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On the stability results for solutions of some fifth order non-linear differential equations

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ABSTRACT

The objective of this paper is to give sufficient conditions for the existence of globally exponentially stable, bounded and periodic and almost periodic for some fifth order non-linear differential equations.

Keywords: Exponential stability, bounded, periodic solutions, almost periodic solutions and frequency-domain.

INTRODUCTION

We shall consider here the fifth order non-linear differential equations of the form;

$$x^{(v)} + ax^{(iv)} + bx^{(iii)} + fx^{(ii)} + gx^{(i)} + ex = P(t) \tag{1.1}$$

Where a, b and e are positive constants and f, g and p are continues functions which depend only on the arguments displayed explicitly.

The problem of concerned here is to determine conditions on these functions under which all solutions of (1.1) are globally exponentially stable, bounded and periodic (or almost periodic). A lot of research study have been done on these properties of solutions for various kind of fifth order non-linear differential equating using Lyapunov's direct method, see [1, 7].

However, our purpose for this study is to use the frequency-domain technique [see 2 – 7, 9, 10] and to study the above mentioned property for the solutions of (1.1). For more exposition on the frequency domain technique, see [1, 8]. My approach in this study has an advantage over [5] and the results obtained in this study generalize the results in [5] and also generalize to the fifth order non-linear differential equations results of Afuwape [4] and Barbalat [6]. The frequency-domain conditions obtained for equation (1.1) are necessary conditions for the existence of a positive definite Lyapunov function of the Lure' Postnikov form with a negative sign derivative. This

study utilizes substantially, the generalized theorem of Yacubovich [9], which is stated without proof.
Generalized Yacubovich's Theorem [7].

Consider the system

$$x' = AX - B\varphi(\sigma) + P(t) = \sigma = C^{*}X \qquad (1.2)$$

Where A is an n×n real matrix, B and C are n×m real matrices with C* as the transpose of C, $\varphi(\sigma) = col\varphi j(\sigma j)$ (j = 1, 2, - - m) and P(t) is an n-vector.

Suppose that in (1.2), the following assumptions are true:

(i) A is a stable matrix;
(ii) P(t) is bounded for all t in R;
(iii) For some constant $\hat{\mu} j \geq 0 \quad (j = 1,2,3 \ldots \ldots . . m)$

$$0 \leq \frac{\varphi j(\sigma j) - \varphi j(\hat{\sigma} j)}{\sigma j - \hat{\sigma} j} \leq \hat{\mu} j, (\sigma j \neq \hat{\sigma} j) \qquad (1.3)$$

(iv) There exist a diagonal matrix D > 0, such that the frequency-domain in equality

$$\pi(\omega) = MD + Re\, DG(i\omega) > 0 \qquad (1.4)$$

holds for all ω in R, where $G(i\omega) = C^{*}(i\omega I - A)^{-1} B$ is the transfer function and

$$M = diag\left(\frac{1}{\hat{\mu} j}\right), \ (j = 1,2,3 \ldots \ldots \ldots . . m)$$

Then, system (1.2) has the following properties

(i) Existence of a bounded solution which is globally exponentially stable;
(ii) Existence of a solution which is Periodic (almost periodic).

1. Formulation of result

The Routh-Hurwitz conditions for stability of Solutions of the linear homogeneous equation of (1.1) will be given as follows

$$a > 0, (ab - c) > 0, (ab - c)c - (ad - e)a > 0$$
$$(ab - c)(cd - be) - (ad - e)^2 > 0, e > 0 \qquad (2.1)$$

The following notations shall be used throughout this study. Equations $v^2a - vc + e = 0$ and $v^2 - vb + d = 0$ have two real positive roots given by v_1, v_2, v_3 and v_4 respectively.
Where,

$$V_1 = 1/2a[c - c^2 - 4ae)^{1/2}] \qquad (2.2)$$
$$V_2 = 1/2a[c + (c^2 - 4ae)^{1/2}] \qquad (2.3)$$
$$V_3 = 1/2[b - (b^2 - 4d)^{1/2}] \qquad (2.4)$$
$$V_4 = 1/2[b + (b^2 - 4d)^{1/2}] \qquad (2.5)$$

Such that,
$b^2 - 4d > 0$, $c^2 - 4ae > 0$ and $0 < v_1 < v_3 < v_2 < v_4$
The main objective of this study is to prove the theorem below.

THEOREM 2.1: Consider (1.1) where the functions f, g and P are continuous with f(0) = g(0) = 0 and P(t) bounded in R. Suppose that there exist positive parameters c, d, μ_1 and μ_2 such that inequality

$$(\mu_1 \mu_2)^2 \leq 16\,(d\mu_2 - e\mu_1)\,(c\mu_1 - b\mu_2) \qquad (2.6)$$

is satisfied and the functions f and g satisfy respectively the following inequalities

$$C \le \frac{f(Z)-f(\hat{Z})}{Z-\hat{Z}} \le C+\mu_1, \ \ (z \ne \hat{z}) \tag{2.7}$$

$$d \le \frac{g(Z)-g(\hat{Z})}{Z-\hat{Z}} \le d+\mu_2, \ \ (z \ne \hat{z}) \tag{2.8}$$

Then equation (1.1) has property (I) and if in addition P(t) is periodic (or almost periodic), then it has property (II)

2. Preliminary Results

The main tool in the proof of our theorem is the function (ω) defined by inequality (1.4). For us to determine the function (ω), we shall, by setting $x_1 = x_2$, reduce (1.1) to system (1.2) with

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}; \ A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -e & -d & -c & -b & -a \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}; \ C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \ P(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P(t) \end{pmatrix};$$

$$\varphi(\sigma) = \begin{pmatrix} \hat{g} & (x_2) \\ \hat{f} & (x_1) \end{pmatrix} \tag{3.1}$$

The transfer function G(iω) = C* (iω1 – A)$^{-1}$ B for system (3.1) becomes

$$G(i\omega) = {}^1\!/_\Delta \begin{pmatrix} i\omega & i\omega \\ -\omega^2 & -\omega^2 \end{pmatrix} \tag{3.2}$$

Where Δ = (ω^4a – ω^2c + e) + iω (ω^4 – bω^2 + d). In order for us to get the function (ω), we shall make use of the generalized theorem of Yacubovich as given in the introduction and this requires the existence of strictly positive number τ_1 and τ_2 such that D = $diaD(Tj)$ *and* $M = diaD(\frac{1}{Nj})$ (i = 1, 2,). After some calculations, we obtain

$$\pi(\omega) = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \tag{3.3}$$

Where,

$$\pi_{11} = \tau_1 \left(\mu_1^{-1} - \omega^2 \frac{(\omega^4 a - \omega^2 c + e)}{|\Delta|^2} \right) \tag{3.4}$$

$$\pi_{22} = \tau_2 \left(\mu_2^{-1} + \omega^2 \frac{(\omega^4 - \omega^2 b + d)}{|\Delta|^2} \right) \tag{3.5}$$

$$\pi_{12} = \frac{1}{2|\Delta|^2} \{\omega^2 \tau_2 (\omega^4 - \omega^2 b + d) - \tau_1(\omega^4 a - \omega^2 c + e) + i\omega[\tau_2(\omega^4 a - \omega^2 c + e) - \tau_1 \omega^2 (\omega^4 - \omega^2 b + d)]\} = \bar{\pi}_{21} \tag{3.6}$$

With $\bar{\pi}_{21}$ as the complex conjugate of $\bar{\pi}_{21}$ *and* $|\Delta|^2 = \Delta\hat{\Delta}$. We shall employ Sylvester's criterion to verify inequality (3.3) and this requires that $л_{11}$, $л_{22}$ and det л(ω) to be positive for all ω in R. These shall be proved by two lemmas.

Lemma 1: Let $s_2(v) = (-v + vb - d) + \frac{(v^2a - vc + c)^2}{v(-v^2 + vb - d)}$
Where $\omega^2 = v$. Then $\pi_{22}(\omega)$ is positive for all $v > 0$, provided that
$\mu_2 < m_2(d) = S_2(v_0) = \min S_2(v)$, and $v_3 < v < v_4$
and,

$$s_2(v_o) = s_2(v_2) = {}^1\!/_{2a}\,(b - {}^c\!/_a)(c + (c^2 - 4ae)^{1/2} - (d - {}^e\!/_a)$$

Where v_0 is the unique real root of $A_2(v) = 0$ with $v_3 < v_0 < v_4$ and $m_2(d)$ is the minimum of $S_2(v)$ and attainable at say $v = v_0$.

Furthermore, if $v_2 \neq b/2$, then, $S_2(v_2) > S_2(b/2)$ and if $v = b/2$ with $e = \frac{2bc - b^2a}{4}$ and $\varepsilon < 2b(b^2 - 4d)(b - c/a)$, $\varepsilon > 0$, then, $S_2(v_2) > S_2(b/2)$.

Proof: For $\pi_{22}(\omega)$ to be positive for all $\omega \in R$, the following inequality must be valid; $(\omega^2 = v)$

$$\mu_2 < (v^2 + vb - d) + \frac{(v^2 - vc + e)^2}{v(-v^2 + vc - d)} \qquad (3.7)$$

Let,

$$\mu_2 < s_2(v) = (-v^2 + vb - d) + \frac{(v^2a - vc + e)^2}{v(-v^2 + vc - d)} \qquad (3.8)$$

On differentiating $S_2(v)$, we have,
$A_2(v) = S_2^1(v).\ v^2(-v^2 + vb - d)^2 = [(b - 2v)\ \{v^2(-v^2 + vb - d)^2 - v(v^2a - vc + e)^2\} + (v^2a - vc + e)(v^2a - vc - e)(-v^2 + vb - d)]$

Obviously, $S_2^1(v)$ can be zero in the interval (v_3v_4) if

$$A_2(v) = 2v^7 + 7(a^2 - 5b)v^6 + (4b^2 - 6a^2b - 12ac + 4d)v^5 + (5a^2d + b^3 + 5c^2 + 10abc - 2ae - 4bd)v^4 + (4b^2d - 4acd + 4bc^2 + 4ce + 5d^2)v^3 + (4d^2 - 3c^2d - 2ade - 2bce - 3e)v^2 - (2be + d^3 - cde)v + de = 0 \qquad (3.9)$$

We note that, the graph of $S_2(v)$ against v, are asymptotes at v_3 and v_4.
On substituting $v = b/2$ into $S_2(v)$, we have

$$s_2\left({}^b\!/_2\right) = \frac{b^2 - 4d}{4} + \frac{b(ab^2 - 2bc + 4e)^2}{2(b^2 - 4d)} \qquad (3.10)$$

Similarly, we obtain

$$S_2(v_1) = 1/2a\ (b - c/a)\ (c - (c^2 - 4ae)\ 1/2) - (d - e/a) \qquad (3.11)$$

and,

$$S_2(v_2) = 1/2a\ (b - c/a)\ (c + (c^2 - 4ae)\ 1/2) - (d - e/a) \qquad (3.12)$$

Let us consider the following cases with the relation

$$e = \frac{2bc - b^2a}{4}$$

(A) If $v_2 = b/2$, then

$$s_2(v_2) = \frac{b^2 - 4d}{4} = s_2({}^b\!/_2) \qquad (3.13)$$

Therefore, $S_2(v_2) = S_2(b/2)$.

(B) If $v_2 > b/2$, then for some $\varepsilon > 0$, $v_2 = b/2 + \varepsilon$, thus

$$s_2(v_2) = \frac{b^2 - 4d}{4} + (b - \frac{c}{a})\varepsilon \qquad (3.14)$$

and,

$$s_2(b/2) = \frac{b^2-4d}{4} + \frac{2\varepsilon^2}{b(b^2-4d)} \quad (3.15)$$

(C) If $v_2 < b/2$, then for some E > 0, $v_2 = b/2 - \varepsilon$.
thus,

$$s_2(v_2) = \frac{b^2-4d}{4} - (b - \frac{c}{a})\varepsilon \quad (3.16)$$

and,

$$s_2(b/2) = \frac{b^2-4d}{4} + \frac{2\varepsilon^2}{b(b^2-4d)} \quad (3.17)$$

On choosing $\varepsilon < 2b\,(b^2 - 4d)\,(b - c/a)$, we obtain the inequality
$S_2(v_2) > S_2(b/2)$

hence,
$m_2(d) = S(v_0) \le S_2(v_2)$ with $A_2(v_0) = 0$.
This completes the proof.

Lemma 2: For all v > 0, det $\pi(\omega) > 0$ ($\omega^2 = v$).

PROOF:

$$\det \pi(\omega) = \pi_{11}\pi_{12} - |\pi_{12}|^2$$

$$= \tau_1\tau_2\left[\frac{1}{\mu_{2\mu}\mu_2} + \frac{v}{|\Delta|^2}\left(\frac{v^2-vb+d}{\mu_1} - \frac{v^2a-vc+e}{\mu_2} - \frac{\tau_2^2+v\tau_1^2}{4\tau_1\tau_2}\right)\right] \quad (3.18)$$

This will be positive for all v > 0 in R, if

$$v^5 + (a^2-2b)v^4 + (b^2-2ac-2d+\mu_2-a\mu_1)v^3 + \left(2ae-c^2-2bd-b\mu_2+c\mu_1-\frac{\tau_1\mu_1\mu_2}{4\tau_2}\right)v^2 + \left(d^2-2ac+d\mu_2-e\mu_1-\frac{\tau_2\mu_1\mu_2}{4\tau_1}\right)v + e^2 > 0 \quad (3.19)$$

If v = 0, then det $\pi(\omega) > 0$. But if $v \neq 0$, then by choosing $\mu_1\mu_2$ and the following inequalities;
$a^2 > 2b$, $b^2 > 2(ac + d)$, $c^2 < 2(ae - bd)$, $d^2 > 2ac$ (3.20)
gives;

$$\frac{\mu_1\mu_2}{4(c\mu_1-b\mu_2} < \frac{\tau_1}{\tau_2} < \frac{4(d\mu_2-e\mu_1)}{\mu_1\mu_2} \quad (3.21)$$

thus,
$\pi_{22}\,\pi_{33} - |\pi_{23}|^2 > 0$ for all v in R, if $(\mu_1\mu_2)^2 < 16(d\mu_2 - e\mu_1)(c\mu_1 - b\mu_2)$.

3. Proofs of the main results

The proofs of the theorem stated in section 2

Proof of theorem 2.1
Let $f(z) = cz + \hat{f}(z)$ and $g(z) = dz + \hat{g}(z)$

Where, c and d are positive parameters. Set $x_1 = x$, the equation becomes
$x^{(v)} + ax^{(iv)} + bx^{(iii)} + f(x^{ii}) + g(x^{i}) + ex = P(t)$,
which reduces to the equivalent form;

$x^1{}_1 = x_2$
$x^1{}_2 = x_3$
$x^1{}_3 = x_4$
$x^1{}_4 = x_5$

$x^1_5 = -ex_1 - dx_2 - cx_3 - bx_4 - ax_4 - \hat{f}(x_3) - \hat{g}(x_2) + P(t)$

Written in vector for gives;
$X^1 = Ax - B\varphi(\sigma) + P(t), \qquad G = C^*X$

With X, A, B, C, P and $\varphi(\sigma)$ in system (3.1). The frequency domain condition reduces the matrix inequality (3.3) which satisfied for all ω in R. This is true by using Lemmas 1 and 2. The conclusions of theorem 2.1 thus follow from the generalized theorem of Yacubovich.

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