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Advances in Applied Science Research, 2010, 1 (2): 65-75

Advances in Applied Science Research

Pelagia Research Library

ISSN: 0976-8610
CODEN (USA): AASRFC

On Transitive and Primitive Dihedral Groups of Degree at most p^2

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ABSTRACT

*Let p be an odd prime number. In this paper an attempt is made to discuss transitivity and primitivity of all the p-subgroups of dihedral groups of degree at most p^2 that have wide range of practical applications especially in engineering, physics and chemistry.The method adopted uses the concepts of p-groups such as Sylow theorems and other Group theoretic concepts including the groups, algorithms and programming (**GAP**) to obtain our results.*

INTRODUCTION

Groups are very important to describe the symmetry of objects, be they geometrical or algebraic. In short, given a geometric figure in the plane the symmetry group of the figure consists of all isometries that transform points on the figure to points on the figure. In particular, chemists use the symmetry groups to describe the symmetry of crystals.

Throughout this paper, Ω is a finite set with n elements and G is a permutation group on Ω. Then $G \leq Sym(\Omega)$ – the symmetric group on Ω.

PRELIMINARIES

The following preliminary results will be required; the first one can be verified quite readily.

Theorem 2.1 (Audu M.S, 2000)

Let $H \leq G$ be groups with $|G: H| = n$. Then there exists $K \trianglelefteq G$ such that

1. $|G: K|$ divides n! ; and
2. $K \leq H$

Theorem 2.2
Let $H \leq G$ be groups. If p is the smallest prime dividing $|G|$ and $|G: H| = p$.
Then $H \trianglelefteq G$.

Proof
By Theorem 2.1, $\exists\, K \trianglelefteq G$ such that $|G:K|$ divides p! and $K \leq H$. But $|G:K| = |G:H|\ |H:K| = p|H:K|$. It follows that $|H: K|$ divides $(p - 1)!$ and so $|H:K|$ is divisible by a prime smaller than p. This contradicts the assumption on p unless H = K. Accordingly, $H = K \trianglelefteq G$.

Theorem 2.3
Let G be a transitive abelian group. Then G is regular.
Proof
Fix $\alpha \in \Omega$. If $\beta \in \Omega\ \exists\, g \in G$ with $\alpha^g = \beta$. Now,

$$G_\beta = G_{\alpha^g} = (G_\alpha)^g = g^{-1}(G_\alpha)g = G_\alpha$$

(since G is abelian). As α,β are arbitrary, we get that $G_\alpha = 1$. Since G is transitive, it is regular.

Theorem 2.4
Let G be a transitive permutation group of prime degree on Ω. Then G is primitive.
Proof
Now since G is transitive, it permutes the sets of imprimitivity bodily and all the sets have the same size. But $\Omega = \cup |\Omega_i|$, Ω_i being the sets of imprimitivity. As $|\Omega|$ is prime we have that either each $|\Omega_i| = 1$ or Ω is the only set of imprimitivity. So, G is primitive.

Theorem 2.5
Let G be a non-trivial transitive permutation group on Ω. Then G is primitive if and only if G_α, $\alpha \in \Omega$ is a maximal subgroup of G or equivalently G is imprimitive if and only if there is a subgroup H of G properly lying between $G_\alpha(\alpha \in \Omega)$ and G.

Proof
Suppose G is imprimitive and ψ a non-trivial subset of imprimitivity of G.

$$\text{Let } H = \{g \in G \mid \psi^g = \psi\}$$

Clearly H is a subgroup of G and a proper subgroup of G because $\psi \subset \Omega$ and G is transitive.
Now choose $\alpha \in \psi$. If $g \in G$ then $\alpha^g = \alpha$, showing that $\alpha \in \psi \cap \psi^g$ and so $\psi = \psi^g$.
Hence $G \leq H$.
Hence $G_\alpha \leq H \leq G$

Since $|\psi| \neq 1$, choose $\beta \in \psi$ such that $\beta \neq \alpha$. By transitivity of G, there exists some $h \in G$ with $\alpha^h = \beta$ so that $h \in G_\alpha$. Now $\beta \in \psi \cap \psi^h$, so $\psi = \psi^h$ and $h \in H - G_\alpha$. Thus $H \neq G_\alpha$.
Hence G_α is not a maximal subgroup.

Conversely, suppose that $G_\alpha < H < G$ for some subgroup H.
Let $\psi = \alpha^H$. Since $H > G_\alpha$, $|\psi| \neq 1$

Now if $\psi = \Omega$, then H is transitive on Ω and hence $|\Omega| = |G: G_\alpha| = |H: G_\alpha|$ showing that H = G, a contradiction.
Hence, $\psi \neq \Omega$.

Now we shall show that ψ is a subset of imprimitivity of G.

Let $g \in G$ and $\beta \in \psi \cap \psi^g$ then $\beta = \alpha^h = \alpha^{h' g}$ for some $h, h' \in H$.

Hence $\alpha^{h' g h^{-1}} = \alpha$. So $h' g h^{-1} \in G_\alpha < H$.

This shows that $g \in H$.
Thus $\psi = \psi^g$. Hence ψ is a non-trivial subset of imprimitivity.
So G is imprimitive.

THE MAIN RESULTS

The main results obtain here are as follows:

Theorem 3.1
Let G be a dihedral group of degree p, p, a prime ≥3. Let N be a Sylow p-subgroup of G. Then
1. G is transitive and primitive,
2. N is transitive, primitive, regular and normal in G.

Proof

$|G| = 2 \times 3$ or $|G| = 2p$, $p > 3$.

Then $|G: N| = 2$, so by Theorem 2.*2* $N \trianglelefteq G$. Also N is of prime order so N is transitive and is cyclic hence abelian. Thus N is regular by Theorem *2.3.*
We have that N is transitive of prime degree and hence N is primitive by Theorem *2.4,* proving (2).
Also any dihedral group is transitive for given α_i, α_j as any two vertices of the regular polygon with i < j, we readily see that $(\alpha_1\alpha_2 \ldots \alpha_i \ldots \alpha_j \ldots \alpha_n)^{j-i}$ is the rotation about the centre of the polygon through angle $2\pi^c/n$, where n is the number of edges of the polygon. Now G is transitive of prime degree. Thus G is primitive by Theorem *2.4,* proving (1).

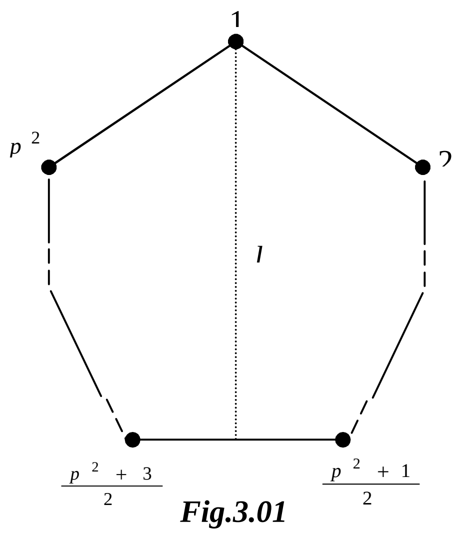

Fig.3.01

Theorem 3.2
Let G be a dihedral group of degree p^2, p a prime ≥3. Let N be a Sylow p-subgroup of G. Then
1. G is transitive and imprimitive,
2. N is transitive, imprimitive, regular and a normal subgroup of G.

Proof
$|G| = 2p^2$, $|N| = p^2$
By Theorem *2.2*, $N \trianglelefteq G$. Again by a well-known result N is either cyclic or elementary abelian of rank 2. In any event, N is abelian.

If N is not cyclic, then $N = C_p \times C_p$ acting regularly. In either case, N is transitive and regular.

By virtue of the orbit formula, $|N| = |N_\alpha| |\alpha^N|$, we get that $N_\alpha = 1$ for all vertices α of the polygon. Certainly, there are subgroups of N properly lying between N_α and N (take for example any other proper subgroup of N). By Theorem *2.5,* N must be imprimitive, proving (2).

As has been shown in Theorem *3.1*, G must be transitive. Next name the vertices of G as 1,2,3, …, p^2, and let ℓ be the line of symmetry joining the vertex 1 and the middle of the vertices $\frac{p^2+1}{2}$ and $\frac{p^2+3}{2}$ so that $a = (2\ \ p^2)(3\ \ p^2-1)(4\ \ p^2-2)\ \ldots \left(\frac{p^2+1}{2}\frac{p^2+3}{2}\right)$ is the reflection in ℓ.Then $G_1 = \{(1), a\}$ is the stabilizer of the point 1. We readily see that G_1 is a non-identity proper subgroup of G which has $H = \{(1), (2\ \ p^2), (3\ \ p^2-1)\ (4\ \ p^2-2)\ \ldots \left(\frac{p^2+1}{2}\frac{p^2+3}{2}\right), a\}$ as a subgroup properly lying between G_1 and G. Thus by virtue of Theorem *2.5,* G is imprimitive, proving (1).

We now validate the results obtained in Section 3 by running the groups, algorithms and programming (**GAP**) on some Groups as shown below.

Validation of result
4.1 Transitive and Primitive Dihedral Groups of Degree p (p=3, 5, 7, 11).

(The Groups, Algorithms and Programming- 4.4 Version).

```
1. gap> # THE DIHEDRAL GROUP OF SYMMETRY, D3
gap>D3:=GroupWithGenerators([(1,2,3),(2,3)]);
Group([ (1,2,3), (2,3) ])
gap> for i in D3 do
> Print(i," ");
> od;
(1) (1,3,2) (1,2,3) (2,3) (1,3) (1,2) gap>
gap> IsTransitive(D3);
true
gap> IsPrimitive(D3);
true
gap> g1:=SylowSubgroup(D3,2);
Group([ (2,3) ])
```

```
gap> for i in g1 do
> Print(i," ");
> od;
(1) (2,3) gap>
gap> IsTransitive(g1);
true
gap> IsPrimitive(g1);
true
gap> IsRegular(g1);
true
gap> IsNormal(D3,g1);
false
gap> g2:=SylowSubgroup(D3,3);
Group([ (1,2,3) ])
gap> for i in g2 do
> Print(i," ");
> od;
(1) (1,3,2) (1,2,3) gap>
gap> IsTransitive(g2);
true
gap> IsPrimitive(g2);
true
gap> IsRegular(g2);
true
gap> IsNormal(D3,g2);
true
```

2. gap> # **THE DIHEDRAL GROUP OF SYMMETRY, D_5.**

```
gap> D5:=GroupWithGenerators([(1,2,3,4,5),(2,5)(3,4)]);
Group([ (1,2,3,4,5), (2,5)(3,4) ])
gap> for i in D5 do
> Print(i," ");
> od;
(1) (1,5,4,3,2) (1,4,2,5,3) (1,3,5,2,4) (1,2,3,4,5) (2,5)(3,4)
(1,5)(2,4) (1,4)(2,3) (1,3)(4,5) (1,2)(3,5) gap>


gap> IsTransitive(D5);
true
gap> IsPrimitive(D5);
true
gap> k1:=SylowSubgroup(D5,2);
Group([ (2,5)(3,4) ])
gap> for i in k1 do
> Print(i," ");
> od;
(1) (2,5)(3,4) gap>
gap> IsTransitive(k1);
```

```
false
gap> IsPrimitive(k1);
false
gap> IsRegular(k1);
false
gap> IsNormal(D5,k1);
false
gap> k2:=SylowSubgroup(D5,5);
Group([ (1,2,3,4,5) ])
gap> for i in k2 do
> Print(i," ");
> od;
(1) (1,5,4,3,2) (1,4,2,5,3) (1,3,5,2,4) (1,2,3,4,5) gap>
gap> IsTransitive(k2);
true
gap> IsPrimitive(k2);
true
gap> IsRegular(k2);
true
gap> IsNormal(D5,k2);
true
```

3. gap> # **THE DIHEDRAL GROUP OF SYMMETRY, D_7.**

```
gap> D7:=GroupWithGenerators([(1,2,3,4,5,6,7),(2,7)(3,6)(4,5)]);
Group([ (1,2,3,4,5,6,7), (2,7)(3,6)(4,5) ])
gap> for i in D7do
> Print(i," ");
> od;
(1) (1,7,6,5,4,3,2) (1,6,4,2,7,5,3) (1,5,2,6,3,7,4)
(1,4,7,3,6,2,5)
(1,3,5,7,2,4,6) (1,2,3,4,5,6,7) (2,7)(3,6)(4,5) (1,7)(2,6)(3,5)
(1,6)(2,5)
(3,4) (1,5)(2,4)(6,7) (1,4)(2,3)(5,7) (1,3)(4,7)(5,6)
(1,2)(3,7)(4,6) gap>
gap> IsTransitive(D7);
true
gap> IsPrimitive(D7);
true
gap> m1:=SylowSubgroup(D7,2);
Group([ (2,7)(3,6)(4,5) ])
gap> for i in m1 do
> Print(i," ");
> od;
(1) (2,7)(3,6)(4,5) gap>
gap> IsTransitive(m1);
false
gap> IsPrimitive(m1);
false
```

```
gap> IsRegular(m1);
false
gap> IsNormal(D7,m1);
false
gap> m2:=SylowSubgroup(D7,7);
Group([ (1,2,3,4,5,6,7) ])
gap> for i in m2 do
> Print(i," ");
> od;
(1) (1,7,6,5,4,3,2) (1,6,4,2,7,5,3) (1,5,2,6,3,7,4)
(1,4,7,3,6,2,5)
(1,3,5,7,2,4,6) (1,2,3,4,5,6,7) gap>
gap> IsTransitive(m2);
true
gap> IsPrimitive(m2);
true
gap> IsRegular(m2);
true
gap> IsNormal(D7,m2);
true
```

4. gap> # **THE DIHEDRAL GROUP OF SYMMETRY, D_{11}.**

```
gap>
D11:=GroupWithGenerators([(1,2,3,4,5,6,7,8,9,10,11),(2,11)(3,10)(
4,9)(5,8) (6,7)]);
Group([ (1,2,3,4,5,6,7,8,9,10,11), (2,11)(3,10)(4,9)(5,8)(6,7)
])
gap> for i in D11 do
> Print(i," ");
> od;
(1) ( 1,11,10, 9, 8, 7, 6, 5, 4, 3, 2) ( 1,10, 8, 6, 4, 2,11, 9,
7, 5, 3)
( 1, 9, 6, 3,11, 8, 5, 2,10, 7, 4) ( 1, 8, 4,11, 7, 3,10, 6, 2,
9, 5)
( 1, 7, 2, 8, 3, 9, 4,10, 5,11, 6) ( 1, 6,11, 5,10, 4, 9, 3, 8,
2, 7)
( 1, 5, 9, 2, 6,10, 3, 7,11, 4, 8) ( 1, 4, 7,10, 2, 5, 8,11, 3,
6, 9)
( 1, 3, 5, 7, 9,11, 2, 4, 6, 8,10) ( 1, 2, 3, 4, 5, 6, 7, 8,
9,10,11)     ( 2,11)( 3,10)( 4, 9)( 5, 8)( 6, 7) ( 1,11)( 2,10)(
3, 9)( 4, 8)( 5, 7)   ( 1,10)( 2, 9)( 3, 8)( 4, 7)( 5, 6) ( 1,
9)( 2, 8)( 3, 7)( 4, 6)(10,11)   ( 1, 8)( 2, 7)( 3, 6)( 4, 5)(
9,11) ( 1, 7)( 2, 6)( 3, 5)( 8,11)( 9,10)    ( 1, 6)( 2, 5)( 3,
4)( 7,11)( 8,10) ( 1, 5)( 2, 4)( 6,11)( 7,10)( 8, 9)   ( 1, 4)(
2, 3)( 5,11)( 6,10)( 7, 9) ( 1, 3)( 4,11)( 5,10)( 6, 9)( 7, 8)
( 1, 2)( 3,11)( 4,10)( 5, 9)( 6, 8) gap>
gap> IsTransitive(D11);
```

```
true
gap> IsPrimitive(D11);
true
gap> u1:=SylowSubgroup(D11,2);
Group([ (2,11)(3,10)(4,9)(5,8)(6,7) ])
gap> for i in u1 do
> Print(i," ");
> od;
(1) ( 2,11)( 3,10)( 4, 9)( 5, 8)( 6, 7) gap>
gap> IsTransitive(u1);
false
gap> IsPrimitive(u1);
false
gap> IsRegular(u1);
false



gap> IsNormal(D11,u1);
false
gap> u2:=SylowSubgroup(D11,11);
Group([ (1,2,3,4,5,6,7,8,9,10,11) ])
gap> for i in u2 do
> Print(i," ");
> od;
(1) ( 1,11,10, 9, 8, 7, 6, 5, 4, 3, 2) ( 1,10, 8, 6, 4, 2,11, 9,
7, 5, 3)
( 1, 9, 6, 3,11, 8, 5, 2,10, 7, 4) ( 1, 8, 4,11, 7, 3,10, 6, 2,
9, 5)
( 1, 7, 2, 8, 3, 9, 4,10, 5,11, 6) ( 1, 6,11, 5,10, 4, 9, 3, 8,
2, 7)
( 1, 5, 9, 2, 6,10, 3, 7,11, 4, 8) ( 1, 4, 7,10, 2, 5, 8,11, 3,
6, 9)
( 1, 3, 5, 7, 9,11, 2, 4, 6, 8,10) ( 1, 2, 3, 4, 5, 6, 7, 8,
9,10,11) gap>
gap> IsTransitive(u2);
true
gap> IsPrimitive(u2);
true
gap> IsRegular(u2);
true
gap> IsNormal(D11,u2);
true
```

4.2 Transitive and Primitive Dihedral Groups of Degree p^2 (p=9, 25, 49).

```
gap> # 1. THE DIHEDRAL GROUP OF SYMMETRY, D9.
```

```
gap>
D9:=GroupWithGenerators([(1,2,3,4,5,6,7,8,9),(2,9)(3,8)(4,7)(5,6)
]);
Group([ (1,2,3,4,5,6,7,8,9), (2,9)(3,8)(4,7)(5,6) ])
gap> IsTransitive(D9);
true
gap> IsPrimitive(D9);
false
gap> a1:=SylowSubgroup(D9,2);
Group([ (2,9)(3,8)(4,7)(5,6) ])
gap> IsTransitive(a1);
false
gap> IsPrimitive(a1);
false
gap> IsRegular(a1);
false
gap> IsNormal(D9,a1);
false
gap> a2:=SylowSubgroup(D9,3);
Group([ (1,2,3,4,5,6,7,8,9), (1,4,7)(2,5,8)(3,6,9) ])
gap> IsTransitive(a2);
true
gap> IsPrimitive(a2);
false
gap> IsRegular(a2);
true
gap> IsNormal(D9,a2);
true
```

2. gap> # **THE DIHEDRAL GROUP OF SYMMETRY,** D_{25}**.**

```
gap>D25:=GroupWithGenerators([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,1
5,16,17,18,19,20,21,22,23,24,25),(2,25)(3,24)(4,23)(5,22)(6,21)(
7,20)(8,19)(9,18)  (10,17)(11,16)(12,15)(13,14)]);
Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22
,23,24,25),(2,25)(3,24)(4,23)(5,22)(6,21)(7,20)(8,19)(9,18)(10,1
7)(11,16) (12,15)(13,14) ])
gap> IsTransitive(D25);
true
gap> IsPrimitive(D25);
false
gap> K1:=SylowSubgroup(D25,2);
```

```
Group([(2,25)(3,24)(4,23)(5,22)(6,21)(7,20)(8,19)(9,18)(10,17)(1
1,16) (12,15)(13,14) ])
gap> IsTransitive(K1);
false
gap> IsPrimitive(K1);
false
gap> IsRegular(K1);
false
gap> IsNormal(D25,K1);
false
gap> K2:=SylowSubgroup(D25,5);
Group([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22
,23,24,25),(1,6,11,16,21)(2,7,12,17,22)(3,8,13,18,23)(4,9,14,19,
24)   (5,10,15,20,25) ])
gap> IsTransitive(K2);
true
gap> IsPrimitive(K2);
false
gap> IsRegular(K2);
true
gap> IsNormal(D25,K2);
true
```

3 . gap> # **THE DIHEDRAL GROUP OF SYMMETRY,** D_{49}.

```
D49:=GroupWithGenerators([(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16
,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,
38,39,40,41,42,43,44,45,46,47,48,49),(2,49)(3,48)(4,47)(5,46)(6,
45)(7,44)(8,43)(9,42)(10,41)
(11,40)(12,39)(13,38)(14,37)(15,36)(16,35)(17,34)(18,33)(19,32)(
20,31) (21,30)(22,29)(23,28)(24,27)(25,26)]);
<permutation group with 2 generators>
gap> IsTransitive(D49);
true
gap> IsPrimitive(D49);
false
gap> u1:=SylowSubgroup(D49,2);
Group([(2,49)(3,48)(4,47)(5,46)(6,45)(7,44)(8,43)(9,42)(10,41)(1
1,40)
(12,39)(13,38)(14,37)(15,36)(16,35)(17,34)(18,33)(19,32)(20,31)(
21,30)(22,29)(23,28)(24,27)(25,26) ])
gap> IsTransitive(u1);
false
gap> IsPrimitive(u1);
false
gap> IsRegular(u1);
false
```

```
gap> IsNormal(D49,u1);
false
gap> u2:=SylowSubgroup(D49,7);
<permutation group with 2 generators>
gap> IsTransitive(u2);
true
gap> IsPrimitive(u2);
false
gap> IsRegular(u2);
true
gap> IsNormal(D49,u2);
true
gap> quit;
```

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