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Some modified Newton's methods withfourth-order convergence

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ABSTRACT

Based on Newton method, we derive three efficient methods of order four for solving nonlinear equations. Per iteration each method requires three evaluations and therefore the efficiency index of the methods is 1.587 which is better than Newton's efficiency index 1.414. Performance of the methods is compared with closest competitors in a series of numerical examples.It is shown by way of illustration these methods are very useful in the applications requiring high precision in computations. Moreover, theoretical order of convergence is verified on the examples.

Keywords: Nonlinear equations; Newton's method; Ostrowski'smethod; Root-finding; Order of convergence.

INTRODUCTION

In this paper, we deal with iteration methods for calculating simple root of a nonlinear equation $f(x)=0$. This problem is prototype for many nonlinear numerical problems [1]. Traub [2] has classified numerical methods into two categories viz. (i) one-point iteration methods with and without memory, and (ii) multipoint iteration methods with and without memory. Kung and Traub [3] have conjectured that multipoint iteration methods without memory based on n evaluations has optimal order 2^{n-1}. In particular, with three evaluations a method of fourth-order can be constructed. The famous Ostrowski's method [4] is an example of fourth-order multipoint methods without memory which is defined as

$$w_i = x_i - \frac{f(x_i)}{f'(x_i)},$$

$$x_{i+1} = w_i - \frac{f(w_i)}{f'(x_i)} \frac{f(x_i)}{f(x_i) - 2f(w_i)}. \qquad (1)$$

The method requires two function f and one derivative f' evaluations per step and is seen to be efficientthan classical Newton's method. Other well-known example of fourth-order multipoint methods with same number of evaluations is King's family of methods [5]. This family is written as

$$w_i = x_i - \frac{f(x_i)}{f'(x_i)},$$

$$x_{i+1} = w_i - \frac{f(w_i)}{f'(x_i)} \frac{f(x_i) + A f(w_i)}{f(x_i) + (A-2) f(w_i)}, \qquad (2)$$

where $A \in \mathbb{R}$ is some parameter.The Ostrowski's method (1) can be seen as a particular case of this family for $A = 0$.

Through this work, we contribute a little more in the development ofthe theory of iteration methods and derive threemultipoint methods of order four. Each methodrequirestwo f and one f' evaluations per iteration and thus the efficiency index(see [6])is same that of Ostrowski's method. These methods are based on Newton's method and consist of two substeps, one Newton substep followed by another generated by quadratic interpolation. For this reason we shall call them modified Newton's methods. We employ new methods to solve some non-linear equations and compare it with well-known methods.

Basic definitions

Definition 1. Let $f(x)$ be a real function with a simple root α and let $\{x_i\}_{i \in N}$ be a sequence of real numbers that converges towards α. We say that the order of convergence of the sequence is *p*, if there exits a $p \in \mathbb{R}^+$ such that

$$\lim_{i \to \infty} \frac{x_{i+1} - \alpha}{(x_i - \alpha)^p} = C \neq 0. \qquad (3)$$

If $p = 2$ or 3, the sequence is said to have quadratic convergence or cubic convergence, respectively.

Definition 2.Let $e_i = x_i - \alpha$ is the error in the *i*th iteration, we call the relation

$$e_{i+1} = C e_i^p + O(e_i^{p+1}), \qquad (4)$$

as the error equation. If we can obtain the error equation for any iterative method, then the value of *p*is its order of convergence.

Definition 3.Suppose that x_{i+1}, x_i and x_{i-1} are three successive iterations closer to the root α. Then, the computational order of convergence ρ (see [7]) is approximated by using (4) as

$$\rho \cong \frac{\ln |(x_{i+1} - \alpha)/(x_i - \alpha)|}{\ln |(x_i - \alpha)/(x_{i-1} - \alpha)|}. \qquad (5)$$

Definition 4. Let ωbe the number of new pieces of information required by a method. A 'piece of information' typically is any evaluation of a function or one of its derivatives. The efficiency of the method is measured by the concept of efficiency index [6] and is defined by

$$E = p^{1/\omega}, \qquad (6)$$

where p is the order of the method.

Development of the methods

Consider the Newton scheme defined by

$$w_i = x_i - \frac{f(x_i)}{f'(x_i)}, \qquad i \geq 0. \tag{7}$$

In what follows, we construct the method for obtaining the approximation x_{i+1} to the root by considering the quadratic function which interpolates f. Let the interpolating function be

$$y(x) = a + bx + cx^2, \tag{8}$$

such that

$$y(x_i) = f(x_i), \tag{9}$$

$$y'(x_i) = f'(x_i), \tag{10}$$

$$y(w_i) = f(w_i), \tag{11}$$

$$y(x_{i+1}) = 0. \tag{12}$$

With the help of (9) – (11) in (8), we can obtain the unknown parameters a, b and c used in (8). Thus, introducing (9), (10) and (11) in (8), we get

$$f(x_i) = a + bx_i + cx_i^2, \tag{13}$$

$$f'(x_i) = b + 2cx_i, \tag{14}$$

$$f(w_i) = a + bw_i + cw_i^2. \tag{15}$$

From (13) – (15) and using Newton iteration (7) for w_i, we may calculate a, b and c as

$$a = f(x_i) - x_i f'(x_i) + cx_i^2,$$

$$b = f'(x_i) - 2cx_i,$$

$$c = \frac{f'^2(x_i) f(w_i)}{f^2(x_i)}. \tag{16}$$

The estimate to the root x_{i+1} is obtained from (8), which implies that

$$a + bx_{i+1} + cx_{i+1}^2 = 0. \tag{17}$$

Using (16) in (17) and solving, we ultimately obtain the iteration formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \frac{2}{1 + \sqrt{1 - 4 f(w_i)/f(x_i)}}, \tag{18}$$

where w_i is the Newton iteration (7).

We further derive iteration formula free from square root term. This can be achieved using the approximation

$$\left[1 - 4\frac{f(w_i)}{f(x_i)}\right]^{1/2} \cong 1 - 2\frac{f(w_i)}{f(x_i)} - 2\frac{f^2(w_i)}{f^2(x_i)}, \tag{19}$$

in formula (18), which yields the iteration method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \frac{1}{1 - f(w_i)/f(x_i) - f^2(w_i)/f^2(x_i)}. \tag{20}$$

Furthermore, the expansion

$$\left[1-\frac{f(w_i)}{f(x_i)}-\frac{f^2(w_i)}{f^2(x_i)}\right]^{-1}\cong 1+\frac{f(w_i)}{f(x_i)}+2\frac{f^2(w_i)}{f^2(x_i)}, \quad (21)$$

suggests us to obtain another elegant formula. This is obtained using (17) in (16) and is given by

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}\left[1+\frac{f(w_i)}{f(x_i)}+2\frac{f^2(w_i)}{f^2(x_i)}\right]. \quad (22)$$

Thus, we derive three modified Newton's methods based on the composition of two substeps, Newton substep (7) and another obtained by quadratic interpolation. It is straightforward to see that per step the methods require two evaluations of f and one of f'. In order to show that the methods (18), (20) and (22) are of order four, we prove the following theorems:

Theorem 1. *Let $f: I\to\mathbb{R}$ denote a real valued function defined on I, where I is a neighborhood of a simple root α of $f(x)$. Assume that $f(x)$ is sufficiently smooth in I, then the method defined by (18) is of order four.*

Proof. Let e_i be the error at i$^{\text{th}}$ iteration, then $e_i=x_i-\alpha$. Denote

$$A_k=(1/k!)\,f^{(k)}(\alpha)/f'(\alpha), \quad k=2,3,\ldots\ldots\ldots\ldots$$

Expanding $f(x_i)$ and $f'(x_i)$ about α and using the fact that $f(\alpha)=0,\ f'(\alpha)\neq 0,$ we have

$$f(x_i)=f'(\alpha)[e_i+A_2e_i^2+A_3e_i^3+A_4e_i^4+O(e_i^5)] \quad (23)$$

and

$$f'(x_i)=f'(\alpha)[1+2A_2e_i+3A_3e_i^2+4A_4e_i^3+O(e_i^4)]. \quad (24)$$

Then,

$$\frac{f(x_i)}{f'(x_i)}=e_i-A_2e_i^2+2(A_2^2-A_3)e_i^3+(7A_2A_3-4A_2^3-3A_4)e_i^4+O(e_i^5). \quad (25)$$

Substitution of (25) in (7) yields

$$w_i-\alpha=A_2e_i^2-2(A_2^2-A_3)e_i^3-(7A_2A_3-4A_2^3-3A_4)e_i^4+O(e_i^5). \quad (26)$$

Expanding $f(w_i)$ about α and using (26), we obtain

$$f(w_i)=f'(\alpha)[A_2e_i^2-2(A_2^2-A_3)e_i^3-(7A_2A_3-5A_2^3-3A_4)e_i^4+O(e_i^5)]. \quad (27)$$

From (23) and (27), we have

$$\frac{f(w_i)}{f(x_i)}=A_2e_i+(2A_3-3A_2^2)e_i^2+(8A_2^3-10A_2A_3+3A_4)e_i^3+O(e_i^4). \quad (28)$$

Furthermore

$$1+\left[1-4\frac{f(w_i)}{f(x_i)}\right]^{1/2}=2-2A_2e_i+4(A_2^2-A_3)e_i^2-2(4A_2^3-6A_2A_3+3A_4)e_i^3+O(e_i^4). \quad (29)$$

From (25) and (29), we get

$$\frac{f(x_i)}{f'(x_i)}\frac{2}{1+\sqrt{1-4\,f(w_i)/f(x_i)}}=e_i+A_2A_3e_i^4+O(e_i^5). \quad (30)$$

Thus using (30) in (18), we get the error as $e_{i+1}=-A_2A_3e_i^4+O(e_i^5).$ (31)

That means the method (18) is of order four. □

Theorem 2. *Under the hypotheses of theorem 1, the method defined by (16) is of order four.*

Proof. Squaring (28) yields

$$\frac{f^2(w_i)}{f^2(x_i)}=A_2^2e_i^2+2A_2(2A_3-3A_2^2)e_i^3+O(e_i^4). \quad (32)$$

From (28) and (32) it follows that

$$1-\frac{f(w_i)}{f(x_i)}-\frac{f^2(w_i)}{f^2(x_i)}=1-A_2e_i-2(A_3-A_2^2)e_i^2-(2A_2^3-6A_2A_3+3A_4)e_i^3+O(e_i^4). \qquad (33)$$

Division of (25) by (33) yields

$$\frac{f(x_i)}{f'(x_i)}\frac{1}{1-f(w_i)/f(x_i)-f^2(w_i)/f^2(x_i)}=e_i+A_2(A_3-2A_2^2)e_i^4+O(e_i^5). \qquad (34)$$

Then, from (20) and (34) the error is given by

$$e_{i+1}=A_2(2A_2^2-A_3)e_i^4+O(e_i^5). \qquad (35)$$

Thus method (20) has order of convergence four. □

Theorem 3. *Under the hypotheses of theorem 1, the method defined by (20) is of order four.*

Proof.Using (28) and (32), we obtain

$$1+\frac{f(w_i)}{f(x_i)}+2\frac{f^2(w_i)}{f^2(x_i)}=1+A_2e_i+(2A_3-A_2^2)e_i^2-(4A_2^3+2A_2A_3-3A_4)e_i^3+O(e_i^4). \qquad (36)$$

Multiplication of (25) and (36) yields

$$\frac{f(x_i)}{f'(x_i)}\left[1+\frac{f(w_i)}{f(x_i)}+2\frac{f^2(w_i)}{f^2(x_i)}\right]=e_i+A_2(A_3-5A_2^2)e_i^4+O(e_i^5). \qquad (37)$$

Then substitution of (37) in (22) yields the error as

$$e_{i+1}=A_2(5A_2^2-A_3)e_i^4+O(e_i^5). \qquad (38)$$

Therefore, the method defined by (22) has order of convergence four. □

Notice that the computational efficiency [6] for these methods is $4^{1/3}\cong 1.587$, which is equal to the efficiency of Ostrowski's method. For real roots, method (18) requires $f(w_i)/f(x_i)\le 0.25$. The binomial approximation (19) is valid only if $|f(w_i)/f(x_i)|<0.25$. Also, the expansion (21) is true only if $|f(w_i)/f(x_i)+f^2(w_i)/f^2(x_i)|<1$, that is, if $-1.62<f(w_i)/f(x_i)<0.62$. These restrictions on $f(w_i)/f(x_i)$ may necessitate the use of multiple precision arithmetic. This is because as x approaches to α, the methods involve the division of quantities that are both approaching to zero.Recall, that theOstrowski's method has similar kind of behaviour. However, numerical experimentation indicates that there is no difficulty in applying the methods in practice.

Numerical examples

We employ the present methods (18), (20) and (22) designated as M_1, M_2 and M_3, respectively to solve some nonlinear equations and compare it with Newton's method (NM), and Ostrowski's method (OM). We accept an approximate solution rather than the exact root, depending on the computer precision $(\in)$. The stopping criteria used for computer program: (a) $|x_{i+1}-x_i|<\in$, (b) $|f(x_{i+1})|<\in$, and so, when the stopping criterion is satisfied, x_{i+1} is taken as the computed root α. The test functions and root α correct up to 16 decimal places are displayed in table 1. Table 2 shows the values of initial approximation x_0 chosen from both ends to the root, the number of iterations (i) required to approximate the root and the computational order of convergence (ρ) defined by (5). For numerical illustrations in table 2, we use fixed stopping criterion $\in=0.5\times10^{-17}$. It is well-known that the convergence of iteration formula is guaranteed only when the initial approximation is sufficiently near to root. In general, it may be divergent when initial approximation is far from the root. However, we can observe from the numerical results that in almost all of the examples, the presented methods appear to be robust. Also the computed order (ρ) agrees with the theoretical order of convergence for every test function.

In table 3, we exhibit the absolute values of the error e_i calculated by costing the same total number of function evaluations (TNFE) by all the methods. The TNFE is counted as sum of the number of evaluations of the function itself plus the number of evaluations of the derivative. Here, TNFE used for all the methods is 12. That means for NM, the error $|e_i|$ is calculated at 6th iteration, whereas for OM, M_1, M_2 and M_3, these are calculated at 4th iteration.

Table 1. Test functions

$f(x)$	Root (α)
$f_1(x)=x^3+4x^2-15$	1.6319808055660635
$f_2(x)=\sin(x)-x/2$	1.8954942670339809
$f_3(x)=e^{-x}+\cos(x)$	1.7461395304080124
$f_4(x)=10xe^{-x^2}-1$	1.6796306104284499
$f_5(x)=\tan^{-1}(x)-x+1$	2.1322677252728851
$f_6(x)=\int_0^x (e^{-t^3/2}-e^{-t^8/2})dt+0.1$	– 0.8805978315532975
$f_7(x)=\int_0^x \{\sin(xt)/t\}dt-0.5$	0.7121746841816167

Table 2. Performance of the methods

$f(x)$	x_0	I					ρ				
		NM	OM	M_1	M_2	M_3	NM	OM	M_1	M_2	M_3
f_1	1	6	3	3	3	4	2.0	4.16	4.16	4.1	4.0
	2.5	6	3	3	3	3	2	4.14	4.17	8	4
f_2	1.5	6	3	3	3	4	2.0	4.08	4.11	4.0	4.0
	2.5	5	3	3	3	3	1	4.07	4.13	8	0
f_3	– 0.5	5	3	3	3	3	2.0	4.16	4.17	4.1	4.1
	2.5	5	3	3	3	3	6	4.20	4.20	6	4
f_4	1	5	3	3	3	3	2.0	4.00	4.03	3.9	3.9
	2	6	3	3	4	4	0	4.00	4.03	7	2
f_5	1	5	3	3	3	3	2.1	4.08	4.07	4.0	4.0
	3	4	3	3	3	3	0	4.04	4.04	6	2
f_6	– 1	5	3	3	3	3	1.9	3.98	3.97	3.9	3.9
	– 0.75	6	3	3	3	3	6	3.98	3.97	7	6
f_7	0.5	5	3	3	3	3	2.0	4.01	4.01	4.0	4.0
	1	5	3	3	3	3	3	4.01	4.02	1	1

Table 3. Accuracy using same TNFE=12 for all methods

$f(x)$	x_0	$\lvert e_i \rvert = \lvert x_i - \alpha \rvert$				
		NM	OM	M_1	M_2	M_3
f_1	1 2.5	2.56E – 31	1.33E – 136	4.37E – 182	4.01E – 82	1.68E – 38
f_2	1.5 2.5	1.54E – 33	1.21E – 127	3.57E – 223	3.94E – 82	2.84E – 42
f_3	– 0.5 2.5	1.55E – 61	3.78E – 171	3.58E – 172	2.59E – 170	9.81E – 169
f_4	1 2	8.46E – 43	4.88E – 116	3.58E – 137	5.68E – 109	8.56E – 100
f_5	1 3	6.26E – 48	1.49E – 133	9.30E – 152	5.20E – 111	3.51E – 79
f_6	– 1 – 0.75	2.55E – 37	1.85E – 170	3.27E – 149	4.65E – 138	4.04E – 132
f_7	0.5 1	5.63E – 53	6.47E – 193	1.93E – 238	1.25E – 225	5.31E – 218

It is quite understood that increasing the order of the method leads us to obtain more precision widening the mantissa. For this reason and for better comparison as well, in table 3 all computations are done with multiprecision arithmetic using 300 significant digits. As shown in table 3, the fourth-order methods $(M_1, M_2 \text{ and } M_3)$ is preferable to second-order (NM) methods in high-precision computations. Moreover, in almost all the problems we consider, the M_1 even works better than OM.

CONCLUSION

In this paper, we have obtained multipoint iterative methods of third and fourth order for finding simple roots of nonlinear equations. The number of function evaluations required per iteration is three in both categories of the methods. These evaluations involve two f and one f', and no higher order derivative evaluations are required. The two important aspects of generating new algorithms are order of convergence and computational efficiency. Therefore, fourth-order methods are the main findings of the present work in terms of speed and efficiency. These facts can be observed from theoretical analysis as well as numerical experimentation. The computational order of convergence (ρ) overwhelmingly supports the theoretical order of convergence for all the methods.

Many numerical applications use high precision in their computations. In these types of applications, numerical methods of higher order are important. The numerical results (Table 3) show that the fourth-order methods associated with a multiprecision arithmetic floating point are very useful, because these methods yield a clear reduction in number of iterations. Finally we conclude that the methods presented in this paper are competitive with other recognized efficient equation solvers, namely Newton and Ostrowski methods.

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