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Oscillatory flow of a jeffrey fluid in an elastic tube of variable cross-section

C. H. Badari Narayana, S. Sreenadh and P. Devaki

Department of Mathematic, Sri Venkateswara University, Tirupati

ABSTRACT

An oscillatory flow of a Jeffery fluid in an elastic tube of variable cross section has been investigated at low Reynolds number. The main concentration is on the excess pressure of the tube. The equations have been solved numerically and investigations are made for different cases on the tube. The results are displayed graphically to study the influence of physical parameters like Jeffrey parameter on the excess pressure velocity and flux. We observe that as the Jeffrey parameter increases the excess pressure decreases. Further effect of excess pressure is more for a tapered tube than straight and locally constricted tube. The theoretical findings may have potential applications in medicine especially in finding remedy for atherosclerosis.

Key words: Jeffrey, Excess Pressure, Oscillatory flow, Elastic tube.

INTRODUCTION

The study of oscillatory flows of a viscous fluid in cylindrical tubes of varying cross-section has received the attention of many research workers as it playss a significant role in understanding the important physiological problems such as the blood flow in an arteriosclerotic blood vessel. Lee and Fung [1] and Manton [2] have considered the steady flow of a viscous fluid in locally constricted, rigid, axisymmetric tubes at low Reynolds number and determined the physically important shear stress distribution on the wall of the tube. Extension of these results to unsteady and oscillatory flows has been considered by Ramachandra Rao and Devanathan [3], Hall [4] and Sehneck and Ostrach [5]. The steady and unsteady flows through constricted or dilated pipes and channels for large Reynolds numbers have been investigated by Smith [6,7]. Womersley [8,9] has considered the oscillatory motion of a viscous fluid in a thin walled elastic tube under a linear approximation for long waves. Morgan and Kiely [10], Anliker and Raman [11] and Rubinow and Keller [12] are some of the few who have considered analytically the flows in elastic tubes in order to understand the blood flow in arteries.

Nevertheless, models of biological ducts demand an inclusion of the solid mechanics of the tube wall, and specifically the muscle action forcing deformation and any material stiffness. A key difficulty with the addition of the tube mechanics is that the shape of the deformable boundary must be determined as part of the solution, rendering the exercise a free-boundary problem. The available literature treating this version of the problem is much focused largely on the dynamics of the ureter (Fung [13]) or tailored to other specific biological applications (Miftakhov & Wingate [14]). In the current article, we approach this problem from a more general perspective, and construct a relatively simple mathematical model that incorporates the solid mechanics of tube of the wall and the biofluid is taken as non-Newtonian Jeffrey fluid. There are some works done on non-Newtonian fluids (Krishna Kumari et al.[15], Kavitha et al.[16]). The article by Tang & Rankin [17] is most closely related some lubrication solutions for a tube modelled as a stretched membrane in which the tension is spatially varied in order to drive a peristaltic motion. Sreenadh et al. [18] studied on the flow of non uniform cross section.

More recently Vajravelu et. al [19] studied the flow of a Herschel-Bulkley fluid in an elastic tube. Also, as the blood is frequently referred to as non-Newtonian fluid, Jeffrey fluid is the simplest form of the non-Newtonian fluid. Now a days there is a favorable interest on wall properties of the tube. (Hemadri et al.[20]). Among several non-

Newtonian fluid models, Jeffrey fluid model is preferred by many authors to describe flow of physiological fluids in tubes and channels. Vajravelu et al. [21] studied the influence of heat transfer on peristaltic transport of Jeffrey fluid in a vertical porous stratum and many authors are now concentrating on this Jeffrey model as it describes closely some physiological and industrial fluids, [22-24]. We note that not much work is done on the flow of Jeffrey fluid in an elastic tube to the knowledge of the authors.

In view of this, the present paper deals with the oscillatory flow of Jeffrey fluid in an elastic tube. Here we are concentrating on the external pressure and velocity of the fluid flow. We find some interesting investigations of different parameters on the velocity and external pressure, which warrant further study on the non-Newtonian fluid phenomena in elastic tubes.

Basic equations

The constitutive equations for an incompressible Jeffrey fluid are

$$\overline{T} = -\overline{p}\,\overline{I} + \overline{s}$$

$$\overline{s} = \frac{\mu}{1+\lambda_1}(\dot{\gamma} + \lambda_2\ddot{\gamma})$$

where $\overline{T}$ and $\overline{s}$ are the Cauchy stress tensor and extra stress tensor respectively. $\overline{p}$ is the pressure, $\overline{I}$ is the identity tensor, λ_1 is the ratio of relaxation to retardation times, λ_2 is the retardation time, γ is shear rate, and dots over the quantities indicate differentiation with respect to time.

Formulation of the Problem:

Consider the flow of a Jeffery fluid in an elastic tube with thin walls of circular cross-section. Here r=0 is the axis of the tube and $r = a(z)$ is the radius of the cross-section of the tube. The equations governing the motion of the fluid in the tube are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r} + \nu\left[\frac{1}{(1+\lambda_1)}\frac{\partial}{\partial r}\left(\frac{u}{r} + \frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right] \quad (1)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{(1+\lambda_1)}\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right] \quad (2)$$

where u, w are the velocity components in r and z directions respectively, ρ_0 is the density of the fluid, ν is the kinematic coefficient of viscosity and p is the pressure. The equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad (3)$$

The other equation for the radial displacement ξ is given by (See Ramachandra Rao[25])

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{1}{h\rho}\left(p - 2\nu\rho_0\frac{\partial u}{\partial r}\right)_{r=a} - \frac{B}{\rho}\frac{\xi}{a^2} \quad (4)$$

where h and ρ are the thickness and density of the material of the tube and $B = \frac{E}{(1-\sigma^2)}$, E is the Young's modulus and σ is the Poisson's ratio. The boundary conditions for the motion of the of the fluid in the tube are

$$u = \frac{\partial \xi}{\partial t}, \qquad w = 0; \qquad on \qquad r = a_0 s(z) \quad (5)$$

where $a_0$the radius of the tube without elasticity, L is is the length of the tube. We assume that $\varepsilon = \frac{a_0}{L}$ is small for a tube with slowly varying cross-section.

The non-dimensional quantities are:

$$\bar{u} = \frac{1}{\varepsilon U_0}u, \quad \bar{w} = \frac{1}{U_0}w, \quad \bar{t} = \omega t, \quad \bar{\xi} = \frac{1}{a_0}\xi, \quad \bar{r} = \frac{1}{a_0}r, \quad \bar{z} = \frac{\varepsilon}{a_0}z, \quad \bar{p} = \frac{\varepsilon a_0}{\rho_0 \nu U_0}p \quad (6)$$

Where U_0 is the characteristic velocity and ω is the frequency of oscillatory flow.

Introducing these non-dimensional quantities in to Eqs. (1) – (5), they reduce to the following form (dropping the bars)

$$\varepsilon^2\alpha^2\frac{\partial u}{\partial t}+\varepsilon^3 R\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial r}+\varepsilon^2\left[\frac{1}{(1+\lambda_1)}\frac{\partial}{\partial r}\left(\frac{u}{r}+\frac{\partial u}{\partial r}\right)\right]+\varepsilon^4\frac{\partial^2 u}{\partial z^2} \quad (7)$$

$$\alpha^2\frac{\partial w}{\partial t}+\varepsilon R(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z})=-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{(1+\lambda_1)}\frac{\partial w}{\partial r}\right)+\varepsilon^2\frac{\partial^2 w}{\partial z^2} \quad (8)$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \quad (9)$$

$$\varepsilon^2\frac{\rho h}{\rho_0 a_0}\frac{\partial^2\xi}{\partial t^2}=\frac{1}{R{S_t}^2}(p-2\varepsilon^2\frac{\partial u}{\partial r})_{r=a}-\frac{1}{\beta}\frac{\xi}{S^2} \quad (10)$$

$$u=S_t\frac{\partial\xi}{\partial t},\qquad w=0;\qquad on\qquad r=S \quad (11)$$

where $\alpha=a_0(\frac{\omega}{\nu})^{1/2}$ (Womersley parameter), $R=\frac{U_0 a_0}{\nu}$ (Reynolds number), $S_t=\frac{\omega a_0}{U_0}$ (Strouhal number), $\beta^2=\frac{\omega^2L^2\rho_0 a_0}{C^2\rho h}$ and $C^2=\frac{B}{\rho}$

Now we neglect the ε and higher order terms completely. Under the assumption of steady oscillation, we take

$$(u,w,p,\xi)=e^{it}(\tilde{u},\tilde{w},\tilde{p},\tilde{\xi}) \quad (12)$$

Then Eqs.(7) – (11), we get

$$\frac{\partial\tilde{p}}{\partial r}=0 \quad (13)$$

$$\frac{\partial^2\tilde{w}}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{w}}{\partial r}+\lambda^2(1+\lambda_1)\tilde{w}=(1+\lambda_1)\frac{\partial\tilde{p}}{\partial z} \quad (14)$$

$$\frac{\partial\tilde{u}}{\partial r}+\frac{\tilde{u}}{r}+\frac{\partial\tilde{w}}{\partial z}=0 \quad (15)$$

$$\tilde{w}=0,\qquad \tilde{u}=iS_t\tilde{\xi};\qquad on\qquad r=s \quad (16a,16b)$$

$$\tilde{\xi}=\frac{\beta^2}{R{S_t}^2}s^2p_e \quad (17)$$

Where $\lambda^2=i\alpha^2$ and p_e is the excess pressure on the wall of the tube.

Here pressure is a function of z only and it is taken as excess pressure p_e .Solving Eq(14) and using (16a) we have

$$\tilde{w}=\frac{1}{\lambda^2}\frac{dp_e}{dz}\left(1-\frac{J_0(\lambda\sqrt{1+\lambda_1}r)}{J_0(\lambda\sqrt{1+\lambda_1}s)}\right) \quad (18)$$

The flow rate in the elastic tube is given by

$$Q=\frac{\pi}{\lambda^2}e^{it}\frac{dp_e}{dz}s^2\frac{J_2(\lambda\sqrt{1+\lambda_1}s)}{J_0(\lambda\sqrt{1+\lambda_1}s)} \quad (19)$$

Using (18) in (15) we get,

$$\tilde{u}=\frac{1}{\lambda^2}\frac{d^2p_e}{dz^2}\left(\frac{J_1(\lambda\sqrt{1+\lambda_1}r)}{J_0(\lambda\sqrt{1+\lambda_1}s)}-\frac{r}{2}\right)+\frac{1}{\lambda^2}\frac{ds}{dz}S^2\frac{J_1(\lambda\sqrt{1+\lambda_1}s)}{{J_0}^2(\lambda\sqrt{1+\lambda_1}s)}\frac{dp_e}{dz}J_1(\lambda\sqrt{1+\lambda_1}s) \quad (20)$$

Using (16b) in (20) we have,

$$\frac{d^2p_e}{dz^2}+\frac{2}{S}\frac{ds}{dz}a_1(\eta s)\frac{dp_e}{dz}-2s\beta^2a_2(\eta s)p_e=0 \qquad (22)$$

Where

$$a_1(\eta s)=\frac{{J_1}^2(\lambda\sqrt{1+\lambda_1}s)}{J_0(\lambda\sqrt{1+\lambda_1}s)J_2(\lambda\sqrt{1+\lambda_1}s)}$$
$$a_2(\eta s)=\frac{J_0(\lambda\sqrt{1+\lambda_1}s)}{J_2(\lambda\sqrt{1+\lambda_1}s)} \qquad (23)$$

where $\eta=\lambda\sqrt{1+\lambda_1}$

For large values of η, the Bessel functions are replaced by their asymptotic series and we get

$$a_1(\eta s)=1-\frac{i}{\eta s}+\cdots,$$
$$a_2(\eta s)=-1+\frac{2i}{\eta s}+\cdots,$$

Here the inertial forces dominate the viscous forces and the Womersley parameter α is greater than 10, which corresponds to arteries with large diameter. The approximate equations for excess pressure is

$$\frac{d^2p_e}{dz^2}+\frac{2}{s}\frac{dS}{dz}(1-\frac{i}{\eta s})\frac{dp_e}{dz}-2s\beta^2(1-\frac{2i}{\eta s})p_e=0 \qquad (24)$$

The excess pressure given by (24) is complex, by writing $p_e=p_r+ip_i$, and equating real and imaginary parts, we obtain two coupled ordinary differential equations of second order for P_r and P_i. These equations are rewritten as four first order equations and are solved using Mathematica by prescribing the initial conditions at some point z of the axial section of the tube. In our problem we have considered from z = 1 to z=10, for all the geometries of the tubes. The modulus of pressure $|p_e|$ has been evaluated for

a) Straight tube given by $s(z)=1,\ 1<z<10,$
b) Tapered tube given by $s(z)=exp\,(-0.025z), 1<z<10,$
c) Locally constricted tube given by $s(z)=1-0.5\,exp\,(-(z-6)^2), 1<z<10.$

The initial conditions are taken as

$$p_r=0.1,\quad p_i=0,\quad \frac{dp_r}{dz}=0,\quad \frac{dp_i}{dz}=-0.01 \qquad at \qquad z=0 \qquad (25)$$

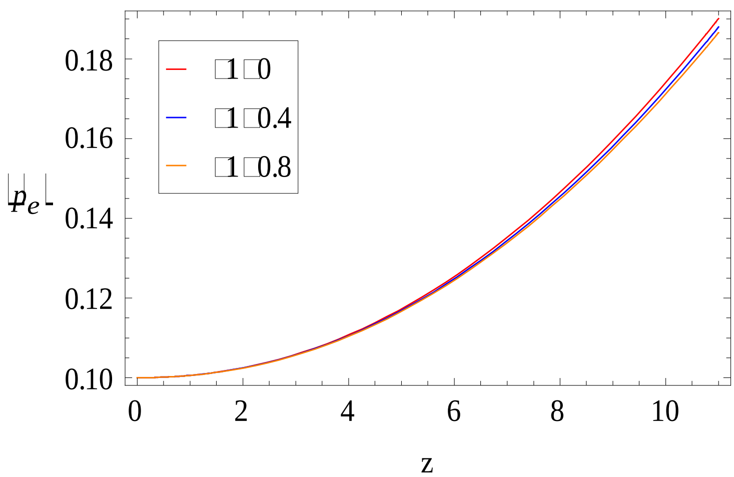

Fig 1: Variation of external pressure with z for different values of the Jeffrey parameter λ_1, for a tapered tube

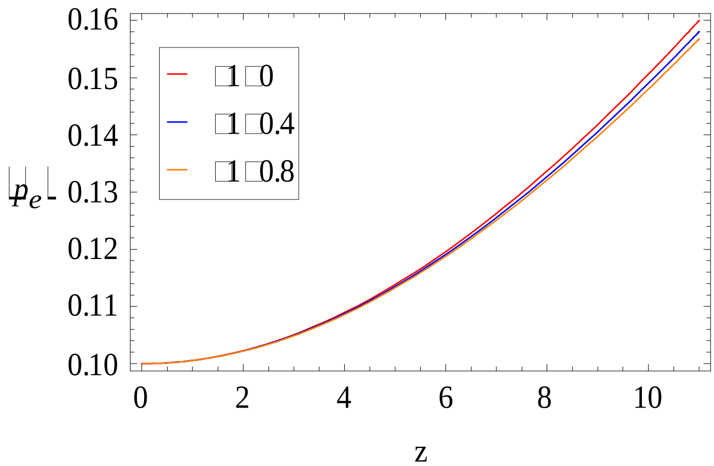

Fig 2: Variation of external pressure with z for different values of the Jeffrey parameter λ_1, for a straight tube

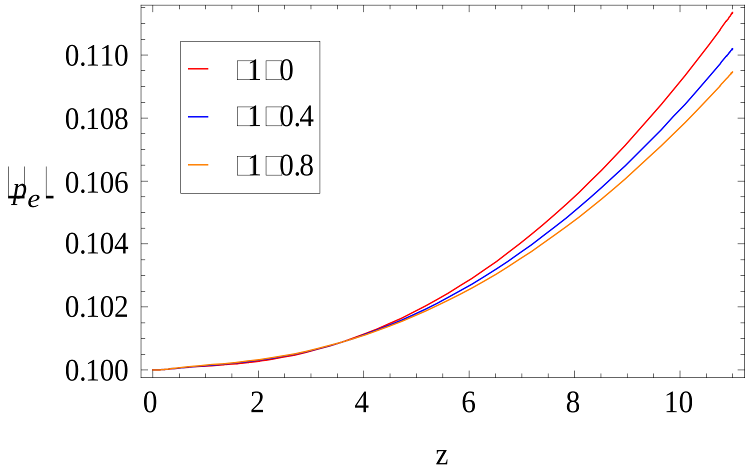

Fig 3: Variation of external pressure with z for different values of the Jeffrey parameterλ_1, for a locally constricted tube

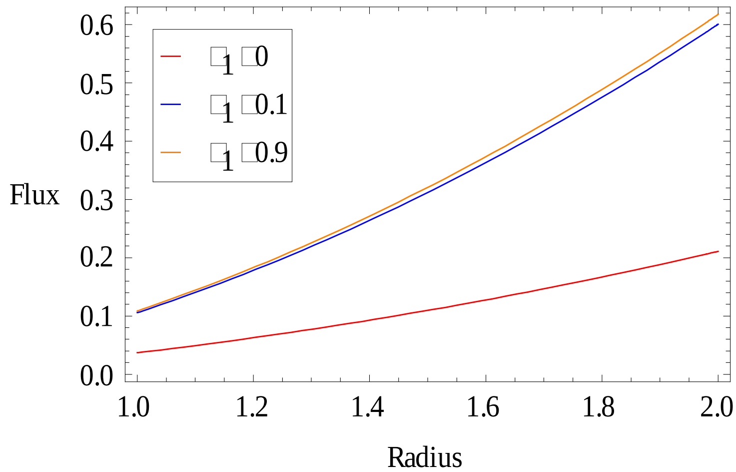

Fig 4: Variation of flux with radius for different Jeffrey parameters.

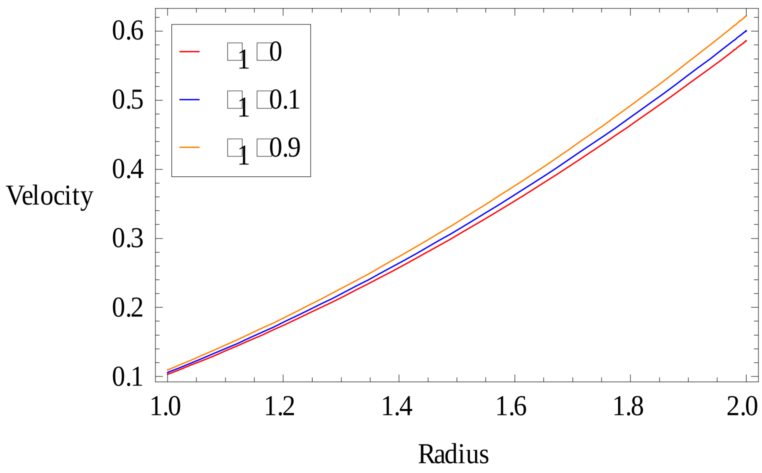

Fig 5: Variation of velocity with radius for different Jeffrey parameters

RESULTS AND DISCUSSION

We study the flow of a Jeffrey fluid with stenosis in an elastic tube. Here the effects of Jeffrey parameter, external pressure, Pressure radius relations and the elastic nature of the tube are explained in detail as follows.

From fig(1) we observe that as the Jeffrey parameter increases , the effect of external pressure is decreasing for a tapered tube. Fig(2) shows that for an increase in the Jeffery parameter, there is a decrease in the effect of external pressure for a linear tube. If the tube is locally constricted, we notice that as the Jeffrey parameter increases, the effect of external pressure on the fluid flow is decreasing, which is shown in fig(3). From fig(4), we observe that flux increases with the increment in the Jeffrey parameter. We notice from fig(5) that as the Jeffrey parameter increases the increasing velocity. From all these figures we can say that as the axial coordinate z increases, the magnitude of the excess pressure is increasing. One of the most important phenomenon to be noticed here is that the magnitude of the excess pressure is high if the tube is tapered when compared with straight and locally constricted tubes.

CONCLUSION

We study the flow of Jeffrey fluid in an elastic tube, with different possibilities and variation in the physical parameters governing the flow. We observe the following:

1. There is considerable effect of non-Newtonian Jeffrey parameter on the Jeffrey fluid flow in tapered, linear and locally constricted tubes. Comparing results in the above configurations, we find that excess pressure p_e is high if the tube is tapered.
2. Excess pressure increases with increasing z for a given Jeffrey fluid.
3. The effect of increasing Jeffrey parameter leads to enhancements in the velocity and the flux.

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