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On the principle of exchange of stabilities in old roydian fluid in porous medium with variable gravity using positive operator method

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ABSTRACT

In the present paper, the problem of thermal convection of a visco-elastic oldroydian fluid in porous medium heated from below with variable gravity is analyzed and it is established by the method of positive operator of Weinberger and uses the positivity properties of Green's function that principle of exchange of stabilities is valid for this problem, when g (z) is nonnegative throughout the fluid layer and the elastic constant of the medium is less than the ratio of permeability to porosity, i.e. $\Gamma < \frac{P_1}{\varepsilon}$ or $\lambda < \frac{k_1}{\varepsilon \upsilon}$.

Keywords: Positive operator, viscoelastic oldroydian fluid, porous, Green's function Kernels, porous medium, principle of exchange of stabilities.

INTRODUCTION

Rayleigh–Bénard convection is a fundamental phenomenon found in many atmospheric and industrial applications. The problem has been studied extensively experimentally and theoretically because of its frequent occurrence in various fields of science and engineering. This importance leads the authors to explore different methods to study the flow of these fluids. Many analytical and numerical methods have been applied to analyze this problem in the domain of Newtonian fluids, including the linearized perturbation method, the lattice Boltzmann method (LBM), which has emerged as one of the most powerful computational fluid dynamics (CFD) methods in recent years.

A problem in fluid mechanics involving the onset of convection has been of great interest for some time. The theoretical treatments of convective problems usually invoked the so-called principle of exchange of stabilities (PES), which is demonstrated physically as convection occurring initially as a stationary convection. This has been stated as "all non decaying disturbances are non oscillatory in time". Alternatively, it can be stated as "the first unstable eigenvalues of the linearized system has imaginary part equal to zero".

Mathematically, if $\sigma_r \geq 0 \Rightarrow \sigma_i = 0$ (or equivalently, $\sigma_i \neq 0 \Rightarrow \sigma_r < 0$), then for neutral stability $(\sigma_r = 0)$, $\sigma = 0$, where σ_r and σ_i are respectively the real and imaginary parts of the complex growth rate σ. This is called the '*principle of exchange of stabilities'* (PES). The establishment of this principle results in the elimination of unsteady terms in a certain class of stability problems from the governing linerized perturbation equations. Further, we know that PES also plays an important role in the bifurcation theory of instability.

First decisive step was taken [11] the in the direction of the establishment of PES in Rayleigh-Benard convection problems in a comprehensive manner. It was proved by [7] an important theorem concerning this problem. He proved that the eigen values of the linearized stability equations will continue to be real when considered as a suitably small perturbation of a self-adjoint problem, such as was considered by Pellew and South well. This was one of the first instances in which *Operator Theory* was employed in hydrodynamic stability theory. As one of several applications of this theorem, he studied Rayleigh-Benard convection with a constant gravity and established PES for the problem. Since then several authors have studied this problem under the varying assumptions of hydromagnetic and hydrodynamics.

Convection in porous medium has been studied with great interest for more than a century and has found many applications in underground coal gasification, solar energy conversion, oil reservoir simulation, ground water contaminant transport, geothermal energy extraction and in many other areas. Also, the importance of non-Newtonian fluids in modern technology and industries is ever increasing and currently the stability investigations of such fluids are a subject matter of intense research. A non –Newtonian fluid is a fluid in which viscosity changes with the applied strain rate and as a result of which the non-Newtonian fluid may not have a well-defined viscosity. Viscoelastic fluids are such fluids whose behavior at sufficiently small variable shear stresses can be characterized by three constants *viz.* a co-efficient of viscosity, a relaxation time and a retardation time, and whose invariant differential equations of state for general motion are linear in stresses and include terms of no higher degree than the second in the stresses and velocity gradients together. The problem of the onset of thermal instability in a horizontal layer of viscous fluid heated from below has its origin in the experimental observation of [3]. For further reviews of the fundamental ideas, methods and results concerning the convective problems from the domain of Newtonian/ non-Newtonian fluids, one may be referred to [3], [7] and [8].

It is clear from the above discussion that the Pellew and South well method is a useful and simple tool for the establishment of PES in convective problems when the resulting eigen value problem, in terms of differential equations and boundary conditions, is having constant coefficients. Thus, the method is not always useful to determine the PES for those convective problems, which are either permeated with some external force fields, such as variable gravity, magnetic field, rotation etc., are imposed on the basic Thermal Convection problems and resulting the eigen value problems contain variable coefficient/s or an implicit function of growth rate, in case of non-Newtonian fluids.

The present work is partly inspired by the above discussions and the works of [6] and the striking features of convection in non-Newtonian fluids in porous medium and motivated by the desire to study the above discussed problems. Our objective here is to extend the analysis of [15] based on the method of positive operator to establish the PES to these more general convective problems from the domain of non-Newtonian fluid. In the present paper, the problem of Thermal convection of a viscoelastic fluid in porous medium heated from below with variable gravity is analyzed and using the positive operator method, it is established that PES is valid for this problem, when g (z) (the gravity field) is nonnegative throughout the fluid layer and the elastic constant of the medium is less than the ratio of permeability to porosity, i.e. $\Gamma < \frac{P_1}{\varepsilon}$ or $\lambda < \frac{k_1}{\varepsilon \upsilon}$.

MATERIALS AND METHODS

(a) Mathematical Formulation of the Physical Problem

Consider an infinite horizontal porous layer of viscoelastic fluid of depth'd' confined between two horizontal planes $z = 0$ and $z = d$ under the effect of variable gravity, $\vec{g}(0,0-g(z))$. Let ΔT be the temperature difference between the lower and upper plates. The fluid is assumed to be viscoelastic and described by the Oldroydian constitutive equations. The porous medium is assumed to have high porosity and hence the fluid flow is governed by Brinkman model with viscoelastic correction. Thus, the governing equations for the Rayleigh-Benard situation in a viscoelastic fluid – saturated porous medium under Boussinesq approximation and under the effect of variable gravity are;

$$\frac{1}{\varepsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial \vec{q}}{\partial t} = \left(1+\lambda\frac{\partial}{\partial t}\right)[-\frac{\nabla p}{\rho_0} + \left(1+\frac{\delta\rho}{\rho_0}\right)\vec{X}] - \left(1+\lambda_0\frac{\partial}{\partial t}\right)\left(\frac{\upsilon}{k_1}\right) \quad (1)$$

$$\nabla.\vec{q} = 0 \tag{2}$$

$$E\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T = K\nabla^2 T \tag{3}$$

$$\rho = \rho_0[1 - \alpha(T - T_0)] \tag{4}$$

In the above equations, ρ, T,K, α λ, λ_0 and υ stand for density ,temperature, thermal diffusivity, coefficient of thermal expansion, the relaxation time and the retardation time and the kinematic viscosity, respectively. Here, E= $\varepsilon + (1-\varepsilon)\dfrac{\rho_s c_s}{\rho_0 c_v}$ is a constant, where $\rho_{s,} c_s$ stand for density and heat capacity of solid (porous matrix) material and $\rho_{0,}, c_v$ for fluid ,respectively . Here, the suffix zero refers to the value at the reference level z=0. This is to mention here that, when the fluid slowly percolates through the pores of the rock, the gross effect is represented by the usual Darcy`s law. As a consequence, the usual viscous terms has been replaced by the resistance term $\left(-\dfrac{\mu}{k_1}\right)\vec{q}$ in the above equations of motion. Here, μ and k_1 are the viscosity and the permeability of the medium and $\vec{q}$ is the filter velocity of the fluid.

Following the usual steps of the linearized stability theory, it is easily seen that the non dimensional linearized perturbation equations governing the physical problem described by equations (1)-(4) can be put into the following forms, upon ascribing the dependence of the perturbations of the form $exp[i(k_x x + k_y y) + \sigma t]$, $(\sigma = \sigma_r + i\sigma_i)$ (*c.f.* Chandrasekhar [1961] and Siddheshwar and Krishna [2001]);

$$\left(\frac{\sigma}{\varepsilon} + \frac{1}{P_l}\left(\frac{1+\Gamma\mu\sigma}{1+\Gamma\sigma}\right)\right)(D^2 - k^2)\,w = -g(z)Rk^2\theta \tag{5}$$

$$(D^2 - k^2 - \sigma E P_r)\theta = -w \tag{6}$$

together with following dynamically free and thermally and electrically perfectly conducting boundary conditions

$$w = 0 = \theta = D^2 w \quad \text{at} \quad z = 0 \text{ and } z = 1 \tag{7}$$

In the forgoing equations, z is the real independent variable, $D \equiv d/dz$ is the differentiation with respect to z, k^2 is the square of the wave number, $Pr = \dfrac{\upsilon}{\kappa}$ is the thermal Prandtl number, $P_l = \dfrac{k_1}{d^2}$ is the dimensionless medium permeability, $\Gamma = \dfrac{\lambda\upsilon}{d^2}$ is elastic constant, E= $\varepsilon + (1-\varepsilon)\dfrac{\rho_s c_s}{\rho_0 c_v}$ is constant, $R^2 = \dfrac{g_0 \alpha\beta d^4}{\kappa\upsilon}$ is the thermal Rayleigh number, $\sigma(= \sigma_r + i\sigma_i)$ is the complex growth rate associated with the perturbations and w, θ are the perturbations in the vertical velocity, temperature, respectively.

The system of equations (5)-(6) together with the boundary conditions (7) constitutes an eigenvalue problem for σ for the given values of the parameters of the fluid and a given state of the system is stable, neutral or unstable according to whether σ_r is negative, zero or positive.

It is remarkable to note here that equations (5)-(6) contain a variable coefficient and an implicit function of σ, hence as discussed earlier the usual method of Pellew and South well is not useful here to establish PES for this general problem. Thus, we shall use the method of positive operator to establish PES.

(b). Abstract Formulation

The method of positive operator

We seek conditions under which solutions of equations (5)-(6) together with the boundary conditions (7) grow. The idea of the method of the solution is based on the notion of a 'positive operator', a generalization of a positive matrix, that is, one with all its entries positive. Such matrices have the property that they possess a single greatest positive eigen value, identical to the spectral radius. The natural generalization of a matrix operator is an integral operator with non-negative kernel. To apply the method, the resolvent of the linearized stability operator is analyzed. This resolvent is in the form of certain integral operators. When the Green's function Kernels for these operators are all nonnegative, the resulting operator is termed positive. The abstract theory is based on the Krein –Rutman theorem, which states that;

"If a linear, compact operator A, leaving invariant a cone $\hbar$, has a point of the spectrum different from zero, then it has a positive eigen value λ, not less in modulus than every other eigen value, and this number corresponds at least one eigen vector $\phi \in \hbar$ of the operator A, and at least one eigen vector $\varphi \in \hbar^*$ of the operator A^*". For the present problem the cone consists of the set of nonnegative functions.

To apply the method of positive operator, formulate the above equations (5) and (6) together with boundary conditions (7) in terms of certain operators as;

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{p_1}\left(\frac{1+\Gamma\mu\sigma}{1+\Gamma\sigma}\right)\right]Mw = g(z)R\theta k^2 \tag{8}$$

$$(M+\sigma E\,Pr)\theta = Rw \tag{9}$$

where,

$Mw = mw, \quad w \in \text{dom}M; \quad M^2 w = m^2 w, \quad w \in \text{dom}(MM);$ and $\quad M\theta = m\theta, \quad w \in \text{dom}M$

The domains are contained in *B*, where

$$B = L^2(0,1) = \left\{\phi \;\middle|\; \int_0^1 |\phi|^2\, dz < \infty\right\},$$

with scalar product $\langle \phi,\varphi\rangle = \int_0^1 \phi(z)\,\overline{\varphi(z)}\,dz$, $\quad \phi,\varphi \in B$; and norm $\|\phi\| = \langle \phi,\phi\rangle^{1/2}$.

We know that $L^2(0,1)$ is a Hilbert space, so, the domain of M is

$$\text{dom}\,M = \{\phi \in B / \; D\phi, m\phi \in B, \;\; \phi(0)=\phi(1)=0\}.$$

We can formulate the homogeneous problem corresponding to equations (5)-(6) by eliminating θ from (8) and (9) as;

$$w = k^2 R^2 M^{-1}\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_l}\left(\frac{1+\Gamma\mu\sigma}{1+\Gamma\sigma}\right)\right)^{-1} g(z)(M+E\,Pr\,\sigma)^{-1} w \tag{10}$$

or $$w = K(\sigma)w \tag{11}$$

$$K(\sigma) = k^2 R^2 T(0)\left(\frac{\sigma}{\varepsilon} + \frac{1}{P_1}\left(\frac{1+\Gamma\mu\sigma}{1+\Gamma\sigma}\right)\right)^{-1} g(z)T(E\Pr\sigma)w \tag{12}$$

Defining, $T(E\Pr\sigma) = (M + E\Pr\sigma)^{-1}$ exists for $\sigma \in T_{\frac{k}{\sqrt{\Pr E}}} = \left\{\sigma \in C \;\middle|\; \mathrm{Re}(\sigma) > \frac{-k^2}{E\Pr}, \mathrm{Im}(\sigma) = 0\right\}$ and for $\mathrm{Re}(\sigma) > -\frac{k^2}{E\Pr}$.

Now, $T(E\Pr\sigma)$ is an integral operator such that for $f \in B$,

$$T(\sigma E\Pr)f = \int_0^1 g(z,\xi;\sigma E\Pr)f(\xi)d\xi,$$

where, $g(z,\xi,E\Pr\sigma)$ is Green's function kernel for the operator $(M + \sigma E\Pr)$, and is given as

$$g(z,\xi,\Pr\sigma) = \frac{\cosh[r(1-|z-\xi|)] - \cosh[r(-1+z+\xi)]}{2r\sinh r}$$

Where, $r = \sqrt{\sigma + \frac{k^2}{E\Pr}}$.

In particular, taking $\sigma = 0$, we have $M^{-1} = T(0)$ is an integral operator.

$K(\sigma)$ defined in (12), which is a composition of certain integral operators, is termed as *linearized stability operator*. K (σ) depends analytically on σ in a certain right half of the complex plane. It is clear from the composition of K (σ) that it contain an implicit function of σ.

We shall examine the resolvent of the K (σ) defined as $[I - K(\sigma)]^{-1}$

$$[I - K(\sigma)]^{-1} = \{I - [I - K(\sigma_0)]^{-1}[K(\sigma) - K(\sigma_0)]\}^{-1}[I - K(\sigma_0)]^{-1} \tag{13}$$

If for all σ_0 greater than some a,

(1) $[I - K(\sigma_0)]^{-1}$ is positive,

(2) $K(\sigma)$ has a power series about σ_0 in $(\sigma_0 - \sigma)$ with positive coefficients; i.e., $\left(-\frac{d}{d\sigma}\right)^n K(\sigma_o)$ is positive for all n, then the right side of (13) has an expansion in $(\sigma_0 - \sigma)$ with positive coefficients. Hence, we may apply the methods of Weinberger [10] and Rabinowitz , to show that there exists a real eigenvalue σ_1 such that the spectrum of $K(\sigma)$ lies in the set $\{\sigma : \mathrm{Re}(\sigma) \le \sigma_1\}$. This is result is equivalent to PES, which was stated earlier as "the first unstable eigenvalue of the linearized system has imaginary part equal to zero."

RESULTS AND DISCUSSION

(a). The principle of exchange of stabilities (PES)

It is clear that $K(\sigma)$ is a product of certain operators. Condition (1) can be easily verified by following the analysis of Herron [2000] for the present operator $K(\sigma)$, i.e. $K(\sigma)$ is a linear, compact integral operator, and has a power series about σ_0 in $(\sigma_0 - \sigma)$ with positive coefficients. Thus, $K(\sigma)$ is a positive operator leaving invariant a cone (set of non negative functions). Moreover, for σ real and sufficiently large, the norms of the operators $T(0)$ and $T(\Pr\sigma)$ become arbitrarily small. So, $\|K(\sigma)\| < 1$. Hence, $[I - K(\sigma)]^{-1}$ has a convergent Neumann series, which implies that $[I - K(\sigma)]^{-1}$ is a positive operator. This is the content of condition (P1).

To verify condition (2), we note that $T(E\Pr\sigma) = (M + E\Pr\sigma)^{-1}$ is an integral operator whose kernel $g(z,\xi,E\,Pr\,\sigma)$ is the Laplace transform of the Green's function $G(z,\xi;t)$ for the initial-boundary value problem

$$\left(-\frac{\partial^2}{\partial z^2} + k^2 + E\Pr\frac{\partial}{\partial t}\right)G = \delta(z-\xi, t), \tag{14}$$

where, $\delta(z-\xi, t)$ is Dirac –delta function in two-dimension,

With boundary conditions $G(0,\xi;t) = G(1,\xi;t) = G(z,\xi;0) = 0$, (15)

Using the similar result proved in Herron [2000] by direct calculation of the inverse Laplace transform, we can have

$T(\sigma E\Pr) = (M + \sigma E\Pr)^{-1}$ is positive operator for all real $\sigma_0 > -\dfrac{k^2}{E\Pr}$, and that $T(\sigma E\Pr)$ has a power series about σ_0 in $(\sigma_0 - \sigma)$ with positive coefficients,

i.e., for all real $\sigma_0 > -\dfrac{k^2}{E\Pr}$, we see that

$$\left(-\frac{d}{d\sigma}\right)^n g(z,\xi,\sigma E\Pr) = \int_0^\infty t^n e^{-\sigma E\Pr t}\, G(z,\xi,t)\,dt \ge 0 \text{ is positive.}$$

In particular, , from the above result, we deduce that $T(0) = (M)^{-1}$ is positive operator for all real $\sigma_0 > -k^2$, taking $\sigma = 0$.

Also, $\left(\dfrac{\sigma}{\varepsilon} + \dfrac{1}{p_1}\left(\dfrac{1+\tau\mu\sigma}{1+\tau\sigma}\right)\right)^{-1} > 0$ for all σ_0 real and $\sigma_0 > \sqrt{\dfrac{p_1+\mu\varepsilon\tau}{2\tau p_1} - \dfrac{\varepsilon}{\tau p_1}} - \dfrac{p_1+\mu\varepsilon\tau}{2\tau p_1}$ and $\Gamma < \dfrac{p_1}{\varepsilon}$ or $\lambda < \dfrac{k_1}{\varepsilon\upsilon}$.

Therefore, for all real $\sigma_0 > \max(-\frac{k^2}{E\Pr}, -k^2, \sqrt{\frac{p_1+\mu\varepsilon\Gamma}{2\Gamma p_1} - \frac{\varepsilon}{\Gamma p_1}} - \frac{p_1+\mu\varepsilon\Gamma}{2\Gamma p_1})$, $g(z) \geq 0$ for all $z \in [0,1]$, by the product rule for differentiation one concludes that $K(\sigma)$, composition of $T(\sigma E \Pr)$, $T(0)$ satisfies condition (2).

Hence, we have the following theorem;

Theorem. PES holds for (5) - (6) together with boundary conditions (7)when g (z) is nonnegative throughout the layer, $\Gamma < \frac{P_1}{\varepsilon}$ or $\Gamma < \frac{k_1}{\varepsilon\upsilon}$ and $\sigma_0 > \max(-\frac{k^2}{E\Pr}, -k^2, \sqrt{\frac{p_1+\mu\varepsilon\Gamma}{2\Gamma p_1} - \frac{\varepsilon}{\Gamma p_1}} - \frac{p_1+\mu\varepsilon\Gamma}{2\Gamma p_1})$.

CONCLUSION

It is concluded from above discussion that when g (z)(the gravity field) is nonnegative throughout the fluid layer and the elastic constant of the medium is less than the ratio of permeability to porosity, i.e. $\Gamma < \frac{P_1}{\varepsilon}$ or $\lambda < \frac{k_1}{\varepsilon\upsilon}$, PES is valid.

In particular, letting $\Gamma = 0$ for Benard Problem, when g (z) (the gravity field) is nonnegative throughout the fluid layer PES is valid.

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