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Advances in Applied Science Research, 2016, 7(5): 46-53

ISSN: 0976-8610
CODEN (USA): AASRFC

On common fixed point theorems for weakly compatible mappings in Menger space

Arihant Jain[1] and Basant Chaudhary[2]

[1]*Department of Applied Mathematics, Shri Guru Sandipani Institute of Technology and Science, Ujjain (M.P.) 456550, India*
[2]*Research Scholar, Department of Mathematics, Mewar University, Chittorgarh (Raj.), India*

ABSTRACT

In this paper, the concept of weak compatibility in Menger space has been applied to prove a common fixed point theorem for six self maps. Our result generalizes and extends the result of Pathak and Verma [8].

Keywords: *Probabilistic metric space, Menger space, common fixed point, compatible maps, weak compatibility.*
AMS Subject Classification: *Primary 47H10, Secondary 54H25.*

INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [6]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [9] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [10] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theory in Menger space.

Recently, Jungck and Rhoades [5] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [11] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [4] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [7]. In the sequel, Pathak and Verma [8] proved a common fixed point theorem in Menger space using compatibility and weak compatibility. Using the concept of compatible mappings of type (A), Jain et. al. [1,2] proved some interesting fixed point theorems in Menger space. Afterwards, Jain et. al. [3] proved the fixed point theorem using the concept of weak compatible maps in Menger space.

In this paper a fixed point theorem for six self maps has been proved using the concept of semi-compatible maps and occasionally weak compatibility which turns out be a material generalization of the result of Pathak and Verma [8].

2. Preliminaries.

Definition 2.1. A mapping $\mathcal{F} : R \rightarrow R^+$ is called a *distribution* if it is non-decreasing left continuous with $\inf \{ F(t) \mid t \in R \} = 0$ and $\sup \{ F(t) \mid t \in R\} = 1$.
We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by $H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$.

Definition 2.2. [7] A mapping $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *t-norm* if it satisfies the following conditions :
(t-1) $t(a, 1) = a$, $t(0, 0) = 0$;

(t-2) $t(a, b) = t(b, a)$;
(t-3) $t(c, d) \geq t(a, b)$; for $c \geq a, d \geq b$,
(t-4) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. [7] A *probabilistic metric space (PM-space)* is an ordered pair (X, F) consisting of a non empty set X and a function $\mathcal{F}: X \times X \to L$, where L is the collection of all distribution functions and the value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$, for all $u,v,w \in X$ and $x, y > 0$.

Definition 2.4. [7] A *Menger space* is a triplet $(X, \mathcal{F}, t)$ where $(X, \mathcal{F})$ is a PM-space and t is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y) \geq t\{F_{u,v}(x), F_{v,w}(y)\}$, for all $u, v, w \in X$, $x, y \geq 0$.

Definition 2.5. [7] A sequence $\{x_n\}$ in a Menger space $(X, \mathcal{F}, t)$ is said to be *convergent* and *converges to a point* x in X if and only if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$F_{x_n, x}(\varepsilon) > 1 - \lambda$ for all $n \geq M(\varepsilon, \lambda)$.

Further the sequence $\{x_n\}$ is said to be *Cauchy sequence* if for $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$F_{x_n, x_m}(\varepsilon) > 1 - \lambda$ for all $m, n \geq M(\varepsilon, \lambda)$.

A Menger PM-space (X, F, t) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

A complete metric space can be treated as a complete Menger space in the following way:

Proposition 2.1. [7] If (X, d) is a metric space then the metric d induces mappings

$\mathcal{F}: X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$, where
$H(k) = 0$, for $k \leq 0$ and $H(k) = 1$, for $k > 0$.

Further if, $t : [0,1] \times [0,1] \to [0,1]$ is defined by $t(a, b) = \min\{a, b\}$. Then $(X, \mathcal{F}, t)$ is a Menger space. It is complete if (X, d) is complete.

The space $(X, \mathcal{F}, t)$ so obtained is called the *induced Menger space.*

Definition 2.6. [8] Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for $x \in X$ implies $ASx = SAx$.

Definition 2.7. [8] Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be *compatible* if $F_{ASx_n, SAx_n}(x) \to 1$ for all $x>0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X, as $n \to \infty$.

Remark 2.1. [8] The concept of weakly compatible mappings is more general than that of compatible mappings.

Lemma 2.1. [8] Let (X, F, *) be a Menger space with t-norm * such that the family $\{*_n(x)\}_{n\in N}$ is equicontinuous at $x = 1$ and let E denote the family of all functions

$\phi : R^+ \to R^+$ such that ϕ is non-decreasing with $\lim_{n\to\infty} \phi^n(t) = +\infty$, $\forall\, t > 0$. If $\{y_n\}_{n\in N}$ is a sequence in X satisfying the condition

$F_{y_n, y_{n+1}}(t) \geq F_{y_{n-1}, y_n}(\phi(t))$,

for all $t > 0$ and $\alpha \in [-1, 0]$, then $\{y_n\}_{n\in N}$ is a Cauchy sequence in X.

Proposition 2.2. Let $\{x_n\}$ be a Cauchy sequence in a Menger space $(X, \mathcal{F}, t)$ with continuous t-norm t. If the subsequence $\{x_{2n}\}$ converges to x in X, then $\{x_n\}$ also converges to x.

Proof. As $\{x_{2n}\}$ converges to x, we have

$$F_{x_n,x}(\varepsilon) \geq t\left(F_{x_n,x_{2n}}\left(\frac{\varepsilon}{2}\right), F_{x_{2n},x}\left(\frac{\varepsilon}{2}\right)\right).$$

Then

$\lim_{n\to\infty} F_{x_n,x}(\varepsilon) \geq t(1,1),$ which gives $\lim_{n\to\infty} F_{x_n,x}(\varepsilon) = 1,$ $\forall\, \varepsilon > 0$ and the result follows.

RESULTS

Theorem 3.1. Let A, B, S, T, P and Q be self mappings on a Menger space $(X, \mathcal{F}, *)$ with continuous t-norm * satisfying :

(3.1.1) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$;

(3.1.2) AB = BA, ST = TS, PB = BP, QT = TQ;

(3.1.3) One of ST(X), Q(X), AB(X) or P(X) is complete;

(3.1.4) The pairs (P, AB) and (Q, ST) are weak compatible;

(3.1.5) $[1 + \alpha F_{ABx, STy}(t)] * F_{Px, Qy}(t)$

$\geq \alpha \min\{F_{Px, ABx}(t) * F_{Qy, STy}(t), F_{Px, STy}(2t) * \quad F_{Qy, ABx}(2t)\}$

$+ F_{ABx, STy}(\phi(t)) * F_{Px,ABx}(\phi(t)) * F_{Qy, STy}(\phi(t)) * F_{Px, STy}(2\phi(t))$

$* F_{Qy, ABx}(2\phi(t))$

for all $x, y \in X$, $t > 0$ and $\phi \in E$.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Suppose $x_0 \in X$. From condition (3.1.1) $\exists\ x_1, x_2 \in X$ such that
$Px_0 = STx_1$ and $Qx_1 = ABx_2$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$y_{2n} = Px_{2n} = STx_{2n+1}$ and $y_{2n+1} = Qx_{2n+1} = ABx_{2n+2}$
for n = 0, 1, 2, ...

Step I. Let us show that $F_{y_{n+2}, y_{n+1}}(t) \geq F_{y_{n+1}, y_n}(\phi(t))$.

For, putting x_{2n+2} for x and x_{2n+1} for y in (3.1.5) and then on simplification, we have

$[1 + \alpha F_{ABx_{2n+2}, STx_{2n+1}}(t)] * F_{Px_{2n+2}, Qx_{2n+1}}(t)$

$\geq \alpha \min\{F_{Px_{2n+2}, ABx_{2n+2}}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{Px_{2n+2}, STx_{2n+1}}(2t)$

$F_{Qx_{2n+1}, ABx_{2n+2}}(2t)\}$

$+ F_{ABx_{2n+2}, STx_{2n+1}}(\phi(t)) * F_{Px_{2n+2}, ABx_{2n+2}}(\phi(t)) * F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t))$

$* F_{Px_{2n+2}, STx_{2n+1}}(2\phi(t)) * F_{Qx_{2n+1}, ABx_{2n+2}}(2\phi(t))$

$[1 + \alpha F_{y_{2n+1}, y_{2n}}(t)] * F_{y_{2n+2}, y_{2n+1}}(t)$

$\geq \alpha \min\{F_{y_{2n+2}, y_{2n+1}}(t) * F_{y_{2n+1}, y_{2n}}(t), F_{y_{2n+2}, y_{2n}}(2t) * F_{y_{2n+1}, y_{2n+1}}(2t)\}$

$+ F_{y_{2n+1}, y_{2n}}(\phi(t))$

$* F_{y_{2n+2}, y_{2n+1}}(\phi(t)) * F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n}}(2\phi(t)) * F_{y_{2n+1}, y_{2n+1}}(2\phi(t))$

$F_{y_{2n+2}, y_{2n+1}}(t) + \alpha F_{y_{2n+1}, y_{2n}}(t) * F_{y_{2n+2}, y_{2n+1}}(t)$

$\geq \alpha \min\{F_{y_{2n+2}, y_{2n}}(2t), F_{y_{2n+2}, y_{2n}}(2t)\} + F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(\phi(t))$
$* F_{y_{2n+2}, y_{2n}}(2\phi(t)) * 1$
$F_{y_{2n+2}, y_{2n+1}}(t) + \alpha F_{y_{2n+1}, y_{2n}}(t) * F_{y_{2n+2}, y_{2n+1}}(t)$
$\geq \alpha F_{y_{2n+2}, y_{2n}}(2t) + F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(2\phi(t))$
$F_{y_{2n+2}, y_{2n+1}}(t) + \alpha F_{y_{2n+2}, y_{2n}}(2t)$
$\geq \alpha F_{y_{2n+2}, y_{2n}}(2t) + F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(\phi(t)) * F_{y_{2n+1}, y_{2n}}(\phi(t))$
$F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(\phi(t))$
or, $F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+1}, y_{2n+2}}(\phi(t)) * F_{y_{2n}, y_{2n+1}}(\phi(t))$
or, $F_{y_{2n+2}, y_{2n+1}}(t) \geq \min\{F_{y_{2n+1}, y_{2n+2}}(\phi(t)), F_{y_{2n}, y_{2n+1}}(\phi(t))\}$.

If $F_{y_{2n+1}, y_{2n+2}}(\phi(t))$ is chosen 'min' then we obtain
$F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+2}, y_{2n+1}}(\phi(t)), \ \forall \ t > 0$
a contradiction as $\phi(t)$ is non-decreasing function.

Thus,
$F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+1}, y_{2n}}(\phi(t)), \ \forall \ t > 0.$

Similarly, by putting x_{2n+2} for x and x_{2n+3} for y in (3.1.5), we have
$F_{y_{2n+3}, y_{2n+2}}(t) \geq F_{y_{2n+2}, y_{2n+1}}(\phi(t)), \ \forall \ t > 0.$

Using these two, we obtain
$F_{y_{n+2}, y_{n+1}}(t) \geq F_{y_{n+1}, y_n}(\phi(t)), \ \forall \ n = 0, 1, 2, \ldots, t > 0.$

Therefore, by lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X.

Case I. ST(X) is complete. In this case $\{y_{2n}\} = \{STx_{2n+1}\}$ is a Cauchy sequence in ST(X), which is complete. Thus $\{y_{2n+1}\}$ converges to some $z \in ST(X)$. By proposition 2.2, we have

$$\{Qx_{2n+1}\} \to z \text{ and } \{STx_{2n+1}\} \to z, \tag{3.1.6}$$

$$\{Px_{2n}\} \to z \text{ and } \{ABx_{2n}\} \to z. \tag{3.1.7}$$

As $z \in ST(X)$ there exists $u \in X$ such that $z = STu$.

Step I. Put $x = x_{2n}$ and $y = u$ in (3.1.5), we get
$[1 + \alpha F_{ABx_{2n}, STu}(t)] * F_{Px_{2n}, Qu}(t)$
$\geq \alpha \min\{F_{Px_{2n}, ABx_{2n}}(t) * F_{Qu, STu}(t), F_{Px_{2n}, STu}(2t) * F_{Qu, ABx_{2n}}(2t)\}$
$+ F_{ABx_{2n}, STu}(\phi(t)) * F_{Px_{2n}, ABx_{2n}}(\phi(t)) * F_{Qu, STu}(\phi(t)) * F_{Px_{2n}, STu}(2\phi(t))$
$* F_{Qu, ABx_{2n}}(2\phi(t))$.

Letting $n \to \infty$ and using (3.1.6), (3.1.7), we get
$[1 + \alpha F_{z, z}(t)] * F_{z, Qu}(t)$
$\geq \alpha \min\{F_{z, z}(t) * F_{Qu, z}(t), F_{z, z}(2t) * F_{Qu, z}(2t)\}$
$+ F_{z, z}(\phi(t)) * F_{z, z}(\phi(t))$
$* F_{Qu, z}(\phi(t)) * F_{z, z}(2\phi(t)) * F_{Qu, z}(2\phi(t))$

$F_{z, Qu}(t) + \alpha F_{z, Qu}(t) \geq \alpha \min\{F_{Qu, z}(t), F_{Qu, z}(2t)\} + F_{Qu, z}(\phi(t)) * F_{Qu, z}(2\phi(t))$
$F_{Qu, z}(t) + \alpha F_{Qu, z}(t) \geq \alpha \min\{F_{Qu, z}(t), F_{Qu, z}(t) * F_{z, z}(t)\} + F_{Qu, z}(\phi(t)) * F_{Qu, z}(\phi(t))$
$* F_{z, z}(\phi(t))$
$F_{Qu, z}(t) + \alpha F_{Qu, z}(t) \geq \alpha F_{Qu, z}(t) + F_{Qu, z}(\phi(t))$
$F_{Qu, z}(t) \geq F_{Qu, z}(\phi(t))$

which is a contradiction by lemma (2.1) and we get
Qu = z and so Qu = z = STu.

Since (Q, ST) is weakly compatible, we have
STz = Qz.

Step III. Put $x = x_{2n}$ and $y = Tz$ in (3.1.5), we have
$[1 + \alpha F_{ABx_{2n}, STTz}(t)] * F_{Px_{2n}, QTz}(t)$
$\geq \alpha \min\{F_{Px_{2n}, ABx_{2n}}(t) * F_{QTz, STTz}(t), F_{Px_{2n}, STTz}(2t) * F_{QTz, ABx_{2n}}(2t)\}$
$+ F_{ABx_{2n}, STTz}(\phi(t)) * F_{Px_{2n}, ABx_{2n}}(\phi(t)) * F_{QTz, STTz}(\phi(t))$
$* F_{Px_{2n}, STTz}(2\phi(t)) * F_{QTz, ABx_{2n}}(2\phi(t))$.
As QT = TQ and ST = TS, we have
QTz = TQz = Tz and ST(Tz) = T(STz) = Tz.
Letting $n \to \infty$, we get
$[1 + \alpha F_{z, Tz}(t)] * F_{z, Tz}(t) \geq \alpha \min\{F_{z, z}(t) * F_{Tz, Tz}(t), F_{z, Tz}(2t) * F_{Tz, z}(2t)\}$
$+ F_{z, Tz}(\phi(t)) * F_{z, z}(\phi(t))$
$* F_{Tz, Tz}(\phi(t)) * F_{z, Tz}(2\phi(t)) * F_{Tz, z}(2\phi(t))$
$F_{z, Tz}(t) + \alpha\{F_{z, Tz}(t) * F_{z, Tz}(t)\} \geq \alpha \min\{1 * F_{Tz, z}(2t)\} + F_{z, Tz}(\phi(t))$
$* 1 * 1 * F_{Tz, z}(2\phi(t))$
$F_{Tz, z}(t) + \alpha F_{Tz, z}(t) \geq \alpha F_{Tz, z}(2t) + F_{Tz, z}(\phi(t)) * F_{Tz, z}(2\phi(t))$
$F_{Tz, z}(t) + \alpha F_{Tz, z}(t) \geq \alpha \{F_{Tz, z}(t) * F_{z, z}(t)\} + F_{Tz, z}(\phi(t))$
$*F_{Tz, z}(\phi(t))*F_{z, z}(\phi(t))$
$F_{Tz, z}(t) + \alpha F_{Tz, z}(t) \geq \alpha F_{Tz, z}(t) + F_{Tz, z}(\phi(t))$
$F_{Tz, z}(t) \geq F_{Tz, z}(\phi(t))$

which is a contradiction and we get Tz = z.

Now, STz = Tz = z implies Sz = z.

Hence, Sz = Tz = Qz = z.

Step IV. As $Q(X) \subseteq AB(X)$, there exists $w \in X$ such that
z = Qz = ABw.

Put x = w and $y = x_{2n+1}$ in (3.1.5), we get
$[1 + \alpha F_{ABw, STx_{2n+1}}(t)] * F_{Pw, Qx_{2n+1}}(t)$
$\geq \alpha \min\{F_{Pw, ABw}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{Pw, STx_{2n+1}}(2t)$
$* F_{Qx_{2n+1}, ABw}(2t)\} + F_{ABw, STx_{2n+1}}(\phi(t)) * F_{Pw, ABw}(\phi(t))$
$* F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t)) * F_{Pw, STx_{2n+1}}(2\phi(t)) * F_{Qx_{2n+1}, ABw}(2\phi(t))$.

Letting $n \to \infty$, we get

$[1 + \alpha F_{z, z}(t)] * F_{Pw, z}(t) \geq \alpha \min\{F_{Pw, z}(t) * F_{z, z}(t), F_{Pw, z}(2t) * F_{z, z}(2t)\}$
$+ F_{z, z}(\phi(t)) * F_{Pw, z}(\phi(t))$
$* F_{z, z}(\phi(t)) * F_{Pw, z}(2\phi(t)) * F_{z, z}(2\phi(t))$
$F_{Pw, z}(t) + \alpha F_{Pw, z}(t) \geq \alpha \min\{F_{Pw, z}(t), F_{Pw, z}(2t)\} + F_{Pw, z}(\phi(t)) * F_{Pw, z}(2\phi(t))$
$F_{Pw, z}(t) + \alpha F_{Pw, z}(t) \geq \alpha \min\{F_{Pw, z}(t), F_{Pw, z}(t) * F_{z, z}(t)\} + F_{Pw, z}(\phi(t))$
$* F_{z, z}(\phi(t))$
$F_{Pw, z}(t) + \alpha F_{Pw, z}(t) \geq \alpha \min\{F_{Pw, z}(t), F_{Pw, z}(t)\} + F_{Pw, z}(\phi(t))$
$F_{Pw, z}(t) + \alpha F_{Pw, z}(t) \geq \alpha F_{Pw, z}(t)\} + F_{Pw, z}(\phi(t))$
$F_{Pw, z}(t) \geq F_{Pw, z}(\phi(t))$

which is a contradiction and hence, we get $Pw = z$.

Hence, $Pz = z = ABz$.

Step V. Put $x = z$ and $y = x_{2n+1}$ in (3.1.5), we have
$[1 + \alpha F_{ABz, STx_{2n+1}}(t)] * F_{Pz, Qx_{2n+1}}(t)$
$\geq \alpha \min\{F_{Pz, ABz}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{Pz, STx_{2n+1}}(2t) * F_{Qx_{2n+1}, ABz}(2t)\}$
$+ F_{ABz, STx_{2n+1}}(\phi(t)) * F_{Pz, ABz}(\phi(t)) * F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t)) * F_{Pz, STx_{2n+1}}(2\phi(t))$
$* F_{Qx_{2n+1}, ABz}(2\phi(t))$.

Letting $n \to \infty$, we get
$[1 + \alpha F_{Pz, z}(t)] * F_{Pz, z}(t)$
$\geq \alpha \min\{F_{Pz, Pz}(t) * F_{z, z}(t), F_{Pz, z}(2t) * F_{z, Pz}(2t)\} + F_{Pz, z}(\phi(t)) * F_{Pz, Pz}(\phi(t))$
$* F_{z, z}(\phi(t)) * F_{Pz, z}(2\phi(t)) * F_{z, Pz}(2\phi(t))$
$F_{Pz, z}(t) + \alpha\{F_{Pz, z}(t) * F_{Pz, z}(t)\}$
$\geq \alpha \min\{1 * 1, F_{Pz, z}(2t) * F_{Pz, z}(2t)\} + F_{Pz, z}(\phi(t)) * 1 * 1 * F_{Pz, z}(2\phi(t))$
$* F_{z, Pz}(2\phi(t))$
$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha \min\{1, F_{Pz, z}(2t)\} + F_{Pz, z}(\phi(t)) * F_{Pz, z}(2\phi(t))$
$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha F_{Pz, z}(2t) + F_{Pz, z}(\phi(t)) * F_{Pz, z}(2\phi(t))$
$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha\{F_{Pz, z}(t) * F_{z, z}(t)\} + F_{Pz, z}(\phi(t)) * F_{Pz, z}(\phi(t)) * F_{z, z}(\phi(t))$
$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha\{F_{Pz, z}(t) * 1\} + F_{Pz, z}(\phi(t)) * 1$
$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha F_{Pz, z}(t) + F_{Pz, z}(\phi(t))$
$F_{Pz, z}(t) \geq F_{Pz, z}(\phi(t))$

which is a contradiction and hence, $Pz = z$

and so $z = Pz = ABz$.

Step VI. Put $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we get
$[1 + \alpha F_{ABBz, STx_{2n+1}}(t)] * F_{PBz, Qx_{2n+1}}(t)$
$\geq \alpha \min\{F_{PBz, ABBz}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{PBz, STx_{2n+1}}(2t)$
$* F_{Qx_{2n+1}, ABBz}(2t)\} + F_{ABBz, STx_{2n+1}}(\phi(t)) * F_{PBz, ABBz}(\phi(t))$
$* F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t)) * F_{PBz, STx_{2n+1}}(2\phi(t)) * F_{Qx_{2n+1}, ABBz}(2\phi(t))$.

As BP = PB, AB = BA so we have
P(Bz) = B(Pz) = Bz and AB(Bz) = B(AB)z = Bz.

Letting $n \to \infty$ and using (3.1.6), we get

$[1 + \alpha F_{Bz,z}(t)] * F_{Bz,z}(t)$
$\geq \alpha \min\{F_{Bz,Bz}(t) * F_{z,z}(t), F_{Bz,z}(2t) * F_{z,Bz}(2t)\}$
$+ F_{Bz,z}(\phi(t)) * F_{Bz,Bz}(\phi(t)) * F_{z,z}(\phi(t)) * F_{Bz,z}(2\phi(t)) * F_{z,Bz}(2\phi(t))$
$F_{Bz,z}(t) + \alpha\{F_{Bz,z}(t) * F_{Bz,z}(t)\}$
$\geq \alpha \min\{1 * 1, F_{Bz,z}(2t)\} + F_{Bz,z}(\phi(t)) * 1 * 1 * F_{Bz,z}(2\phi(t))$
$F_{Bz,z}(t) + \alpha F_{Bz,z}(t) \geq \alpha F_{Bz,z}(2t) + F_{Bz,z}(\phi(t)) * F_{Bz,z}(2\phi(t))$
$F_{Bz,z}(t) + \alpha F_{Bz,z}(t) \geq \alpha \{F_{Bz,z}(t) * F_{z,z}(t)\} + F_{Bz,z}(\phi(t)) * F_{Bz,z}(\phi(t)) * F_{z,z}(\phi(t))$
$F_{Bz,z}(t) + \alpha F_{Bz,z}(t) \geq \alpha \{F_{Bz,z}(t) * 1\} + F_{Bz,z}(\phi(t)) * 1$
$F_{Bz,z}(t) + \alpha F_{Bz,z}(t) \geq \alpha F_{Bz,z}(t) + F_{Bz,z}(\phi(t))$
$F_{Bz,z}(t) \geq F_{Bz,z}(\phi(t))$

which is a contradiction and we get Bz = z and so
z = ABz = Az.

Therefore, Pz = Az = Bz = z.

Combining the results from different steps, we get
Az = Bz = Pz = Qz = Tz = Sz = z.

Hence, the six self maps have a common fixed point in this case.

Case when P(X) is complete follows from above case as $P(X) \subseteq ST(X)$.

Case II. AB(X) is complete. This case follows by symmetry. As $Q(X) \subseteq AB(X)$, therefore the result also holds when Q(X) is complete.

Uniqueness :
Let z_1 be another common fixed point of A, B, P, Q, S and T. Then
$Az_1 = Bz_1 = Pz_1 = Sz_1 = Tz_1 = Qz_1 = z_1$, assuming $z \neq z_1$.

Put x = z and y = z_1 in (3.1.5), we get

$[1 + \alpha F_{ABz,STz_1}(t)] * F_{Pz,Qz_1}(t)$
$\geq \alpha \min\{F_{Pz,ABz}(t) * F_{Qz_1,STz_1}(t), F_{Pz,STz_1}(2t) * F_{Qz_1,ABz}(2t)\}$
$+ F_{ABz,STz_1}(\phi(t)) * F_{Pz,ABz}(\phi(t)) * F_{Qz_1,STz_1}(\phi(t)) * F_{Pz,STz_1}(2\phi(t))$
$* F_{Qz_1,ABz}(2\phi(t))$
$[1 + \alpha F_{z,z_1}(t)] * F_{z,z_1}(t)$
$\geq \alpha \min\{F_{z,z}(t) * F_{z_1,z_1}(t), F_{z,z_1}(2t) * F_{z_1,z}(2t)\} + F_{z,z_1}(\phi(t)) * F_{z,z}(\phi(t))$
$* F_{z_1,z_1}(\phi(t)) * F_{z,z_1}(2\phi(t)) * F_{z_1,z}(2\phi(t))$
$F_{z,z_1}(t) + \alpha\{F_{z,z_1}(t) * F_{z,z_1}(t)\} \geq \alpha \min\{1, F_{z,z_1}(2t)\} + F_{z,z_1}(\phi(t)) * F_{z,z_1}(2\phi(t))$
$F_{z,z_1}(t) + \alpha F_{z,z_1}(t) \geq \alpha F_{z,z_1}(2t)\} + F_{z,z_1}(\phi(t)) * F_{z,z_1}(\phi(t)) * F_{z,z}(\phi(t))$
$F_{z_1,z}(t) + \alpha F_{z_1,z}(t) \geq \alpha\{F_{z_1,z}(t) * F_{z,z}(t)\} + F_{z_1,z}(\phi(t)) * 1$
$F_{z_1,z}(t) + \alpha F_{z_1,z}(t) \geq \alpha F_{z_1,z}(t) + F_{z_1,z}(\phi(t))$

$F_{z_1, z}(t) \geq F_{z_1, z}(\phi(t))$

which is a contradiction.

Hence $z = z_1$ and so z is the unique common fixed point of A, B, S, T, P and Q.

This completes the proof.

Remark 3.1. If we take B = T = I, the identity map on X in theorem 3.1, then condition (3.1.2) is satisfied trivially and we get

Corollary 3.1. Let A, S, P and Q be self mappings on a Menger space (X, F, *) with continuous t-norm * satisfying :

(i) $P(X) \subseteq T(X)$, $Q(X) \subseteq A(X)$;
(ii) One of S(X), Q(X), A(X) or P(X) is complete;
(iii) The pairs (P, A) and (Q, S) are weak compatible;
(iv) $[1 + \alpha F_{Ax, Sy}(t)] * F_{Px, Qy}(t)$
$\geq \alpha \min\{F_{Px, Ax}(t) * F_{Qy, Sy}(t), F_{Px, Sy}(2t) * F_{Qy, Ax}(2t)\}$
$+ F_{Ax, Sy}(\phi(t)) * F_{Px,Ax}(\phi(t)) * F_{Qy, Sy}(\phi(t)) * F_{Px, Sy}(2\phi(t))$
$* F_{Qy, Ax}(2\phi(t))$

for all $x, y \in X$, $t > 0$ and $\phi \in E$.

Then A, S, P and Q have a unique common fixed point in X.

Remark 3.2. In view of remark 3.1, corollary 3.1 is a generalization of the result of Pathak and Verma [8] in the sense that condition of compatibility of the first pair of self maps has been restricted to weak compatibility and we have dropped the condition of continuity in a Menger space with continuous t-norm.

REFERENCES

[1]. Jain, Arihant and Singh, Bijendra, Common fixed point theorem in Menger space through compatible maps of type (A), Chh. J. Sci. Tech. 2 (2005), 1-12.
[2]. Jain, Arihant and Singh, Bijendra, A fixed point theorem in Menger space through compatible maps of type (A), V.J.M.S. 5(2), (2005), 555-568.
[3]. Jain, Arihant and Singh, Bijendra, Common fixed point theorem in Menger Spaces, The Aligarh Bull. of Math. 25 (1), (2006), 23-31.
[4]. Jungck, G., Compatible mappings and common fixed points, Internat. J. Math. and Math. Sci. 9(4), (1986), 771-779.
[5]. Jungck, G. and Rhoades, B.E., Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 29(1998), 227-238.
[6]. Menger, K., Statistical metrics, Proc. Nat. Acad. Sci. USA. 28(1942), 535 -537.
[7]. Mishra, S.N., Common fixed points of compatible mappings in PM-spaces, Math. Japon. 36(2), (1991), 283-289.
[8]. Pathak, H.K. and Verma, R.K., Common fixed point theorems for weakly compatible mappings in Menger space and application, Int. Journal of Math. Analysis, Vol. 3, 2009, No. 24, 1199-1206.
[9]. Schweizer, B. and Sklar, A., Statistical metric spaces, Pacific J. Math. 10 (1960), 313-334.
[10]. Sehgal, V.M. and Bharucha-Reid, A.T., Fixed points of contraction maps on probabilistic metric spaces, Math. System Theory 6(1972), 97- 102.
[11]. Sessa, S., On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. Beograd 32(46), (1982), 146-153.