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On the Ellipsoid Method for Systems of Linear Inequalities

B. O. Adejo[1] and A. M. Okutachi[2]

Department of Mathematical Sciences, Kogi State University, Anyigba, Nigeria

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ABSTRACT

In this article, we give an overview of the basic ellipsoid method, its antecedents and modifications made to improve its rate of convergence. Furthermore, its importance is highlighted.

Keywords: Polynomial time, feasible solution, perturbed system

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INTRODUCTION

In 1947, G. B. Dantzig formulated the Linear Programming (LP) problem and provided the simplex method to solve it. The simplex method is still the widely used method to solve LP problems, although polynomial-time algorithms have emerged that work on the relative interior of the feasible region in their search for solutions to the LP problem.

The simplex method was made available in 1951. It is an iterative (i.e algebraic) procedure which either solves a LP problem in a finite number of steps or gives an indication that there is an unbounded solution or infeasible solution to the problem under consideration. The simplex method can be viewed as an exterior point method since it 'crawls' along the edges (vertices) of the feasible region in its search for solution to the LP problem.

In 1979, Khachian L. G [11] resolved an open question of whether linear programming problems belonged to the P-class (i. e. class of problems solvable in polynomial time) or to the NP-class (i. e. class of problems not solvable in polynomial-time). He adapted the ellipsoid method used in convex optimization developed independently by Shor [19] and Iudin and Nemirovskii [9] to give a polynomial-time algorithm for LP. In 1984, Karmarkar [10] introduced the first ever interior point projective algorithm for LP problems. For his work, he was awarded with the Fulkerson prize of the American Mathematical Society and the Mathematical Programming Society [12, 22]. Karmarkar's [10] algorithm has led to the development of other interior point algorithms for LP which compare favourably with the simplex method, especially for problems with large sizes.

2. The Basic Ellipsoid Method in R^n

We discuss how the ellipsoid algorithm can be used to determine the feasibility or otherwise of the system of linear inequalities with integer data in polynomial-time.

Suppose, we want to determine an n-tuple $(x_1,\ x_2, \ldots, x_n)$ that satisfies the following system of linear inequalities:

$$a_1 x = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le \beta_1$$

$$a_2 x = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le \beta_2$$

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$$a_m x = a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le \beta_m$$

which can be written in matrix notation as $A^T x \le b,$ where A is an mxn matrix, x is an n – vector and b is an m – vector.

Furthermore, suppose that the column vectors corresponding to the outward drawn normal of the constraints are $a_1, a_2, \ldots, a_n$ and the components of b are $\beta_1, \beta_2 \ldots, \beta_{m,}$ then $A^T x \le b,$ can be restated as

$$a_i^T x \le \beta_i \quad \ldots (2.1)$$

we assume $n > 1$

Roughly speaking, the basic idea of the ellipsoid method is to start with an initial ellipsoid containing the solution set of (2.1). The centre of the ellipsoid is in each step a candidate for a feasible point of the problem. After checking whether this point satisfies all linear inequalities, one either produced a feasible point and the algorithm terminates, or one found a violated inequality. This is used to construct a new ellipsoid of smaller volume and with a different centre. Now, this procedure is repeated until either a feasible point is found or a maximum number of iterations $(i.e. 6n^2 L)$ is reached. In the latter case, this implies that the inequality set has no feasible region.

The Basic Iteration:

The ellipsoid method involves construction of a sequence of ellipsoids $E_1, E_2, \ldots, E_k, \ldots$ each of which contains a point that satisfies the system (2.1) if one exists. On the $(k+1)^{st}$ iteration, the method checks whether the centre x_k of the current ellipsoid E_k satisfies the constraint (2.1). If it does, then the method stops. However, if it does not satisfy the constraint (2.1), then some constraints that are violated by x_k i. e

$$a_i^T x_k > \beta_i \text{ for some } i = 1 \le i \le m \quad \ldots (2.2)$$

are chosen and the ellipsoid of minimum volume that contains the half – ellipsoid

$$\left\{ x \in E_k \,\middle|\, a_i^T x \le a^T x_k \right\} \quad \ldots (2.3)$$

is constructed.

The new ellipsoid and its centre are denoted by E_{k+1} and x_{k+1} respectively

$$x_{k+1} = x_k - \tau \left(\frac{B_k a}{\sqrt{a^T B_k a}} \right) \quad \ldots (2.4)$$

$$B_{k+1} = \delta \left[B_k - \frac{\sigma (B_k a)(B_k a)^T}{a^T B_k a} \right] \quad \ldots (2.5)$$

where $\tau = \frac{1}{n+1}$, $\sigma = \frac{2}{n+1}$ and $\delta = \frac{n^2}{n^2 - 1}$

τ is known as the step parameter, while σ and δ are the dilation and expansion parameters respectively.

The ellipsoid method gives a possibly infinite iterative scheme for determining the feasibility of the system (2.1). However, this problem can be solved by initialization.

Khachian [11] have shown that the feasibility or non-feasibility of the system of linear inequalities (2.1) can be determined within a pre-specified number of iterations i. e $i.e. 6n^2 L$ by performing any of the following:

(i) modifying the algorithm to account for finite precision arithmetic
(ii) applying the algorithm to a suitable perturbation of the system (2.1) and
(iii) choosing E_0 appropriately

The system of inequalities is feasible if and only if termination occurs with a feasible solution of the perturbed system within the prescribed number of iterations.

[3] Precursors of the Ellipsoid Method:
In this section, we present some antecedents to the ellipsoid method.

(i) The Relaxation Method
This method for linear inequalities was introduced simultaneously by Agmon [2] and Motzkin and Schoenberg [16] in 1954. They considered the problem (2.1) i. e

$a_i^T x \leq \beta_i$

and produced a sequence $\{x_k\}$ of iterates. At the $(k+1)^{st}$ iteration, if x_k is not feasible, then a violated constraint (say (2.2)) is chosen and we set

$$x_{k+1} = x_k - \lambda_k a \left(\frac{a^T x_k - \beta}{a^T a} \right) \qquad \ldots(3.1)$$

where λ_k is called the relaxation parameter. For Motzkin and Schoenberg's [16] method, they considered $\lambda_k = 2$, while Agmon [2] chose $0 < \lambda_k < 2$.

Agmon [2] showed that each iterate came closer by some fixed ratio to the set of feasible solution than its predecessor.

The difference between the relaxation method and the basic ellipsoid method is that the ratio in the former depends on the data of the problem rather than the dimension. Bounds on the ratio have been provided by Agmon [2], Hoffman [8], Goffin [5, 6] and Todd [21].

(ii) The Subgradient and Space Dilation Method:
The subgradient method for minimizing a convex (not necessarily differentiable) function

$f : R^n \to R$

was first introduced by Shor [19].
The method has the general form:

$$x_{k+1} = x_k - \mu_k \frac{g_k}{\|g_k\|} \qquad \ldots (3.2)$$

where g_k is a subgradient of the function f at x_k

To solve (2.1), we can minimize

$f(x) = \max \left\{ \max \left(a_i^T x - \beta_i \right), 0 \right\}$

then $a = a_i$ is a subgradient of f at x_k, if $a_i^T x > \beta_i$, is the most violated constraint for (2.1). The choice of μ_k that ensures global convergence are given in Ermolev [4] and Polyak [18]. For instance, $\mu_k \to 0$ and $\sum \mu_k = \infty$. Suffice, however with very slow convergence results. (See Bland et al [3]).

(iii) The Method of Central Sections:
The method was developed by Levin [15] and Newman [17], where they addressed the problem of minimizing a convex function f over a bounded polyhedron $P_0 \subseteq R^n$.

The method produces a sequence of iterates $\{x_k\}$ and polytope $\{P_k\}$ by choosing x_k and $P_{k+1} = \left\{ x \in P_k \middle| g_k^T x \leq g_k^T x_k \right\}$ as the centre of gravity of P_k and where g_k is a subgradient of f at x_k.

Now, since f is convex, P_{k+1} contains all points of P_k whose objective function value is not greater than the value of x_k. In this case, the volume of P_{k+1} is at most $\left(1 - e^{-1}\right) x$ volume of P_k i.e volume of

$P_{k+1} \leq \left(1-e^{-1}\right) x$ .volume of P_k. Calculating the centre of gravity of polytopes with many facets in high dimensional spaces is a very difficult task. Hence, Levin [15] proposed $n = 2$ for simplification.

[4] <u>Modifications of the Ellipsoid Method</u>:

Here, we discuss some modifications, as presented by Bland et al [3] and polynomial equivalence conditions of Adejo [1], that could be made on the ellipsoid algorithm in trying to increase its rate of convergence.

(i) **Deep cuts (i. e. violated inequalities):**

Shor and Gershovich [20] were the first to propose the use of deep cuts to speed up the ellipsoid method.

Suppose x_k violates $a_i^T x \leq \beta_i$. The ellipsoid E_{k+1} is determined by formulae for x_{k+1} (2.4) and B_{k+1}(2.5) and it contains the half ellipsoid $\left\{x \in E_k \,\middle|\, a^T x \leq a^T x_k \right\}$. Since it is only required that E_{k+1} contains the smaller portions of E_k i. e $\left\{x \in E_k \,\middle|\, a^T x \leq \beta\right\}$, it seems obvious that we can obtain an ellipsoid of smaller volume by using "deep cut" $\left\{a^T x \leq \beta\right\}$ instead of the cut $a^T x \leq a^T x_k$ which passes through the centre of E_k as shown in figures $(1(a))$ and $(1(b))$.

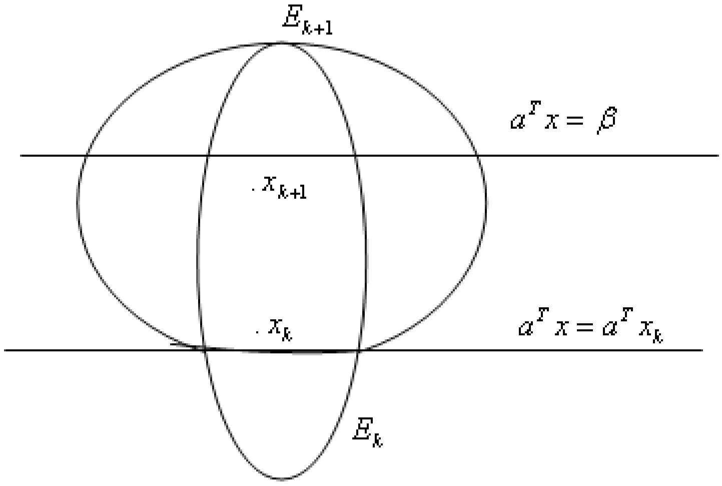

Fig 1(a). The ellipsoid method without deep cuts

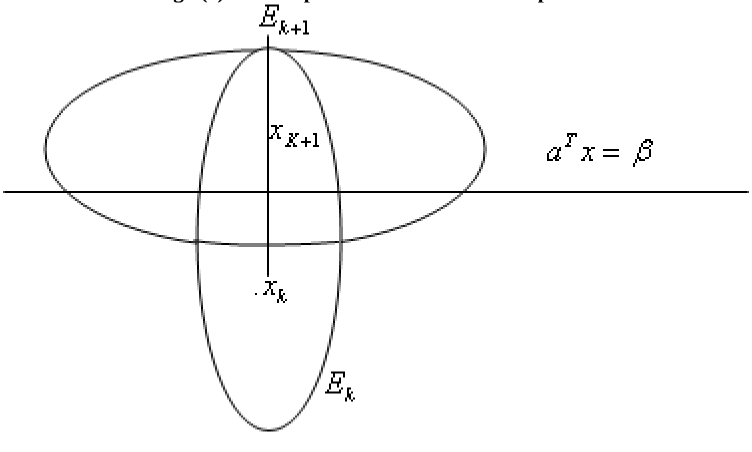

Fig 1(b). The ellipsoid method with deep cuts

The smallest of such ellipsoids is given by:

$$x_{k+1} = x_k - \tau\left(\frac{B_k a}{\sqrt{a^T B_k a}}\right)$$

$$B_{k+1} = \delta\left[B_k - \sigma\left(\frac{(B_k a)(B_k a)^T}{\sqrt{a^T B_k a}}\right)\right]$$

where $\tau = \left(\frac{1+n\alpha}{n+1}\right)$, $\sigma = \frac{2(1+n\alpha)}{(n+1)(1+\alpha)}$, $\delta = \left(\frac{n^2}{n^2-1}\right)(1-\alpha^2)$

and $\alpha = \frac{a^T x_k - \beta}{\sqrt{a^T B_k a}}$

Now, by computing α for each inequality in (2.1) we can select the deepest cut possible i. e, the cut that corresponds to the largest possible volume of α. If the value of α is greater than one, then the system (2.1) is infeasible.

(ii) <u>Surrogate cut</u>:
The basic idea here is that if inequalities (2.1) are combined, it may be possible to obtain cuts that are "deeper" than any cut generated by a single constraint of the inequalities (2.1). Any cut of the form:

$$u^T A^T x \le u^T b$$
$$i.e. a^T x \le \beta$$

with $a = A\,u$ and $\beta = u^T b$

is valid as long as $u \ge 0$. Since no point that satisfy (2.1) is cut off by this inequalities.

Goldfarb and Todd [7] introduced the term 'surrogate cuts', while Krol and Mirman [14], proposed the idea of using surrogate cuts in the ellipsoid method. Bland et al [3] have stated that the best or deepest surrogate cut is that which can be obtained by solving

$$\max_{u \ge 0} \frac{u^T\left(A^T x_k - b\right)}{\sqrt{u^T A^T B A u}}$$

which is equivalent to solving a quadratic programming problem. Let $\overline{A}^T x \le \overline{b}$ be any subset of constraints (2.1), where the columns of $\overline{A}$ are linearly independent and at least one of the constraints is violated by x_k. It has been observed in Goldfarb and Todd [7] that if

$$\overline{u} = \left(\overline{A}^T B_k \overline{A}\right)^{-1}\left(\overline{A} x_k - \overline{b}\right) \qquad \ldots(3.3)$$

is non-negative, then the surrogate cut

$$\overline{u}^T \overline{A}^T x \le \overline{u}^T \overline{b}$$

is deepest with respect to that subset.

The price to obtain the deepest or nearly deepest surrogate cut is too high a price to pay in solving quadratic programming problem or computing $\overline{u}$ by (3.3). Hence, Goldfarb and Todd [7] suggested that only surrogate cuts which can be generated from two constraints need be considered.

(iii) **Parallel cuts**
If (2.1) contains a parallel pair of constraints:

$$a^T x \le \beta$$

and $-a^T x \le -\hat{\beta}$

then, it is possible to use both constraints simultaneously to generate a new ellipsoid E_{k+1}.

Let $$\alpha = \frac{a^T x_k - \beta}{\sqrt{a^T B_k a}}$$

$$\hat{a} = \frac{\hat{\beta} - a^T x_k}{\sqrt{a^T B_k a}}$$

$\alpha\hat{\alpha} < \frac{1}{n}$ and $\alpha \leq -\hat{\alpha} \leq 1$

then formulae (2.4) and (2.5) with

$$\tau = \left(\frac{\alpha - \hat{\alpha}}{2}\right)\delta$$

$$\sigma = \frac{1}{n+1}\left[n + \frac{2}{(\alpha - \hat{\alpha})^2}\left(1 - \alpha\hat{\alpha} - \frac{p}{2}\right)\right]$$

$$\delta = \frac{n^2}{n^2 - 1}\left[1 - \frac{\left(\alpha^2 + \hat{\alpha}^2 - p/n\right)}{2}\right]$$

and $P = \sqrt{4(1-\alpha^2)\left(1-\hat{\alpha}^2\right) + n^2\left(\hat{\alpha}^2 - \alpha^2\right)^2}$

generates an ellipsoid that contains the slice $\left\{x \in E_k \middle| \hat{\beta} \leq a^T x \leq \beta\right\}$of E_k when $\beta = \hat{\beta}$, i. e $a^T x = \beta$ for all feasible x,

$\hat{\alpha} = -\alpha$

and we obtain

$$\tau = \left(\frac{\alpha - \hat{\alpha}}{2}\right)\sigma \; = \left(\frac{\alpha + \hat{\alpha}}{2}\right).1 = \alpha$$

$$\sigma = \frac{1}{n+1}\left[n + \frac{1}{2\alpha^2}\left(2\alpha^2\right)\right] = 1$$

$$\delta = \frac{n^2}{n^2 - 1}\left[1 - \frac{1}{2}\left(\alpha^2 + \hat{\alpha}^2 - \frac{p}{n}\right)\right]$$

$$= \frac{n^2}{n^2 - 1}\left[1 - \frac{1}{2}\left(\alpha^2 + \hat{\alpha}^2 - \frac{2\left(1-\alpha^2\right)}{n}\right)\right] = \frac{n\left(1-\alpha^2\right)}{n-1}$$

that is, rank $\left(B_{k+1}\right). = rank \; \left(B_k\right) - 1 \; and \; E_{k+1}$ becomes flat in the directive of a.

Like in the case of deep cuts, Shor and Gershovich [20] were the first to suggest the use of parallel cuts and provided formulae for implementing them. They also derived formulae for circumscribing an ellipsoid (of close to minimum volume) about the region of a unit ball and whose normal are mutually obtuse.

The formulae for parallel cuts were also derived independently by Konig and Pallaschke [13], and Todd [21] with proofs that they give the ellipsoid of minimum volume contained there-in.

(iv) **Polynomial Equivalence Conditions**:
Adejo [1] have proposed polynomial equivalence conditions for the relaxation and ellipsoid methods, as well as for the relaxation and the subgradient space dilation methods respectively. With the aim of increasing the rate of convergence of the ellipsoid method as follows:

For the relaxation method from (3.1)

$$x_{k+1} = x_k - \lambda_k a\left(\frac{a^T x_k - \beta}{a^T a}\right) \qquad \text{… (4.1)}$$

while for the ellipsoid method from (2.4)

$$x_{k+1} = x_k - \tau\left(\frac{B_k a}{\sqrt{a^T B_k a}}\right) \quad \text{… (4.2)}$$

For both (4.1) and (4.2) to coincide

$$\tau = \lambda_k a\left(\frac{a^T x_k - \beta}{a^T a}\right)\frac{\sqrt{a^T B_k a}}{(B_k a)} \quad \text{… (4.3)}$$

For the subgradient/space dilation method from (3.2)

$$x_{k+1} = x_k - \frac{\mu_k g_k}{\| g_k \|} \quad \text{…(4.4)}$$

For both (4.2) and (4.4) to coincide

$$\tau = \frac{\mu_k g_k \sqrt{a^T B_k a}}{(\beta_k a)\| g_k \|} \quad \text{…(4.5)}$$

We note from (4.3) that $\tau \alpha \lambda_k$, while from (4.5), $\tau \alpha \mu_k$. Adejo [1] proposes the use of τ as given in (4.3) and (4.5), instead of the one used in the basic ellipsoid method (2.4).

CONCLUSION

The ellipsoid method have been used to resolve open questions of whether linear programming problems belonged to the P-class or not. Furthermore, it had also been used to show that certain combinatorial optimization problems belong to the P-class while others are NP-hard. Relatively, little computational experience with the ellipsoid algorithm is available but the general consensus is that it is not a practical alternative to the simplex nor Karmarkar's algorithms.

Although the ellipsoid method has shown some insurmountable difficulties in its practical applicability, its overall impact on theoretical developments, combinatorial optimization and in handling problems with an exponential number of constraints cannot be denied. It is strongly believed that the dilemma of considering the ellipsoid method as theoretically significant but practically impoverished indicates the need for more reconsideration of its various complexity measures.

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