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On *para*-Kenmotsu manifolds satisfying certain conditions on the curvature tensors

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ABSTRACT

The objective of this paper is to study certain properties of para-Kenmotsu manifolds satisfying the conditions $R(X,Y).R=0$, $R(X,Y).S=0$ *and* $P(X,Y).S=0$ *where R(X, Y) is the Riemannian curvature tensor, S(X, Y) is the Ricci curvature tensor and P(X, Y) is the Weyl projective curvature tensor. It is shown that a* p *-Kenmotsu manifold satisfying the conditions* $R(X,Y).R=0$ *is flat, and* $R(X,Y).S=0$ *is an Einstein manifold. Finally, we also proved that if a* p *-Kenmotsu manifold satisfies the condition* $P(X,Y).S=0$, *then the structure vector* ξ *is normal to the tangent space of the manifold.*

Mathematics Subject Classification: 53C25
Keywords: *Para Kenmotsu manifold, Curvature tensor, Ricci tensor, Scalar curvature, Einstein Manifold*

INTRODUCTION

The notion of almost para contact structure was introduced by Sato [8]. Later, Adati and Matsumoto [1] defined and studied p -Sasakian and sp -Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, Kenmotsu [6] defined a class of almost contact Riemannian manifolds. In 1995, Sinha and Sai Prasad [9] have defined a class of almost para contact metric manifolds namely para- Kenmotsu (p -Kenmotsu) and special para Kenmotsu (sp -Kenmotsu) manifolds as analogues of p -Sasakian and sp -Sasakian manifolds.

A Riemannian manifold M_n is locally symmetric if its curvature tensor R satisfies ∇ R = 0, where ∇ is Levi-Civita connection of the Riemannian metric [4]. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M_n is said to be semi-symmetric if its curvature tensor R satisfies $R(X,Y).R=0$ where $R(X,Y)$ acts on R as a derivation [10]. Locally symmetric and semi-symmetric *p*-Sasakian manifolds are widely studied by many geometers [2, 5, 7].

In this paper, we consider p -Kenmotsu manifolds satisfying the conditions on the Riemannian curvature tensor R, the Ricci curvature tensor S and Weyl projective curvature tensor P and studied their properties, for the first time.

PRELIMINARIES

Let M_n be an n-dimensional differentiable manifold equipped with structure tensors (Φ, ξ, η) where Φ is a tensor of type (1, 1), ξ is a vector field, η is a 1-form such that

$$\eta(\xi) = 1 \tag{2.1}$$

$$\Phi^2(X) = X - \eta(X)\xi; \overline{X} = \Phi X \tag{2.2}$$

Then M_n is called an almost para contact manifold [8].

Let g be the Riemannian metric satisfying, such that, for all vector fields X and Y on M_n,

$$g(X, \xi) = \eta(X) \tag{2.3}$$

$$\Phi\xi = 0, \eta(\Phi X) = 0, \text{rank } \Phi = n-1 \tag{2.4}$$

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.5}$$

Then the manifold M_n [8] is said to admit an almost para contact Riemannian structure (Φ, ξ, η, g).

A manifold of dimension n with Riemannian metric g admitting a tensor field Φ of type (1, 1), a vector field ξ and a 1-form η satisfying (2.1), (2.3) along with

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \tag{2.6}$$

$$(\nabla_X \nabla_Y \eta)Z = [-g(X,Z) + \eta(X)\eta(Z)]\eta(Y) + [-g(X,Y) + \eta(X)\eta(Y)]\eta(Z) \tag{2.7}$$

$$\nabla_X \xi = \Phi^2 X = X - \eta(X)\xi \tag{2.8}$$

$$(\nabla_X \Phi)Y = g(\Phi X, Y)\xi - \eta(Y)\Phi X \tag{2.9}$$

is called a para Kenmotsu manifold (or) p-Kenmotsu manifold [9].

A p-Kenmotsu manifold admitting a 1-form η satisfying

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) \tag{2.10}$$

$$g(X, \xi) = \eta(X) \text{ and } (\nabla_X \eta)Y = \varphi(\overline{X}, Y), \text{ where } \varphi \text{ is an associate of } \Phi \tag{2.11}$$

is called as special p-Kenmotsu manifold (or) an sp-Kenmotsu manifold [9].

It is known that [9] in a p-Kenmotsu manifold the following relations hold:

$$S(X, \xi) = -(n-1)\eta(X) \; where \; g(QX, Y) = S(X,Y) \tag{2.12}$$

$$g[R(X,Y)Z, \xi] = \eta[R(X,Y,Z)] = g(X,Z)\eta(Y) - g(Y,Z)\eta(X) \tag{2.13}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi \tag{2.14}$$

$$R(X,Y,\xi) = \eta(X)Y - \eta(Y)X; \text{ when } X \text{ isorthogonalto } \xi \tag{2.15}$$

where S is the Ricci tensor, r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor and R is the Riemannian curvature.

If the Ricci curvature tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.16}$$

where a and b are functions on M_n, then M_n is called as an $\eta - Einstein$ manifold and if $b = 0$ then it is an Einstein manifold.

Moreover, it is also known that if a p-Kenmotsu manifold is projectively flat then it is an Einstein manifold and the scalar curvature has a negative constant value $-n(n-1)$ [9].

In this case,

$$S(Y,Z) = -(n-1)g(Y,Z) \tag{2.17}$$

and hence

$$S(\Phi Y, \Phi Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z) \tag{2.18}$$

Also if a p-Kenmotsu manifold is of constant curvature, we have

$$'R(X,Y,Z,P) = \frac{1}{(n-1)}[S(Y,Z)g(X,P) - S(X,Z)g(Y,P)] \tag{2.19}$$

The above results will be used further in the next sections.

RESULTS

3.1 p-Kenmotsu Manifolds satisfying $R(X,Y).R = 0$

In this section, we consider semisymmetric p-Kenmotsu manifolds, i.e., p-Kenmotsu manifolds satisfying the conditions $R(X,Y).R = 0$, $R(X,Y).S = 0$ and $P(X,Y).S = 0$ where $R(X,Y)$ is considered as a derivation of tensor algebra at each point of the manifold for tangent vectors X and Y. Now

$$\begin{aligned}(R(X,Y)\cdot R)(U,V)W &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &\quad - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W.\end{aligned} \tag{3.1}$$

Then using $R(X,Y).R = 0$, the eqn (3.1) can be written as

$$\begin{aligned}&g(R(X,Y)R(U,V)W,\xi) - g(R(R(X,Y)U,V)W,\xi) \\ &\quad - g(R(U,R(X,Y)V)W,\xi) - g(R(U,V)R(X,Y)W,\xi) = 0.\end{aligned} \tag{3.2}$$

Using (2.13), eqn (3.2) can be written as

$$\begin{aligned}&'R(U,V,W,X)\eta(Y) - {}'R(U,V,W,Y)\eta(X) + g(V,W)g(X,U)\eta(Y) \\ &\quad - g(U,W)g(X,V)\eta(Y) - g(V,W)g(Y,U)\eta(X) + g(U,W)g(Y,V)\eta(X) = 0.\end{aligned} \tag{3.3}$$

Putting $Y = \xi$ in (3.3) and on using (2.13), we have

$$\begin{aligned}&'R(U,V,W,X)-g(U,W)\eta(V)\eta(X)+g(V,W)\eta(U)\eta(X)+g(V,W)g(X,U)\\&\quad -g(U,W)g(X,V)-g(V,W)\eta(U)\eta(X)+g(U,W)\eta(V)\eta(X)=0.\end{aligned} \tag{3.4}$$

For $X=\xi$, the above equation becomes

$$\begin{aligned}&g(U,W)\eta(V)-g(V,W)\eta(U)-g(U,W)\eta(V)\\&\quad +g(V,W)\eta(U)-g(V,W)\eta(U)+g(U,W)\eta(V)=0.\end{aligned} \tag{3.5}$$

or,

$$g(U,W)\eta(V)-g(V,W)\eta(U)=0. \tag{3.6}$$

which is nothing but $\eta(R(U,V,W))=0$ in view of (2.13). This shows that either $R(U,V,W)=0$, i.e., either the manifold is flat or ξ is normal to the curvature tensor. Hence we have the following.

Theorem 3.1: A semisymmetric p-Kenmotsu manifold is either flat or the structure vector ξ is normal.

Proof: The proof follows, immediately, from (3.6).

3.2 p-Kenmotsu Manifolds satisfying $R(X,Y).S=0$

Now, we suppose that p-Kenmotsu manifold is Ricci-symmetric, i.e., $R(X,Y).S=0$. Then we have

$$S(R(X,Y,U),W)+S(U,R(X,Y,W))=0. \tag{3.7}$$

Putting $U=\xi$ in the above expression and using (2.15), we get

$$S[\eta(X)Y-\eta(Y)X,W)]+S(\xi,R(X,Y,W))=0. \tag{3.8}$$

Again on using (2.12), we get

$$\eta(X)S(Y,W)-\eta(Y)S(X,W)-(n-1)\eta(R(X,Y,W))=0. \tag{3.9}$$

Using eqn (2.13), we get

$$\eta(X)S(Y,W)-\eta(Y)S(X,W)-(n-1)g(X,W)\eta(Y)+(n-1)g(Y,W)\eta(X)=0. \tag{3.10}$$

Now, putting X = ξ in the above equation and on using (2.17), we get

$$S(Y,W)+(n-1)\eta(Y)\eta(W)-(n-1)\eta(W)\eta(Y)+(n-1)g(Y,W)=0. \tag{3.11}$$

or,

(a) S(Y, W) = - (n-1)g(Y, W), which on contraction gives the scalar curvature constant as

(b) r = - n (n-1).

On using (3.6) in (3.9), we get

(c) S(Y, W) = - (r/n) g(Y, W).

Thus, we have

Theorem 3.2: A p-Kenmotsu manifold, being Ricci symmetric, is an Einstein manifold and hence it is an sp-Kenmotsu manifold.

Proof: The proof of the theorem is an immediate consequence of the equations (3.11) (a), (3.11) (b) and (3.11) (c), because a p-Kenmotsu manifold of constant scalar curvature is an sp-Kenmotsu manifold.

3.3 p-Kenmotsu Manifold satisfying $P(X,Y).S = 0$

Now, we consider p-Kenmotsu manifolds satisfying the the condition $P(X,Y).S = 0$, where P(X, Y) denotes the Weyl projective curvature tensor [11] defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY]. \tag{3.12}$$

Now, we have

$$(P \cdot S)(Z,U,X,Y) = -S(P(X,Y,Z),U) - S(Z,P(X,Y,U) = 0 \tag{3.13}$$

Using (3.13)and (2.12), we get

$$'P(X,Y,Z,KU) + 'P(X,Y,U,KZ) = 0 \tag{3.14}$$

where $'R(X,Y,Z,U) = g(R(X,Y)Z,U)$.

Then, in view of (3.12), we have

$$\begin{aligned}(a)\quad & 'R(X,Y,Z,KU) + 'R(X,Y,U,KZ) + \frac{1}{n-1}[g(X,Z)S(Y,QU) \\ & - g(Y,Z)S(X,QU) + g(X,U)S(Y,QZ) - g(Y,U)S(X,QZ)] = 0.\end{aligned}$$

(or) (3.15)

$$\begin{aligned}(b)\quad & 'R(Z,KU,X,Y) + 'R(U,KZ,X,Y) + \frac{1}{n-1}[g(X,Z)S(Y,QU) \\ & - g(Y,Z)S(X,QU) + g(X,U)S(Y,QZ) - g(Y,U)S(X,QZ)] = 0.\end{aligned}$$

Now putting Y = ξ and using (2.13),(2.12) and (2.3), we get

$$\begin{aligned} & g(Z,X)\eta(QU) - g(QU,X)\eta(Z) + g(U,X)\eta(QZ) \\ & - g(QZ,X)\eta(U) + \frac{1}{n-1}[-(n-1)g(X,Z)\eta(QU) \\ & - \eta(Z)S(X,QU) - (n-1)g(X,U)\eta(QZ) - \eta(U)S(X,QZ)] = 0.\end{aligned} \tag{3.16}$$

On putting X = ξ in (3.16) and on using (2.12), we have

$$\begin{aligned} & \eta(Z)\eta(QU) + (n-1)\eta(U)\eta(Z) - (n-1)\eta(Z)\eta(U) \\ & + (n-1)\eta(Z)\eta(U) + \eta(Z)\eta(QU) - \eta(U)\eta(QZ) + \eta(QZ)\eta(U) = 0\end{aligned} \tag{3.17}$$

which on simplication implies $\eta(U)\,\eta(Z) = 0$ for $n > 1$, shows that ξ is normal to the tangent space of M_n.

Also, for $Y = \xi$ in (3.15) (b), we have

$$S(U,X)\eta(Z) + S(Z,X)\eta(U) + \frac{1}{n-1}[S(X,QU)\eta(Z) + S(X,QZ)\eta(U)] = 0. \tag{3.18}$$

For $Z = \xi$, we have

$$S(U,X)-(n-1)\eta(X)\eta(U)+\frac{1}{n-1}[S(X,QU)+g(QX,Q\xi)\eta(U)]=0. \quad (3.19)$$

which on using eqn (2.12), reduces to

$$\begin{aligned} &S(X,QU)=-(n-1)S(U,X) \\ &(or)\ Q^2X=-(n-1)QX \end{aligned} \quad (3.20)$$

Hence we have the following result.

Theorem 3.3: In a p -Kenmotsu manifold if the condition $(P\cdot S)(Z,U,X,Y)=0$ is satisfied then we have,
(i) either the structure tensor ξ is normal to T(M_n), or

(ii) $Q(QX)=Q^2X=-(n-1)QX$, for $n>1$.

Proof: The proof of the theorem is obvious in view of the results obtained in (3.17) and (3.20).

CONCLUSION

In this paper, we had studied three new properties of para-Kenmotsu manifolds, satisfying the conditions $R(X,Y).R=0$, $R(X,Y).S=0$ and $P(X,Y).S=0$. The results obtained here are similar to the findings reported earlier for para-Sasakian manifolds.

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