Available online at www.pelagiaresearchlibrary.com

Pelagia Research Library

Advances in Applied Science Research, 2013, 4(5):173-183

ISSN: 0976-8610
CODEN (USA): AASRFC

Numerical solution of Laplace's equation in a cracked polygon

Ouigou M. Zongo[1], Sie Kam[2], Kounhinir Some[1] and Alioune Ouedraogo[3]

[1]*Université de Ouagadougou, BP 7021, Ouagadougou 03, Burkina Faso. Département de Mathématiques, UFR-SEA, Laboratoire d'Analyse Numérique, d'Informatique et de Biomathématique (LANIBIO)*
[2]*Université de Ouagadougou, Département de Physique, UFR-SEA, Laboratoire d'Energies Thermiques Renouvelables (LETRE)*
[3]*Université de Ouagadougou, Département de Physique, UFR-SEA, Laboratoire de Chimie Organique et de Physique Appliquée (LCOPA)*

ABSTRACT

This study reports the numerical solution of Laplace's equation in the first membrane of Giraud, using large singular finite elements method. We compare these results with those obtained using the conventional method of finite elements. Both methods provide results that align quite well everywhere except near the singularities where significant differences exist. The results deviate near the singularity and we obtain the classical Gibbs phenomenon. The comparisons are based on u solution values, those of its first derivatives and Laplacian's.

Keywords: large elements, finite elements, singularities, cracks.

INTRODUCTION

The Laplace equation and more generally Poisson's equation is used in several problems in engineering, physics and other disciplines. This equation appears in electromagnetism [1], fluid dynamics [2], stationary heat conduction [3], electrostatics [4] and in elasticity [5-6]. When the problem is singular at the vertices of a polygon, the digital processing of the Laplace equation is very difficult and usual methods of finite elements give unsatisfactory results when used in their standard form. These methods as demonstrated by various authors [7-12] can be significantly improved if they take the analytical form of the solution near the singularities into account. In addition, the resolution of the Dirichlet's problem with Laplace's equation with cracks is particularly difficult. The main difficulty comes from the singularities which are located at their ends. Indeed, at these points σ_i, the series corresponding to the solution of the Laplace's equation are $\sum_{k=1}^{\infty} a_{ik} r_i^{\frac{k}{2}} \sin \frac{k}{2}\theta_i$ and their first term which is proportional to $r_i^{\frac{1}{2}}$, presents derivatives that tend to infinity near the end of the crack [13]. Large singular finite elements method (LSFEM) was designed to overcome the "due to singularities" that gives very satisfactory results all over the study field while the finite elements method (FEM) gives good results only on areas located far from singularities. This demonstrates the power, efficiency and accuracy of this method for a number of coefficients that are less important than finite elements method.

MATERIALS AND METHODS

Let determine the stationary field temperature in a crack polygon which has the shape of a billiard or a Chapman-Giraud membrane. It is a domain that was built by Chapman [14] and studied by Giraud [15]. The domain consists of a rectangle whose sides of reduced length are respectively 4 and 2 amputated of a square with a reduced unit side.

In addition, it has an internal boundary formed by two perpendicular line segments of unit length. This internal boundary is the equivalent of a crack in the domain. We assume that the external boundary of the domain and its internal boundary are respectively maintained at different fixed temperatures 0 and 1 in reduced variables. We need to solve the Laplace's equation with discontinued conditions at the Dirichlet boundaries in a domain that has re-entrant angles of $3\pi/2$ and 2π. The problem has five real singularities σ. Two of them are combined geometrically and are due to temperature variations that go from 0 to 1 and from 1 to 0 if it runs through the boundary of the domain clockwise. Two other singularities exist because of the presence of one re-entrant angle of $3\pi/2$. The fifth is located at the end of the internal crack with a re-entrant angle whose value is 2π (Figure 1).

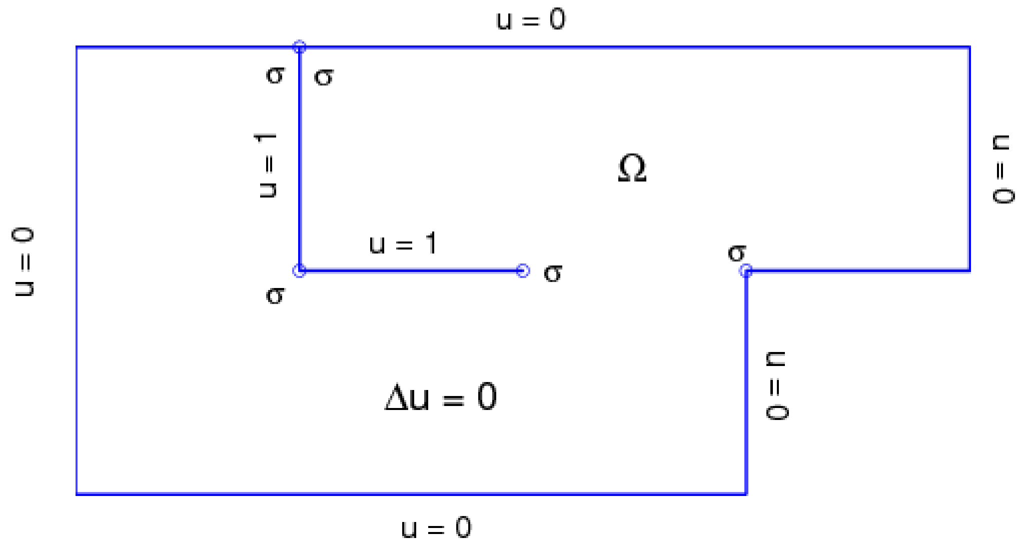


Figure-2: Field of Chapman-Giraud. Dirichlet boundary conditions

Large singular finite elements method (LSFEM) which is due to Tolley [11], comprises three steps:

Step 1: Decomposition of the domain

A division of the field that does not take into account the real singularities is irrelevant to get a good approximate solution of the Dirichlet problem. The entire domain must be covered by the union of disks of convergence associated with singularities. Naturally, we need to introduce pseudo additional singularities to obtain a partition of the computational domain into sixteen sub-domains separated in pairs by twenty-two sub-borders (Figure 2).

- Five identical rectangles: Ω_2 , Ω_3 , Ω_8, Ω_9 and Ω_{16}
- Five identical squares with no cracks Ω_1 , Ω_4 , Ω_6 , Ω_7 and Ω_{15}
- Three identical squares: Ω_{10} and Ω_{14} with a crack at one side and Ω_{11} which forms the crack on both sides of the right angle.
- A unit side square Ω_{12} with also a crack on half a median with an opening angle 2π at point L
- Two L-shaped domains : Ω_5 and Ω_{13} where the opening of the re-entrant angle is $3\pi/2$

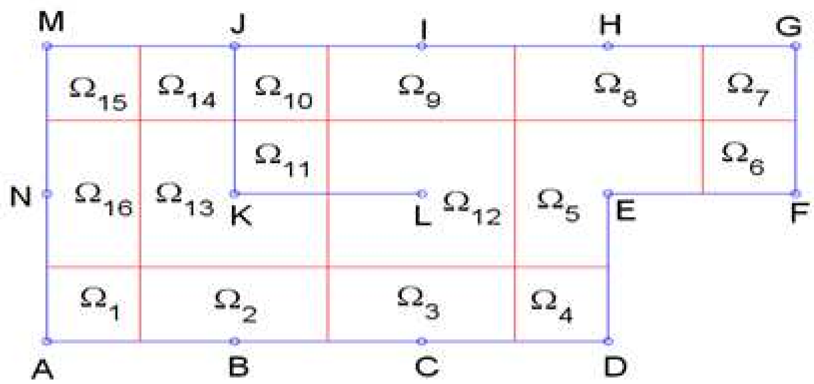


Figure-2 Division of the Chapman Giraud domain and singularities at J, K and L

Step 2: Resolution of auxiliary problems

The second step involves solving auxiliary problems. It raises as many problems as the existing sub-domains. To each sub-domain Ω_i is associated an origin P_i a singularity, an angle α_i which is the opening angle in P_i and a local system of polar coordinates (r_i, θ_i).

For the five identical rectangular sub-domains $\Omega_2, \Omega_3, \Omega_8$, Ω_9 and Ω_{16}, we solve the following identical auxiliary problems:

$$\Delta u(r_i, \theta_i) = 0 \qquad (r_i, \theta_i) \in \Omega_i \tag{1-a}$$

$$u(r_i, 0) = 0 \tag{1-b}$$

$$u_i(r_i, \pi) = 0 \tag{1-c}$$

with i taking respectively the values 2, 3, 8, 9 and 16.

And also in the five identical square sub-domains $\Omega_1, \Omega_4, \Omega_6$, Ω_7 and Ω_{15} with no cracks, we solve the following identical auxiliary problems :

$$\Delta u_j(r_j, \theta_j) = 0 \qquad (r_j, \theta_j) \in \Omega_j \tag{2-a}$$

$$u_j(r_j, 0) = 0 \tag{2-b}$$

$$u_j(r_j, \pi/2) = 0 \tag{2-c}$$

j is respectively 1, 4, 6, 7 and 15.

In the square sub-domain Ω_{10} with the crack on one side, we solve the Laplace's equation with conditions at the Dirichlet boundary:

$$\Delta u_{10}(r_{10}, \theta_{10}) = 0 \qquad (r_{10}, \theta_{10}) \in \Omega_{10} \tag{3-a}$$

$$u_{10}(r_{10}, 0) = 1 \tag{3-b}$$

$$u_{10}(r_{10}, \pi/2) = 0 \tag{3-c}$$

In the square sub-domain Ω_{11} with the crack on both sides of the right angle, we solve the Laplace's equation with conditions at the Dirichlet boundaries.

$$\Delta u_{11}(r_{11}, \theta_{11}) = 0 \qquad (r_{11,} \theta_{11}) \in \Omega_{11} \tag{4-a}$$

$$u_{11}(r_{11}, 0) = 1 \tag{4-b}$$

$$u_{11}(r_{11}, \pi/2) = 1 \tag{4-c}$$

In the square sub-domain Ω_{12} with also a crack on half a median with an opening angle 2π, we solve:

$$\Delta u_{12}(r_{12}, \theta_{12}) = 0 \qquad (r_{12}, \theta_{12}) \in \Omega_{12} \tag{5-a}$$

$$u_{12}(r_{12}, 0) = 1 \tag{5-b}$$

$$u_{12}(r_{12}, 2\pi) = 1 \tag{5-c}$$

In the square sub-domain Ω_{14} which has a crack on one side, we solve:

$$\Delta u_{14}(r_{14}, \theta_{14}) = 0 \qquad (r_{14}, \theta_{14}) \in \Omega_{14} \tag{6-a}$$

$$u_{14}(r_{14}, 0) = 0 \tag{6-b}$$

$$u_{14}(r_{14}, \pi/2) = 1 \tag{7-c}$$

Finally for the two L-shaped sub-domains Ω_5 and Ω_{13}, we respectively solve:

$$\Delta u_5(r_5,\theta_5)=0 \qquad (r_5,\theta_5)\in\Omega_5 \tag{7-a}$$
$$u_5(r_5,0)=0 \tag{7-b}$$
$$u_5(r_5,3\pi/2)=0 \tag{7-c}$$
$$\Delta u_{13}(r_{13},\theta_{13})=0 \quad (r_{13},\theta_{13})\in\Omega_{13} \tag{8-a}$$
$$u_{13}(r_{13},0)=1 \tag{8-b}$$
$$u_{13}(r_{13},3\pi/2)=1 \tag{8-c}$$

The problems raised in identical sub-domains with the same boundary conditions will have identical auxiliary solutions. The undetermined solutions of the sixteen auxiliary problems taking into account the boundary conditions can be written as follows [13]:

- In the case of square sub-domains bearing no cracks on their sides, we have:

$$u_i(r_i,\theta_i)=\sum_{n=1}^{\infty}a_{in}r_i^{2n}\sin 2n\theta_i \ \text{ with } i=2,4,6,7,15 \tag{9-a}$$

- In the case of squares with cracks, we have the following four solutions:

$$u_{10}(r_{10},\theta_{10})=1-2\theta_{10}/\pi+\sum_{n=1}^{\infty}a_{10n}r_{10}^{2n}\sin 2n\theta_{10} \tag{10-a}$$

$$u_{11}(r_{11},\theta_{11})=1+\sum_{n=1}^{\infty}a_{11n}r_{11}^{2n}\sin 2n\theta_{11} \tag{11-a}$$

$$u_{12}(r_{12},\theta_{12})=1+\sum_{p=1}^{\infty}a_{12p}r_{12}^{\frac{p}{2}}\sin(p\theta_{12}/2) \tag{12-a}$$

$$u_{14}(r_{14},\theta_{14})=2\theta_{14}/\pi+\sum_{n=1}^{\infty}a_{14n}r_{14}^{2n}\sin 2n\theta_{14} \tag{13-a}$$

In the case of the rectangular domains with j = 2, 3, 8, 9 et 16,

$$u_j(r_j,\theta_j)=\sum_{m=1}^{\infty}a_{jm}r_j^{m}\sin m\theta_j \tag{14-a}$$

In the case of L-shaped domains, we have:

$$u_5(r_5,\theta_5)=\sum_{l=1}^{\infty}a_{5l}\,r_5^{\frac{2l}{3}}\sin(\frac{2l}{3}\theta_5) \tag{15-a}$$

$$u_{13}(r_{13},\theta_{13})=1+\sum_{k=1}^{\infty}a_{13k}\,r_{13}^{\frac{2k}{3}}\sin(\frac{2k}{3}\theta_{13}) \tag{16-a}$$

The a_{kl} coefficients used in various auxiliary solutions from (9-a) to (16-a) remain arbitrary.

Step 3: Connecting auxiliary solutions.
The connection of auxiliary solutions is done by requiring the continuity of the function u and its normal derivative along the border Γ_{ij} in between two adjacent sub-domains Ω_i and Ω_j. This continuity will be imposed on the least squares sense. In practice, we obtain the approximate solutions by keeping a finite number of coefficients in the auxiliary solutions from (9-a) to (16-a). The sensible choice is to take this number of coefficients proportional to the value of the opening of the angle at the origin of the local coordinate system [11-12].

- In the case of square domains with no cracks on their sides, we have these solutions:

$$u_i(r_i,\theta_i)=\sum_{n=1}^{N}a_{in}r_i^{2n}\sin 2n\theta_i \ \text{ with } i=2,4,6,7,15 \tag{9-b}$$

- In the case of square domains with cracks, we have these four solutions:

$$u_{10}(r_{10},\theta_{10}) = 1 - 2\theta_{10}/\pi + \sum_{n=1}^{N} a_{10n} r_{10}^{2n} \sin 2n\theta_{10} \tag{10-b}$$

$$u_{11}(r_{11},\theta_{11}) = 1 + \sum_{n=1}^{N} a_{11n} r_{11}^{2n} \sin 2n\theta_{11} \tag{11-b}$$

$$u_{12}(r_{12},\theta_{12}) = 1 + \sum_{p=1}^{4N} a_{12n} r_{12}^{\frac{p}{2}} \sin(p\theta_{12}/2) \tag{12-b}$$

$$u_{14}(r_{14},\theta_{14}) = 2\theta_{14}/\pi + \sum_{n=1}^{N} a_{14n} r_{14}^{2n} \sin 2n\theta_{14} \tag{13-b}$$

- In the case of rectangular domains with j = 2, 3, 8, 9 et 16; we have these solutions:

$$u_j(r_j,\theta_j) = \sum_{m=1}^{2N} a_{jm} r_j^{m} \sin m\theta_j \tag{14-b}$$

- In the case of L-shaped domains, we have:

$$u_5(r_5,\theta_5) = \sum_{l=1}^{3N} a_{5l} r_5^{\frac{2l}{3}} \sin(\frac{2l}{3}\theta_5) \tag{15-b}$$

$$u_{13}(r_{13},\theta_{13}) = 1 + \sum_{k=1}^{3N} a_{13k} r_{1''}^{\frac{2k}{3}} \sin(\frac{2k}{3}\theta_{13}) \tag{16-b}$$

In order to obtain the solution of the initial problem from solutions of auxiliary problems, we "just need" to make a "good choice" of arbitrary coefficients a_{in} . According to Tolley [11], the good choice is obtained by imposing the continuity of auxiliary functions and those of their normal derivatives in the sense of least squares along the sub borders Γ_{ij} .

$$I(a_{mn}) = \sum_{i<j} \int_{\Gamma_{ij}} \left[(u_i(a_{ik}) - u_j(a_{jl}))^2 + \left(\frac{\partial u_i(a_{ik})}{\partial n_i} + \frac{\partial u_j(a_{jl})}{\partial n_j} \right)^2 \right] ds_{ij} \tag{17}$$

If I is a function of all coefficients a_{kl} involved in various approximate solutions. The method of least squares aims at minimizing the integral $I(a_{mn})$ with respect to unknown coefficients involved in approximate solutions, i.e. to write that

$$\frac{\partial I(a_{mn})}{\partial a_{kl}} = 0 \tag{18}$$

which gives as many equations as unknown coefficients. Then, we get a non homogeneous algebraic system of the unknown coefficients that can be solved.

The accuracy of approximate solutions is directly linked to the quality of the connection of auxiliary solutions. It is therefore natural to characterize this precision by measuring the imperfections of continuity conditions. For this, we will use the overall error η definite by (19):

$$\eta = \sum_{k<l} \frac{1}{S_{kl}} \int_{\Gamma_{kl}} \left[(u_k - u_l)^2 + \left(\frac{\partial u_k}{\partial \upsilon_k} + \frac{\partial u_l}{\partial \upsilon_l} \right)^2 \right] ds_{kl} \tag{19}$$

where ds_{kl} is the element of arc length of Γ_{kl} , S_{kl} its length and υ_k and υ_l the normals to the sub-boundary separating both two adjacent sub-domains. If the overall error is null, the approximate solution coincides with the exact solution.

RESULTS AND DISCUSSION

Results obtained by LSFEM.

Our numerical results show that the convergence of the method of large singular finite elements is exponential as the logarithm to base ten of the overall error decreases linearly with $28 \times N$ as shown in Figure 3. If we limit the total number of coefficients a_{kl} to 616 (for N = 22) the overall error is 10^{-10} while it is close to 10^{-12} if we keep 700 coefficients a_{kl} (for N = 25).

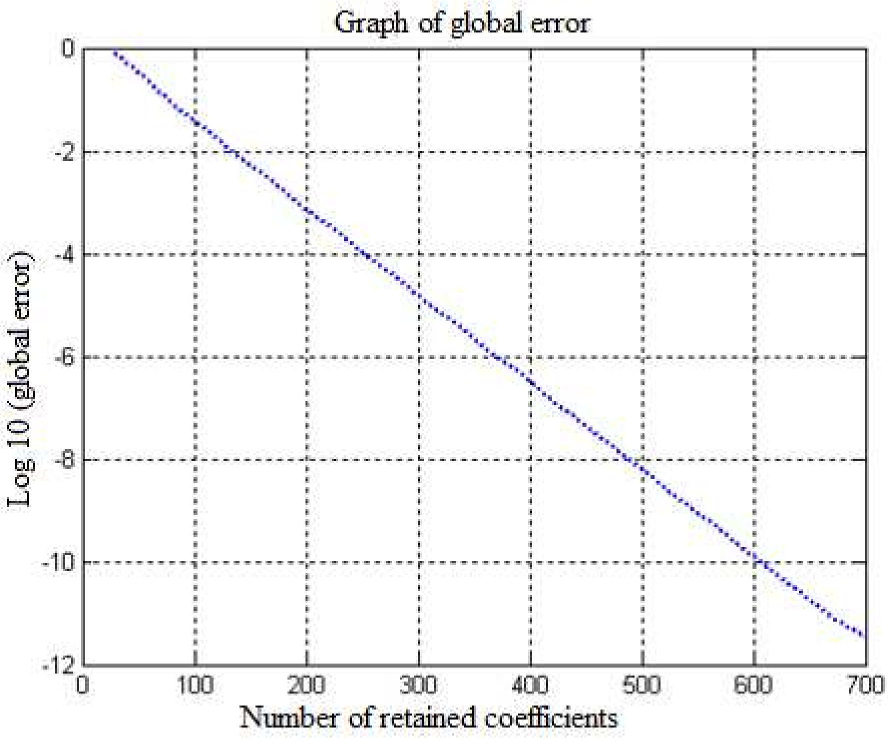


Figure 3 Domain of Chapman-Giraud. Evolution of the overall error according to the number of coefficients a_{kl} conserved.

Comparison of results obtained by the LSFEM with those by the FEM**.**

The comparison of the results obtained with those supplied by LSFEM with those given by FEM is the value of u and that of its partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. We examine the evolution of the approximations obtained by both methods on concentric circles around singularities σ_5 and σ_{13} whose radius is becoming gradually smaller. They correspond to the re-entrant angles $3\pi/2$. Near these points, the large singular elements method is favored towards the method of finite elements.

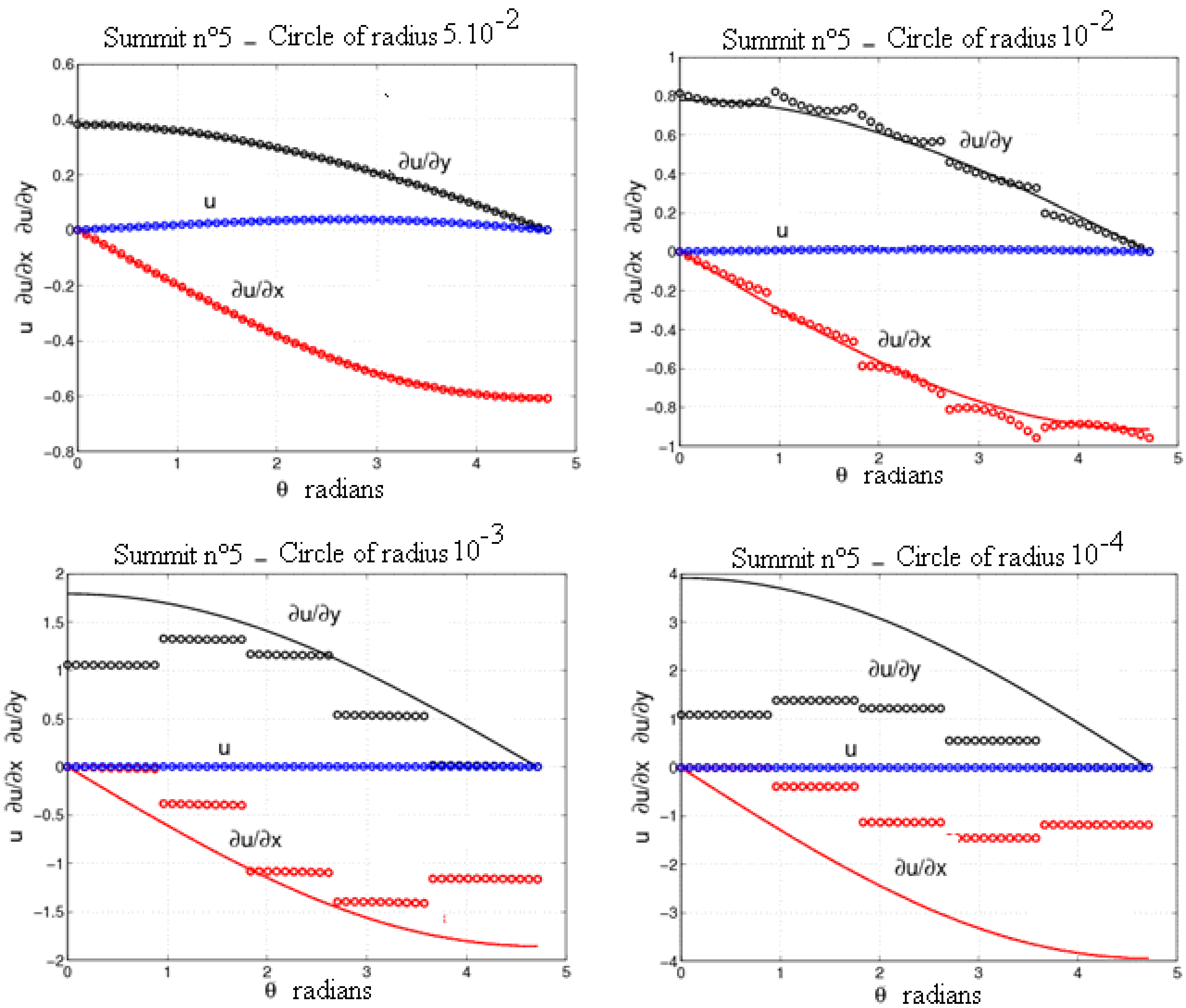
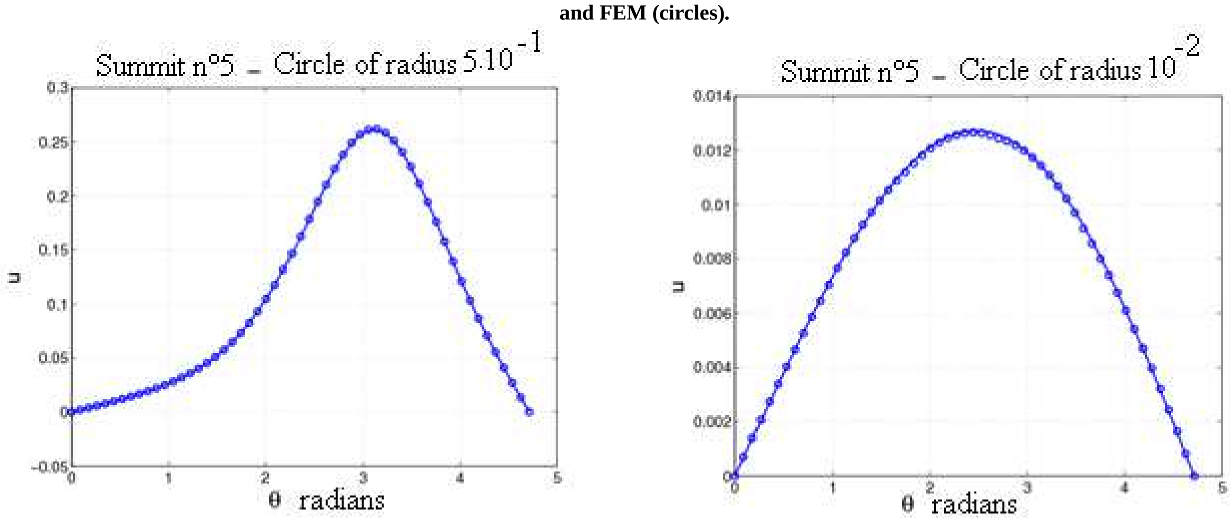


Figure 4. Domain of Chapman-Giraud: comparison of the values of u (blue), $\frac{\partial u}{\partial x}$ (red), $\frac{\partial u}{\partial y}$ (black) obtained by LSFEM (continuous lines) and FEM (circles).



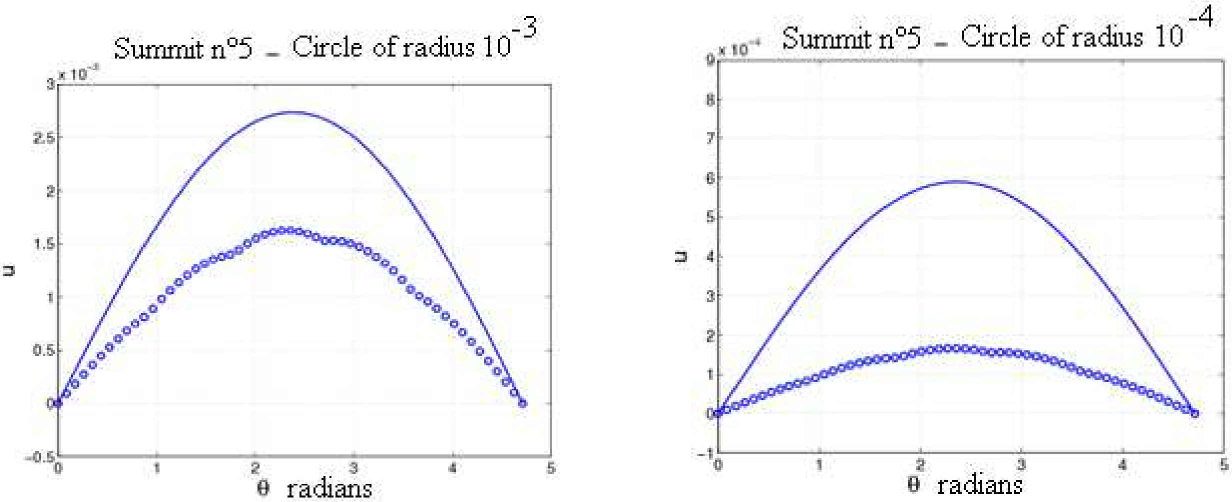


Figure 5. Domain of Chapman-Giraud: comparing the values of u obtained by LSFEM (continuous lines) and FEM (circles).

Figures 6 and 7 are similar to the previous two and refer to the singularity σ_{13} of the sub-domain Ω_{13}. They can allow us to make the same observation as above.

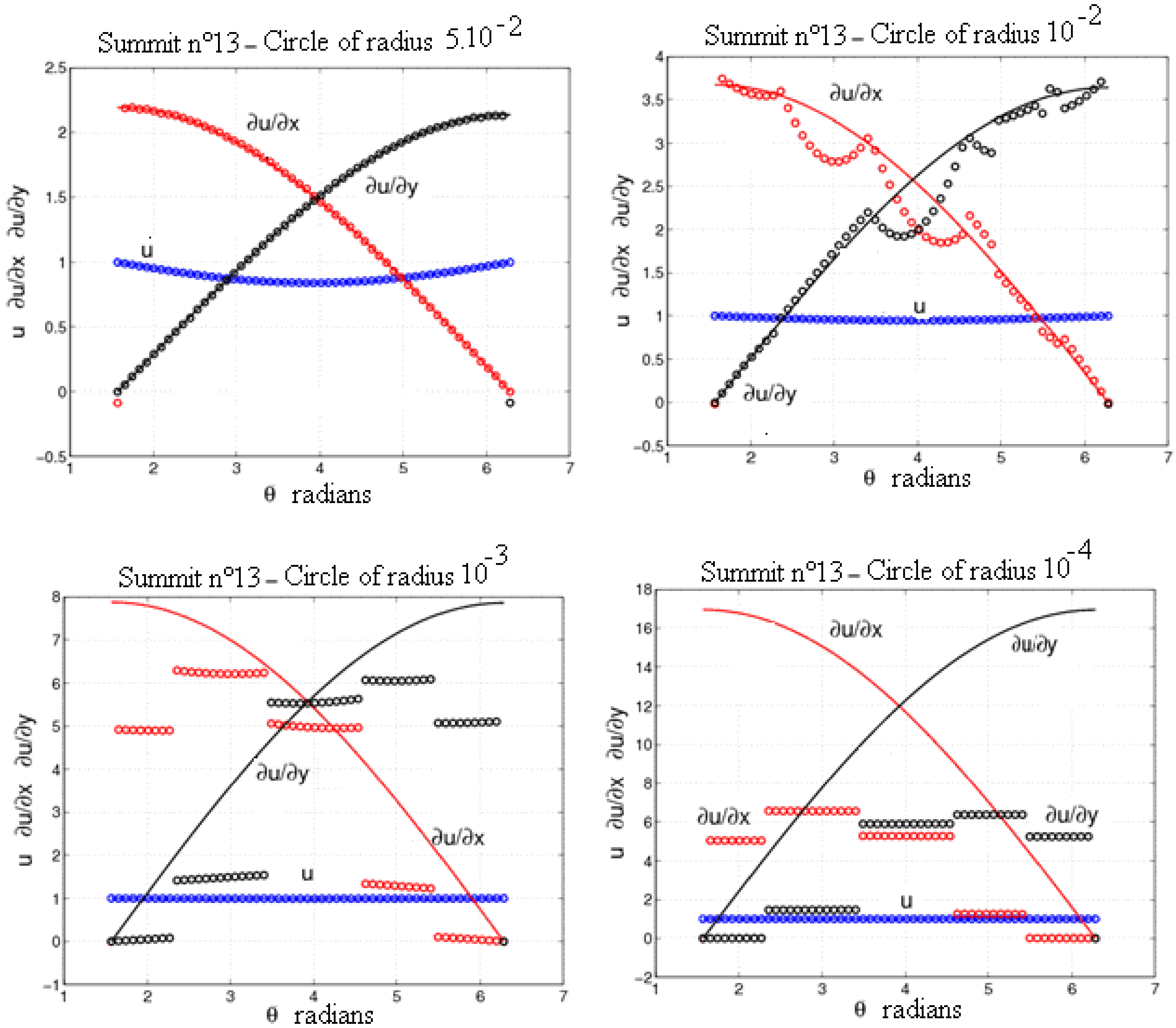


Figure 6. Domain of Chapman-Giraud: comparison of the values of u (blue) $\frac{\partial u}{\partial x}$ (red), $\frac{\partial u}{\partial y}$ (black) obtained by LSFEM (continuous lines) and FEM (circles).

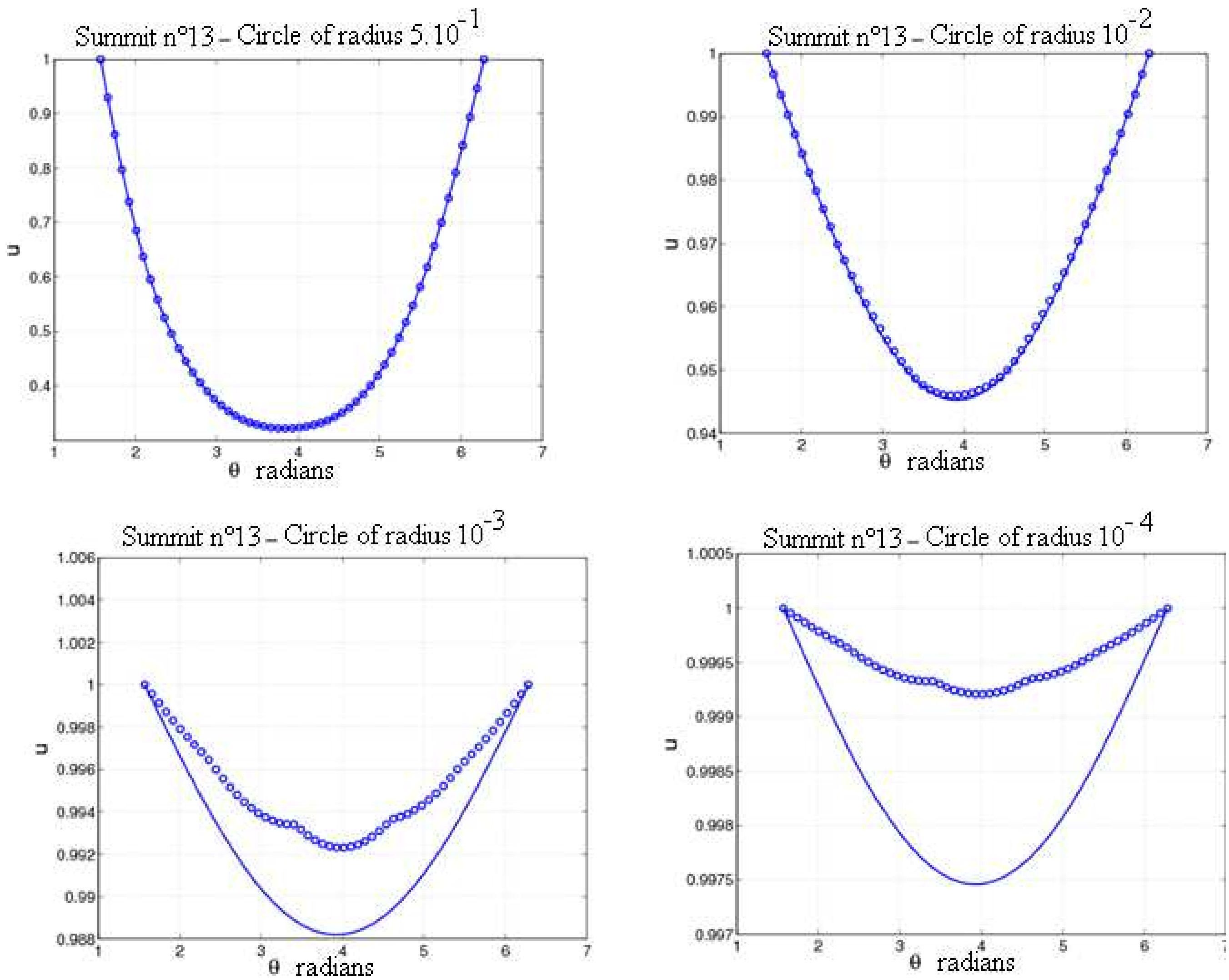


Figure 7. Domain of Chapman-Giraud: comparison of the values of u, obtained by LSFEM (continuous lines) and FEM (circles).

To complete our comparison of results obtained by both methods, we evaluated Δu near the coordinate point (1, 2) (singularities σ_{10} and σ_{14}) on circles of radii, $2\,10^{-2}$, $5\,10^{-2}$, 10^{-1} and $2.5\,10^{-1}$.The results of LSFEM are not represented since the very principle of the method implies that $\Delta u = 0$ on all the circles. As for values obtained by FEM, they are given in like polar diagrams in Figure 8. Although these values are obtained with the finer grid, we can see that results are becoming worse as the radius of the circle becomes smaller and that Δu takes values up to 1000!

The study of cracked polygons obtained through translations, symmetries and rotations from a basic equilateral triangle using the LSFEM gives also satisfactory results throughout the study area except at the end of the cracks where there are large variations of u and its first derivatives [16-17].

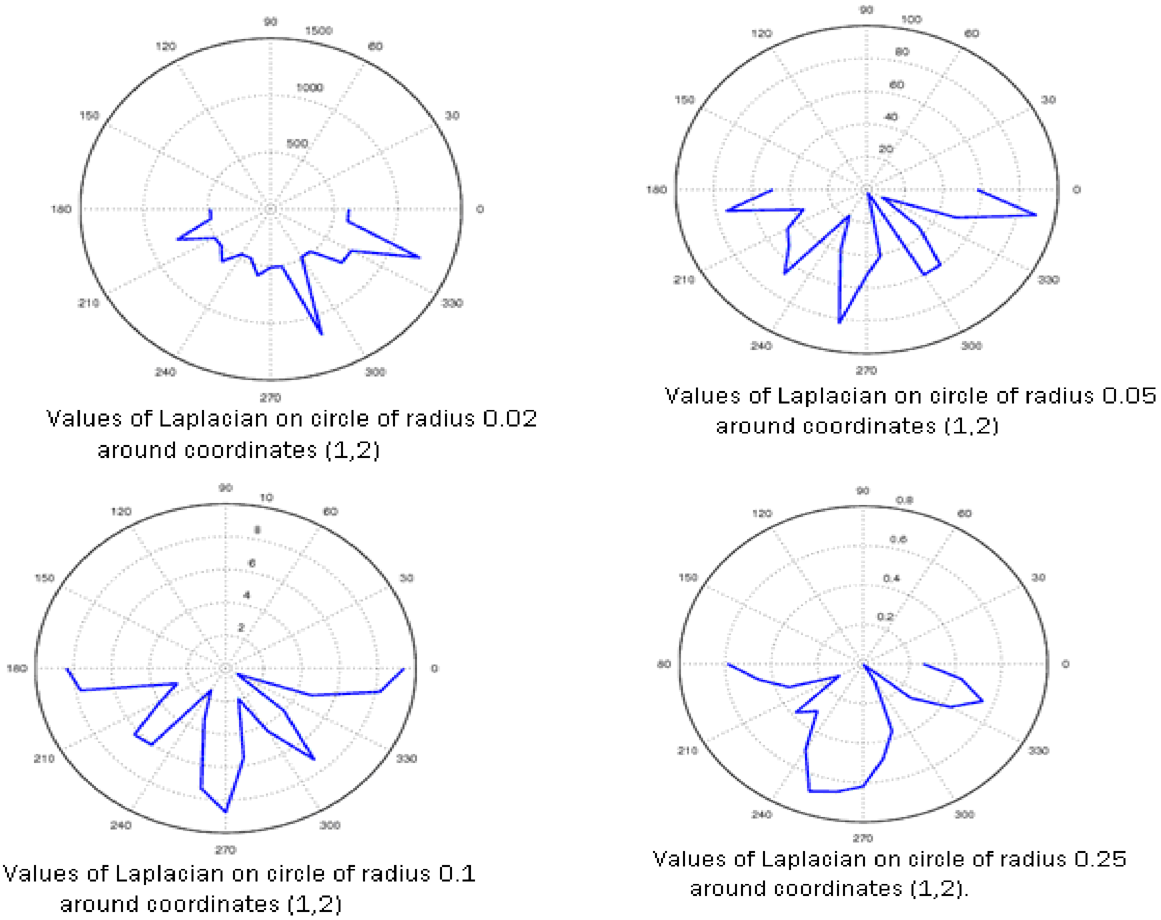


Figure 8. Domain of Chapman-Giraud : values of Laplacian obtained by finite element method (217,857 degrees of freedom) on the circles around the coordinates (1, 2).

CONCLUSION

Large singular finite elements method gives very satisfactory results with singularities and the points far from them. Solutions to problems are sought in analytical form, which provides all the derived quantities with the same accuracy as the key item, without further formulation. By comparing the results obtained with the conventional finite elements method, this gives the advantage to those of large finite singular elements method because it leads to much more accurate results, especially near singularities. It gives accurate results in all parts of the domain, Δu is everywhere zero because of the analytical form of the solution. Although the values obtained with the finer grid (217,857 degrees of freedom), we can see that the results are even worse when the radius of the circle is small and the laplacian takes values that ranging over 1000. Results deteriorate near the singularity and we obtain the classical Gibbs phenomenon.

Acknowledgements
The authors would like to thank CUD, Belgium, for the grant (CIUF, P2, Physics) and the provision of MATLAB Software to Pr O. Michel Zongo and Professors M.D. Tolley and B. Haut for receiving them at Université Libre de Bruxelles (ULB).

REFERENCES

[1] O. Maurice, A. Reinex, P. Hoofmann, B. Bernard and P. Pouliguen, *Adv. in Appl. Sci. Res*. **2011**, 2 (5), 439.
[2] D. S. Chauhan and V. Kumar, *Adv. in Appl. Sci. Res*. **2012,** 3 (1),75.
[3] V. Sri Hari Babu and G. V. Ramana Reddy, *Adv. in Appl. Sci. Res*. **2011,** 2 (4),138.
[4] M. Andrews, *J. Electrostat*. **2006**, 64, 664.
[5] Z. Yosibash, S. Shannon, M. Dauge, M. Costabel, *Int. J. Fracture*, 2**011**, 168, 31.
[6] O. M. Zongo, S. Kam, S. and A. Ouedraogo, *GJPAS.* **2012**, 18 (1&2), 75.
[7] A. F. Emery, *Trans. ASME* (C), **1973**, 95, 344.

[8] Z. C. Li and T.T. Lu,. *Math. Comput. Model*. **2000**, 31, 97.
[9] H. Motz, *Quart. Appl. Math*. **1946**, 4, 371.
[10]G. Strang and G. Fix, An analysis of the finite element method. Prentice Hall. **1973**.
[11] M. D. Tolley, PhD Thesis, Université Libre de Bruxelles, (Bruxelles, Belgium), **1977.**
[12**]** O. M. Zongo, S. Kam, K. Palm and A. Ouedraogo, *Adv. in Appl. Sci. Res.* **2012**, 3 (3), 1572.
[13] Z. C. Li, T. T. Lu, H.Y. Hu and A. H. D Cheng, *Eng. Anal. Bound. Elem* **. 2005**, 29, 59.
[14] S. J. Chapman, *Amer. Math. Monthly*, **1995**, 102, 124.
[15] O. Giraud, *J. Phys. A.* **2004**, 37, 2751.
[16] O. M. Zongo, S. Kam, P. F. Kieno and A. Ouedraogo, *Adv. in Appl. Sci. Res.* **2012**, 3 (5), 2716.
[17] O. M. Zongo , PhD Thesis, Université de Ouagadougou, (Ouagadougou, Burkina Faso), **2012.**