t; x, y)$. Hence the trivial solution from the above expression is asymptotically stable.

From equation 3.12, $\dot{V}(t;\ x, y) \le -\ M_3\ (x^2+y^2) +\ M_4\ (x^2+y^2)^{\frac{1}{2}}$

$P(t, x, y)$ and also from 3.5, we have $(x^2+y^2)^{\frac{1}{2}} \le (\frac{2V}{M1})^{\frac{1}{2}}$

hence equation 3.12 becomes;

$$\frac{dv}{dt} \le\ M_6V +\ M_7\ V^{\frac{1}{2}}\ |P(t)| \qquad 4.1$$

It shall be observed that $M_3\ (x^2+y^2) =\ M_3\frac{V}{K_1}$ and

$\frac{dv}{dt} \leq M_6 V + M_7 V^{\frac{1}{2}} |P(t)|$ 4.2

Where $M_6 = \frac{M_3}{M_2}$ and $M_7 = \frac{M_5}{M_2^{\frac{1}{2}}}$

Which implies that $V^{\cdot} \leq - M_6 \mathrm{V} + M_7 ½ |\mathrm{P}(t)|$ also can be written as

$V^{\cdot} \leq - 2M_8 \mathrm{V} + M_7 \mathrm{V}^{½} |\mathrm{P}(t)|$ 4.3

when $M_8 = ½ M_6$

Implies $\dot{V} + M_8 V \leq -M_8 \mathrm{V} + M_7 \mathrm{V}^{½} |\mathrm{P}(t)|$ 4.4

$\dot{V} + M_8 V \leq M_7 \mathrm{V}^{½} \{|\mathrm{P}(t)| - M_9 \mathrm{V}^{½}\}$ 4.5

Where $M_9 = \frac{M_8}{M_7}$, equation 4.5 becomes

$\dot{V} + M_8 V \leq M_7 \mathrm{V} \frac{1}{2} \mathrm{V}^{\mathrm{b}}$ 4.6

Where $\mathrm{V}_b = |\mathrm{P}(t)| - M_9 \mathrm{V}^{½}$ 4.7

$\leq \mathrm{P}|(t)|$ 4.8

When $\mathrm{P}|(t)| \leq M_9 \mathrm{V}^{½}, \; \mathrm{V}_b \leq 0$ 4.9

and when $|\mathrm{P}(t)| - M_9 \mathrm{V}^{½}, \mathrm{V}_b \leq |\mathrm{P}(t)| \frac{1}{M_9}$ 4.10

Substituting equation 4.10 into 4.5, we have;

$\dot{V} + M_8 \mathrm{V} \leq \mathrm{M}_{10} \mathrm{V}^{½} |\mathrm{P}(t)|$

Where $\mathrm{M}_{10} = \frac{\mathrm{M}_7}{\mathrm{M}_9}$

This becomes $\mathrm{V}^{-½} \mathrm{V} + M_8 \mathrm{V}^{½} \leq \mathrm{M}_{10} |\mathrm{P}(t)|$ 4.11

Multiplying both sides of 4.11 by $e^{½} Mst$ we have

$e^{½} Mst \{\mathrm{V}^{-½} V + M_8 V½\} \leq e^{½} Mst \; \mathrm{M}_{10} |\mathrm{P}(t)|$ 4.12

Which implies that;

$\partial \frac{d}{dt} \{V½ \, e^{½} Mst\} \leq e^{½} Mst \; \mathrm{M}_{10} |\mathrm{P}(t)|$ 4.13

Integrating both sides of equation 4.13 from t_o to t gives; $\{\mathrm{V}^{½} \, e^{½} Mst\}_{to}^{t} \leq \int_{to}^{t} ½ \, e^{½} Mst \; \mathrm{M}_{10} |\mathrm{P}(t)| dt =$ $\{\mathrm{V}^{½}(t)\} \, e^{½} Mst \leq V½(to) \, e^{½} Msto + \frac{1}{2} \mathrm{M}_{10} \int_{to}^{t} |\mathrm{P}(t)| \; e^{½} Mst \cdot dt$ 4.14

Using equation 3.5 and 3.7 we have;

$\mathrm{K}_1 (x^2(t) + \dot{x}^2 \, (t) \leq e^{-½} Mst \left\{ M_2 (x^2(t_o) + \dot{\mathrm{x}}^2(t_o)) \, e^{½} Mst_o + \frac{1}{2} \mathrm{M}_{10} \int_{to}^{t} |\mathrm{P}(t)| \; e^{½} Mst \; dt \right\}^2$ 4.15

for all $\leq t_o$, thus;

$$x^2(t) + \dot{x}^2(t) \leq \frac{1}{M_1}\left\{ e^{-½Mst}\left\{M_2\, x^2(t_o) + \dot{x}^2(t_o)\, e^{½Mst_o} + \frac{1}{2}\, M_{10} \int_{to}^{t} |P(t)|\ \ e^{½Mst}\, dt\right\}^2\right\} \leq \left\{ e^{½Mst}\left\{A_1 + A_2 \int_{to}^{t} |P(t)|\right\} e^{½Mst}\, dt\right\}^2 \qquad 4.16$$

By substituting $M_8 = M$ in equation 4.16, we have;

$$x^2(t) + \dot{x}(t) \leq\ e^{-½\mu t}\left\{A_1 + A_2 \int_{to}^{t} |P(t)|\ \ e^{-½\mu t} dt\right\}^2 \qquad 4.17$$

Equation 4.17 is the completion of the proof.

REMARK: From the proof of the theorem, below corollary can be pointed out as the direct consequence of the theorem.

Corollary 4.1: If $P(t; x, y)\ \leq (|x| + |y|)\ \varphi(t)$, where $\varphi(t)$ is a non-negative and continuous function of (t) and satisfies $\int_{to}^{t} \varphi\,(s)NS \leq M < \infty\ and\ M, a\ positive\ constant.$

Then, there exists a constant K which depends on $M, K_1K_2$and to such that every solution $x(t)$ of equation 1.1 satisfies $|x(t)| \leq K_{0,}\ |\dot{x}(t)| \leq K_0$ for sufficiently target.

Corollary4.2: If $P(t; x, y) = 0$, equation 4.17 becomes $x^2(t) + x^2(\dot{t}) \leq\ e^{-½\, MtA_1}$,
and as $t \to \infty, x^2(t) + \dot{x}^2(t) \to 0$ which implies that the trivial solution of equation 1.1 is globally asymptotically stable.

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