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Planetary equations based upon Newton's Equations of motion and Newton's gravitational field of a static homogeneous oblate Spheroidal Sun

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ABSTRACT

The expression for the equations of motion for a particle in Cartesian cylindrical and spherical coordinates and their applications in mechanics are well known. It is however, now well known that the planets, the sun and all rotating astronomical bodies are more precisely Spheroidal in geometry and the motions of test particles in them require spheroidal coordinates. Consequently, in this paper we derive the expression for the equations of motion for any test particle in oblate spheroidal coordinates to pave way for the corresponding extension of the well-known mechanics of spherical bodies.

Keywords: speroidal coordinates, equations of motion, mechanics, test particles

INTRODUCTION

In Newtonian mechanics, the motion of particles (such as projectile, satellites and pendulli) in Earth's atmosphere are treated under the assumption that the Earth is a perfect sphere [10]. Similarly, in the solar system the motion of bodies (such as planets, cornets and asteroids) are treated under the assumption that the sun is a perfect sphere [2,9]. Also in Einstein's theory, the motion of bodies (such as planets) and particles (such as photons) are treated under the assumption that the sun is a perfect sphere (Schwarz child's space-time). It is well known that the only reason for these restrictions are mathematical convenience and simplicity. The real fact of Nature is that all rotating planets, stars, and galaxies in the universe are spheroidal [5,6,7]. It is obvious that their spheroidal geometry will have corresponding consequences and effects in the motion of all particles in their gravitational field. These effects will exist in both Newtonian mechanics and in Einstein's theory. Consequently, we hereby prepare the way for the solution of the equation of motions of spheroidal bodies by deriving the equations of motion for oblate sphroidal bodies.

Mathematical Formulations

Consider a homogenous oblate spheroidal body of rest massM_0.Let S be the reference frame whose origin O coincides with the centre of the body as shown in fig.1 below

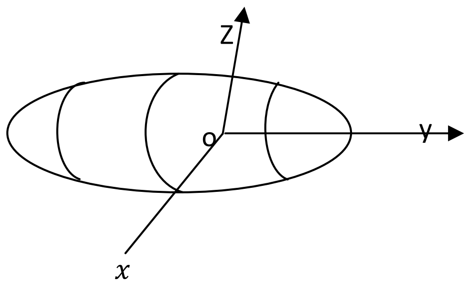

Fig.1 Oblate spheroid

Then the oblate spheroidal coordinates (η, ξ, ϕ) are defined in terms of the Cartesian (x, y, z) by [1,4]:

$$x = a(1-\eta^2)^{\frac{1}{2}}(1+\xi^2)^{\frac{1}{2}}cos\theta \quad (1)$$

$$y = a(1-\eta^2)^{\frac{1}{2}}(1+\xi^2)^{\frac{1}{2}}sin\theta \quad (2)$$

$$z = a\eta\xi \quad (3)$$

where a is a constant and

$$0 < \xi < \infty, -1 \le \eta \le 1, 0 \le \phi \le 2\pi \quad (4)$$

Also in the spheroidal coordinates, the surface of the spheroid is given by:

$$\xi = \xi_0 \quad (5)$$

where ξ_0 is a constant

Now if the body is homogeneous, its density, ρ, is given by:

$$\rho = \rho_0 \,;\, \xi \le \xi_0 \quad (6)$$

$$\rho_{(r)} = 0 \,;\, \xi > \xi_0 \quad (7)$$

where ρ_0 is the constant density of rest mass. It is well known that the Newton's gravitational field equation for the gravitational scalar potential F due to a distribution of mass density ρ is given by:

$$\nabla^2 F = 4\pi G\rho \quad (8)$$

where G is the universal gravitational constant. It follows from the explicit expression for the Laplacian operator in oblate spheroidal coordinates that the interior and exterior gravitational scalar potential, f^+ and f^- respectively, satisfy the equation.

$$\frac{1}{a2_{[\eta^2+\xi^2]}}\left[\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}\right] + \frac{\partial}{\partial\xi}\left[(1+\xi^2)\frac{\partial}{\partial\xi}\right] + \frac{\eta^2+\xi^2}{(1-\eta^2)(1+\xi^2)}\frac{\partial^2}{r\emptyset^2}\right]F^-(\eta\xi\phi)$$
$$= 4\pi G\rho_0 \quad (9)$$

and

$$\frac{1}{a2_{[\eta^2+\xi^2]}}\left[\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}\right] + \frac{\partial}{\partial\xi}\left[(1+\xi^2)\frac{\partial}{\partial\xi}\right] + \frac{\eta^2+\xi^2}{(1-\eta^2)(1+\xi^2)}\frac{\partial^2}{r\emptyset^2}\right]F^-(\eta\xi\phi) = 0 \quad (10)$$

By the symmetry of the distribution of the mass about the polar axis, the potential will be independent of the azimuthal angle. Hence (9) and (10) becomes:

$$\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}f^-(\eta\xi)\right] + \frac{\partial}{\partial\xi}\left[(1+\xi^2)\frac{\partial}{\partial\xi}f^-(\eta\xi)\right] = 4\pi G\rho_0 a^2{}_{(\eta^2+\xi^2)} \quad (11)$$

And

$$\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}f^+(\eta\xi)\right] + \frac{\partial}{\partial\xi}\left[(1+\xi^2)\frac{\partial}{\partial\xi}f^+(\eta\xi)\right] = 0 \quad (12)$$

General complementary solution of (11) is given as:

$$f_c^-(\eta,\xi) = \sum_{l=0}^{\infty}[A_l P_l(i\xi) + B_l Q_l(i\xi)][C_l P_l(\eta) + D_l Q_l(\eta)] \quad (13)$$

We seek particular solution of (11) as:

$$f_p(\eta\xi) = T(-\eta^2+\xi^2) \quad (14)$$

By using (14) in (11), we have:

$$T = \frac{2}{3}a^2{}_{\pi}G\rho_0 \tag{15}$$

Hence general solution of (11) and (12) are giving as:

$$f^-(\eta,\xi) = \sum_{l=0}^{\infty}[A_lP_l(i\xi) + B_lQ_l(i\xi)][C_lP_l(\eta) + D_lQ_l(\eta)] + T(\xi^2 - \eta^2) \tag{16}$$

and

$$f^+(\eta,\xi) = \sum_{l=0}^{\infty}[H_lP_l(-i\xi) + J_lQ_l(i\xi)][K_lP_l(\eta) + M_lQ_l(\eta)] \tag{17}$$

where $A_lB_l, C_l, D_l and H_l, J_l, K_l, M_l$ are arbitrary constants,$P_l and Q_l$ are the two linearly independent Legendre functions of order $l = 0, 1, 2, ...$ Now since the interior and exterior region both contain the Coordinate $\eta = 0$ which is a singularity of Q_l, we choose:

$$D_l \equiv M_l \equiv 0\,;\ l = 0, 1, 2\ ... \tag{18}$$

in the general solutions of (16) and (17). Also since $\xi = 0$ is a singularity ofQ_l, we choose:

$$B_l \equiv 0;\ l = 0, 1, 2, ... \tag{19}$$

Also, since P_l is not defined for $\xi \to \infty$ in the exterior region, we choose:

$$H_l \equiv 0;\ l = 0, 1, 2, ... \tag{20}$$

It follows that (16) and (17) becomes:

$$f^- = \sum_{l=0}^{\infty} A_lP_l(i\xi)C_lP_l(\eta) + T(\xi^2\eta^2)$$

$$= \sum_{l=0}^{\infty} A_lP_l(i\xi)P_l(\eta) + T(\xi^2\eta^2) \tag{21}$$

and

$$f^+ = \sum_{l=0}^{\infty} J_lQ(i\xi)K_lP_l(\eta) = \sum_{l=0}^{\infty} B_lQ(i\xi)P_l(\eta) \tag{22}$$

where $A_l and\ B_l$are arbitrary constant.

Consequently by the conditions of the continuity of the potentials and their normal derivatives at the $\xi = \xi_0$ boundary of the spheroid, it follows that:

$$A_0 = \frac{T\left\{\left[2Q_0\xi_0 - \left(\xi_0^2 - \frac{1}{3}\right)\left[\frac{d}{d\xi}Q_0(i\xi)\right]_{\xi=\xi_0}\right]\right\}}{\left[P_0((i\xi))\frac{d}{d\xi}Q_0(i\xi) - Q_0(i\xi)\frac{d}{d\xi}P_0(i\xi)\right]_{\xi=\xi_0}} \tag{23}$$

$$B_0 = \frac{T\left[\left(\xi_0^2 - \frac{1}{3}\right)\frac{d}{d\xi}P_0(i\xi) + 2P_0(i\xi)\xi\right]_{\xi=\xi_0}}{\left[P_0((i\xi))\frac{d}{d\xi}Q_0(i\xi) - Q_0(i\xi)\frac{d}{d\xi}P_0(i\xi)\right]_{\xi=\xi_0}} \tag{24}$$

$$A_1 = B_1 = 0 \tag{25}$$

$$A_2 = \frac{-\frac{2}{3}T\left[\frac{d}{d\xi}Q_2(i\xi)\right]_{\xi=\xi_0}}{\left[Q_2((i\xi))\frac{d}{d\xi}P_2(i\xi) - P_2(i\xi)\frac{d}{d\xi}Q_2(i\xi)\right]_{\xi=\xi_0}} \tag{26}$$

$$A_2 = \frac{-\frac{2}{3}T\left[\frac{d}{d\xi}P_2(i\xi)\right]_{\xi=\xi_0}}{\left[Q_2\big((i\xi)\big)\frac{d}{d\xi}P_2(i\xi) - P_2(i\xi)\right]} \quad (27)$$

and

$$A_l = B_l = 0; l = 3,4,5 \ldots \quad (28)$$

Consequently the final solutions are:

$$F^{+}(\eta,\xi) = A_0P_0(\eta)P_0(i\xi) + A_2P_2(\eta)P_2(i\xi) + T(\xi^2 - \eta^2) \quad (29)$$

And

$$F^{+}(\eta,\xi) = B_0P_0(\eta)Q_0(i\xi) + B_2P_2(\eta)Q_2(i\xi) \quad (30)$$

These are the Newtonian interior and exterior gravitational scalar potentials of the oblate spheroid in terms of its constant rest mass density ρ_0, surface coordinate ξ_0 and parameter a [5].

Newton's equations of motion, in oblate spheroidal coordinate are defined as:

$$\underline{a}^{-} = -(\nabla F^{-})(\eta,\xi,\phi) \quad (31)$$

$$\underline{a}^{+} = -(\nabla F^{+})(\eta,\xi,\phi) \quad (32)$$

And $\underline{a}$ is the instantaneous acceleration in terms of oblate spheroidal coordinates as:

$$\underline{a} = a_\eta\hat{\eta} + a_\xi\hat{\xi} + a_\phi\hat{\phi} \quad (33)$$

where

$$a_\eta = \frac{a(\eta^2+\xi^2)^{\frac{1}{2}}}{(1-\eta^2)^{\frac{1}{2}}}\left\{\ddot{\eta} + \frac{2\xi}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} + \frac{\eta(1-\xi^2)}{(1-\eta^2)(\eta^2\xi^2)}\dot{\eta}^2 - \frac{\eta(1-\eta^2)}{(1+\eta^2)(\eta^2+\xi^2)}\dot{\xi}^2 + \frac{\eta(1-\eta^2)(1+\xi^2)}{(\eta^2+\xi^2)}\dot{\phi}^2\right\} \quad (34)$$

$$a_\xi = \frac{a(\eta^2+\xi^2)^{\frac{1}{2}}}{(1-\eta^2)^{\frac{1}{2}}}\left\{\ddot{\xi} + \frac{2\eta}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} - \frac{\xi(1+\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 + \frac{\xi(1-\eta^2)}{(1+\eta^2)(\eta^2+\xi^2)}\dot{\xi}^2 - \frac{\xi(1-\eta^2)(1+\xi^2)}{(\eta^2+\xi^2)}\dot{\phi}^2\right\} \quad (35)$$

and

$$a_\phi = a(1-\eta^2)^{\frac{1}{2}}(1-\xi^2)^{\frac{1}{2}}\left\{\ddot{\phi} - \frac{2\eta}{1-\eta^2}\dot{\eta}\dot{\phi} + \frac{2\xi}{1+\xi^2}\dot{\xi}\dot{\phi}\right\} \quad (36)$$

Also, the del or nebla ($\underline{\nabla}$) is expressed interm of oblate spheroidal coordinate as:

$$\underline{\nabla}(\eta,\xi,\phi) = \frac{\hat{\eta}(1-\eta^2)^{\frac{1}{2}}}{a(\eta^2+\xi^2)^{\frac{1}{2}}}\frac{\partial}{\partial\eta} + \frac{\hat{\xi}(1+\xi^2)^{\frac{1}{2}}}{a(\eta^2+\xi^2)^{\frac{1}{2}}}\frac{\partial}{\partial\xi} + \hat{\phi}\frac{1}{a[(1-\eta^2)(1+\xi^2)]^{\frac{1}{2}}}\frac{\partial}{\partial\phi} \quad (37)$$

Then;

$$\underline{a}^{-}(\eta) = -(\nabla F^{-})(\eta)$$

$$\ddot{\eta} + \frac{2\xi}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} + \frac{\eta(1-\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 - \frac{\eta(1-\eta^2)}{(1+\eta^2)(\eta^2+\xi^2)}\dot{\xi}^2 + \frac{\eta(1-\eta^2)(1+\xi^2)}{(\eta^2+\xi^2)} + \eta(3A_2P_2(i\xi) - 2T) = 0 \quad (38)$$

$$\underline{a}^{-}(\xi) = -(\nabla F^{-})(\xi)$$

$$\ddot{\xi} + \frac{2\xi}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} + \frac{\xi(1-\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 + \frac{\xi(1-\eta^2)}{(1+\xi^2)(\eta^2+\xi^2)}\dot{\xi}^2 - \frac{\xi(1-\eta^2)(1+\xi^2)}{(\eta^2+\xi^2)} + A_0P_0^1(i\xi) + \frac{1}{2}A_2(3\eta^2-1)P_2^1(i\xi) + 2T\xi = 0 \quad (39)$$

$\underline{a}^-(\phi) = -(\nabla F^-)(\phi)$

$$\ddot{\phi} - \frac{2\xi}{(1-\eta^2)}\dot{\eta}\dot{\phi} + \frac{2\xi}{(1+\xi^2)}\dot{\xi}\dot{\phi} = 0 \quad (40)$$

$\underline{a}^+(\eta) = -(\nabla F^+)(\eta)$

$$\ddot{\eta} + \frac{2\xi}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} + \frac{\eta(1-\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 - \frac{\eta(1-\eta^2)}{(1+\eta^2)(\eta^2+\xi^2)}\dot{\xi}^2 + \frac{\eta(1-\eta^2)(1+\xi^2)\phi^2}{(\eta^2+\xi^2)} + 3B_3Q(i\xi) = 0 \quad (41)$$

And

$$\underline{a}^+(\xi) = -(\nabla F^+)(\xi)\ddot{\xi} + \frac{2\xi}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} - \frac{\xi(1-\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 + \frac{\xi(1-\eta^2)}{(1+\xi^2)(\eta^2+\xi^2)}\dot{\xi} - \frac{\xi(1-\eta^2)(1+\xi^2)\phi^2}{(\eta^2+\xi^2)} + B_0Q_0^1(i\xi) + \frac{1}{2}B_2Q_2^1(i\xi)(3\eta^2-1) = 0 \quad (42)$$

And

$\underline{a}^+(\phi) = -(\nabla F^+)(\phi)$

$$\ddot{\phi} - \frac{2\xi}{(1-\eta^2)}\dot{\eta}\dot{\phi} + \frac{2\xi}{(1+\xi^2)}\dot{\xi}\dot{\phi} = 0 \quad (43)$$

Equation (42) integrates exactly to yield

$$\ddot{\phi} = \frac{L}{(1-\eta^2)(1-\xi^2)} \quad (44)$$

Where L is a constant. It follows that (37),(38),(40) and (41) becomes:

$$\ddot{\eta} + \frac{2\xi}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} + \frac{\eta(1-\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 - \frac{\eta(1-\eta^2)}{(1+\eta^2)(\eta^2+\xi^2)}\dot{\xi}^2 + \frac{\eta L^2}{(\eta^2+\xi^2)(1-\eta^2)(1+\xi^2)} + \eta(3A_2P_2(i\xi) - 2T) = 0 \quad (45)$$

$$\ddot{\xi} + \frac{2\eta}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} - \frac{\xi(1-\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 + \frac{\xi(1-\eta^2)}{(1+\eta^2)(\eta^2+\xi^2)}\dot{\xi}^2 - \frac{\xi L^2}{(\eta^2+\xi^2)(1-\eta^2)(1+\xi^2)} + A_0P_0^1(i\xi) + \frac{1}{2}A_2(3\eta^2-1)P_2 1_{(i\xi)} + 2T\xi = 0 \quad (46)$$

And

$$\ddot{\eta} + \frac{2\xi}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} + \frac{\eta(1-\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 - \frac{\eta(1-\eta^2)}{(1+\eta^2)(\eta^2+\xi^2)}\dot{\xi}^2 + \frac{\eta L^2}{(\eta^2+\xi^2)(1-\eta^2)(1+\xi^2)} + 3\eta B_2Q_2(i\xi) = 0 \quad (47)$$

$$\ddot{\xi} + \frac{2\xi}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} - \frac{\xi(1-\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 + \frac{\xi(1-\eta^2)}{(1+\eta^2)(\eta^2+\xi^2)}\dot{\xi}^2 - \frac{L^2}{(\eta^2+\xi^2)(1-\eta^2)(1+\xi^2)} + B_0Q_0^1(i\xi) + \frac{1}{2}B_2Q_2^1(i\xi)(3\eta^2-1) = 0 \quad (48)$$

This is the completion of the equation of motions of oblate speroidal coordinate system.

RESULTS AND DISCUSSION

In this paper we have derived the Newton's equations of motion for the interior and exterior scalar gravitational potential in oblate spheroidal coordinates as (45),(46),(47) and (48) respectively.

These equations (45) to (48) extend Newton's theory of classical mechanics from the well-known spherical bodies to those of spheroidal bodies, and hence spheroidal effects. Consequently, the door is now open for the theoretical solution of these equations of motion of non-zero rest masses in the gravitational fields of spheroidal bodies such as; the planets, comet, asteroids in the solar system and satellites in earth orbits.

CONCLUSION

Finally, the work in this paper is an excellent demonstration of an application in gravitation theory for orthogonal curvilinear coordinate systems other than the usual Cartesian, cylindrical, and spherical coordinates.

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