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Interval Valued Intuitionistic Fuzzy Positive Implicative Hyper BCK - Ideals of Hyper BCK – Algebras

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ABSTRACT

In this paper, we introduce the notions of an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of types-1,2,3,…,8. Then we present some theorems which characterize the above notions according to the interval-valued level subsets. Also we obtain certain relationships among these notions, interval-valued intuitionistic fuzzy (strong, weak) hyper BCK-ideals and interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideals of types-1,2,3,..,8 and obtain some related results.

Keywords: Hyper BCK-algebras; Fuzzy positive implicative ideal; Interval valued Intuitionistic Fuzzy positive implicative hyper BCK-ideals

INTRODUCTION

The hyper structure theory was introduced in1934 by Marty [1] at the 8th science congress of Scandinavian Mathematicians. In the following years, several authors have worked on this subject notably in France, United States, Japan, Spain, Russia and Italy. Hyper structures have many applications in several sectors of both pure and applied sciences. In Jun et al. [2], applied hyper structures to BCK-algebras and introduced notion of Hyper BCK-algebras which is generalization of a BC-algebra. The notion of interval-valued fuzzy sets was first introduced by Zadeh [3] as an extension of fuzzy sets. An interval-valued fuzzy set is a fuzzy set whose membership function is many-valued and form an interval in the membership scale. This idea gives the simplest method to capture the imprecision of the membership grade for a fuzzy set. On the other hand, Atanassov [4] introduced the notion of intuitionistic fuzzy sets as an extension of fuzzy set which not only gives a membership degree, but also a non-membership degree. Atanassov and Gargov [5] introduced the notion of interval-valued intuitionistic fuzzy sets which is a generalization of both intuitionistic fuzzy sets and interval-valued fuzzy sets. In this paper we apply the concept of interval-valued intuitionistic fuzzy sets to positive implicative hyper BCK-ideals of hyper BCK-algebras and obtain interval valued intuitionistic positive implicative hyper BCK-ideals of type-1,2,3,…,8 and also we obtain certain relationships among these notions.

PRELIMINARIES

The **B, C, K** system is a variant of combinatory logic that takes as primitive the combinators **B, C, K**. This system was discovered by Curry in his doctoral thesis *Grundlagen der kombinatorischen Logik*, whose results are set out in Curry.

The combinators are defined as follows:

- **B** $x\ y\ z = x\ (y\ z)$

- **C** $x\,y\,z=x\,z\,y$
- **K** $x\,y=x$.

Let H be a nonempty set endowed with hyper operation that is, "o" is a function from $H \times H$ to $P^{*}(H)=P(H)\setminus\{\varnothing\}$. For any two subsets A and B of H, denoted by $A \circ B$ the set $\bigcup_{a\in A, b\in B} aob$. We shall use $x \circ y$ instead of $xo\{y\}$, $\{x\}oy$, or $\{x\}o\{y\}$.

Definition 2.1

By a hyper BCK-algebra $(\mathrm{H}, \circ, 0)$, we mean a nonempty set H endowed with a hyper operation "o" and a constant 0 satisfying the following axioms:

(HK-1) $(x\circ z)\circ(y\circ z) \ll x\circ y$,

(HK-2) $(x\circ y)\circ z=(x\circ z)\circ y$,

(HK-3) $x\circ H \ll \{x\}$,

(HK-4) $x \ll y$ and $y \ll x \Rightarrow x=y$ for all $x,y,z\in H$.

We can define a relation "$\ll$" on H by letting $x \ll y$ if and only if $0\in x\circ y$ and for every $A,B\subseteq H$, $A\ll B$ is defined by $\forall a\in A, \exists b\in B$ such that $a \ll b$. In such case, we call "$\ll$" the hyper order in H **[6]**.

In any hyper BCK-algebra H the following hold

(P1) $\mathrm{x}\circ\mathrm{x}\ll\{\mathrm{x}\}$ (P2) $x\circ 0\ll\{x\}$, $0\circ x\ll\{x\}$ and $0\circ 0\ll\{0\}$,

(P3) $(A\circ B)\circ C=(A\circ C)\circ B$, $A\circ B\ll A$ and $0\circ A\ll\{0\}$, (P4) $0\circ 0=\{0\}$, (P5) $0\ll x$

(P6) $x\ll x$, (P7) $A\ll A$, (P8) $A\subseteq B \Rightarrow A\ll B$, (P9) $0\circ x=\{0\}$, (P10) $x\circ 0=\{x\}$,

(P11) $0\circ A=\{0\}$, (P12) $A\ll\{0\}\Rightarrow A=\{0\}$, (P13) $A\circ B\ll A$, (P14) $x\in x\circ 0$,

(P15) $x\circ 0\ll\{y\}\Rightarrow x\ll y$, (P16) $y\ll z\Rightarrow x\circ z\ll x\circ y$

(P17) $x\circ y=\{0\}\Rightarrow(x\circ z)\circ(y\circ z)=\{0\}$ and $x\circ z\ll y\circ z$, (P18) $A\circ\{0\}=\{0\}\Rightarrow A=0$ for all $x,y,z\in H$ and for all non-empty sets A, B and C of H.

Let I be a non-empty subset of a hyper BCK-algebra H and $0\in I$. Then

I is called a hyper BCK-sub algebra of H, if $x\circ y\subseteq I$ for all $x,y\in I$.

A weak hyper BCK-ideal of H if $x\circ y\subseteq I$ and $y\in I$ implies $\mathrm{x}\in\mathrm{I}$, $\forall$ $\mathrm{x,y}\in\mathrm{H}$

A hyper BCK-ideal of H, if $x\circ y\ll I$ and $y\in I$ implies $\mathrm{x}\in\mathrm{I}$ for all $x,y\in H$.

A strong hyper BCK-ideal of H, if $x\circ y\cap I\neq\varnothing$ and $y\in I$ implies $\mathrm{x}\in\mathrm{I}$, $\forall$ $\mathrm{x,y}\in\mathrm{H}$

I is said to be reflexive if $x\circ x\subseteq I$ for all $x\in H$.

S-reflexive if it satisfies $(x\circ y)\cap I\neq\varnothing$ implies that $x\circ y\ll I$, for all $x,y\in H$.

Closed, if $x\ll y$ and $y\in I$. implies that $x\in I$ for all $x\in H$.

It is easy to see that every S-reflexive sub-set of H is reflexive.

Definition 2.2

Let H be a hyper BCK-algebra then H is said to be a positive implicative hyper BCK-algebra, if for all $x,y,z\in H$, $(x\circ y)\circ z=(x\circ z)\circ(y\circ z)$ **[7]**.

Definition 2.3 [7]

Let I be a nonempty subset of H and $0\in I$. Then I is said to be a positive implicative hyper BCK-ideal of

(i) type 1, if $(x\circ y)\circ z\subseteq I$ and $y\circ z\subseteq I \Rightarrow x\circ z\subseteq I$,

(ii) type 2, if $(x\circ y)\circ z\ll I$ and $y\circ z\subseteq I \Rightarrow x\circ z\subseteq I$,

(iii) type 3, if $(x\circ y)\circ z\ll I$ and $y\circ z\ll I \Rightarrow x\circ z\subseteq I$,

(iv) type 4 , if $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I \Rightarrow x \circ z \subseteq I$,

(v) type 5, if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I \Rightarrow x \circ z \ll I$,

(vi) type 6, if $(x \circ y) \circ z \ll I$ and $y \circ z \ll I \Rightarrow x \circ z \ll I$,

(vii) type 7, if $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I \Rightarrow x \circ z \ll I$,

(viii) type 8, if $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I \Rightarrow x \circ z \ll I$ for all $x, y, z \in H$.

The determination of maximum and minimum between two real numbers is very simple, but it is not simple for two intervals. Biswas [8] described a method to find max/sup and min/inf between two intervals and set of intervals. By an interval number $\tilde{a}$ on [0,1], we mean an interval $\left[a^-, a^+\right]$, where $0 \le a^- \le a^+ \le 1$. The set of all closed subintervals of [0,1] is denoted by $D[0,\ 1]$. The interval $[a,\ a]$ is identified with the number $a \in [0,\ 1]$. For an interval numbers $\tilde{a}_i = \left[a_i^-, b_i^+\right] \in D[0,1],\ i \in I$.

We define $\inf \tilde{a}_i = \left[\min\limits_{i \in I} a_i^-,\ \min\limits_{i \in I} b_i^+\right]$, $\sup \tilde{a}_i = \left[\max\limits_{i \in I} a_i^-,\ \max\limits_{i \in I} b_i^+\right]$

And put

(i) $\tilde{a}_1 \wedge \tilde{a}_2 = \min(\tilde{a}_1, \tilde{a}_2) = \min\left(\left[a_1^-,\ b_1^+\right],\ \left[a_2^-,\ b_2^+\right]\right) = \left[\min\left\{a_1^-, a_2^+\right\}, \min\left\{b_1^-, b_2^+\right\}\right]$

(ii) $\tilde{a}_1 \vee \tilde{a}_2 = \max(\tilde{a}_1, \tilde{a}_2) = \max\left(\left[a_1^-,\ b_1^+\right],\ \left[a_2^-,\ b_2^+\right]\right) = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{b_1^+, b_2^+\right\}\right]$

(iii) $\tilde{a}_1 + \tilde{a}_2 = \left[a_1^- + a_2^- - a_1^-.a_2^-,\ b_1^+ + b_2^+ - b_1^+.b_2^+\right]$

(iv) $\tilde{a}_1 \le \tilde{a}_2 \Leftrightarrow a_1^- \le a_2^-$ and $b_1^+ \le b_2^+$

(v) $\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$,

$mD = m[a]m\left[a_1^-, \mathrm{b}_1^+\right] = \left[ma_1^-, m\,\mathrm{b}_1^+\right]$ where $0 \le \mathrm{m} \le 1$.

Obviously $(D[0,1],\ \le, \vee, \wedge)$ form a complete lattice with [0,0] as its least element and [1,1] as its greatest element.

People observed that the determination of membership value is a difficult task for a decision maker. In Zadeh [3] defined another type of fuzzy set called interval-valued fuzzy sets (i-v FSs). The membership value of an element of this set is not a single number, it is an interval and this interval is a sub-interval of the interval [0,1]. Let $D[0,1]$ be the set of a subintervals of the interval [0, 1].

Let L be a given nonempty set. An interval-valued fuzzy set (briefly, i-v fuzzy set) B on L is defined by $B = \left\{\left(X, \left[\mu_B^-(x), \mu_B^+(x)\right]\right) : x \in L\right\}$, Where $\mu_{\mathrm{B}}^-(x)$ and $\mu_{\mathrm{B}}^+(x)$ are fuzzy sets of L such that $\mu_{\mathrm{B}}^-(x) \le \mu_B^-(x)$ for all $x \in L$. Let $\tilde{\mu}_B(x) = \left[\mu_{\mathrm{B}}^-(x),\ \mu_{\mathrm{B}}^+(x)\right]$, then $\mathrm{B} = \left\{\left(x,\ \tilde{\mu}_B(x)\right) : x \in L\right\}$ where $\tilde{\mu}_B : L \to D[0,\ 1]$.

A mapping $A = (\tilde{\mu}_A, \tilde{\lambda}_A) : L \to D[0,\ 1] \times D[0,\ 1]$ is called an interval-valued intuitionistic fuzzy set (i-v IF set, in short) in L if $0 \le \mu_{\mathrm{A}}^+(x) + \lambda_{\mathrm{A}}^+(x) \le 1$ and $0 \le \mu_{\mathrm{A}}^-(x) + \lambda_{\mathrm{A}}^-(x) \le 1$ for all $x \in L$ (that is, $A^+ = \left(X,\ \mu_{\mathrm{A}}^+,\ \lambda_{\mathrm{A}}^+\right)$ and $A^- = (X, \mu_A^-, \lambda_A^-)$ are intuitionistic fuzzy sets), where the mappings $\tilde{\mu}_A(x) = [\mu_{\mathrm{A}}^-(x),\ \mu_{\mathrm{A}}^+(x)] : L \to D[0,\ 1]$ and $\tilde{\lambda}_A(x) = \left(\lambda_A^-(x),\ \lambda_A^+(x)\right) : L \to D[0,1]$ denote the degree of membership (namely $\tilde{\mu}_A(x)$) and degree of non-membership (namely $\tilde{\lambda}_A(x)$) of each element $x \in L$ to A respectively.

INTERVAL-VALUED INTUITIONISTIC FUZZY POSITIVE IMPLICATIVE HYPER BCK-IDEALS OF HYPER BCK-ALGEBRAS

Definition 3.1

Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy subset of H and $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x), \tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(y)$ for all $x, y \in H$. Then $\tilde{A}$ is said to be an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 1, if for all $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \min\left\{\inf_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \inf_{b \in y \circ z} \tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\sup_{c \in (x \circ y) \circ z} \tilde{\lambda}_A(c), \sup_{d \in y \circ z} \tilde{\lambda}_A(d)\right\}.$$

(ii) type 2, if for all $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \min\left\{\sup_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \inf_{b \in y \circ z} \tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\inf_{c \in (x \circ y) \circ z} \tilde{\lambda}_A(c), \sup_{d \in y \circ z} \tilde{\lambda}_A(d)\right\}$$

(iii) type 3, if for all $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \min\left\{\sup_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \sup_{b \in y \circ z} \tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\inf_{c \in (x \circ y) \circ z} \tilde{\lambda}_A(c), \inf_{d \in y \circ z} \tilde{\lambda}_A(d)\right\}.$$

(iv) type 4, if for all $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \min\left\{\inf_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \sup_{b \in y \circ z} \tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\sup_{c \in (x \circ y) \circ z} \tilde{\lambda}_A(c), \inf_{d \in y \circ z} \tilde{\lambda}_A(d)\right\} \text{ for all } x, y, z \in H$$

Definition 3.2

Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy sub-set of H. Then $\tilde{A}$ is said to be an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of

(i) type 5, if there exists $t \in x \circ z$ such that

$$\tilde{\mu}_A(t) \geq \min\left\{\inf_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \inf_{b \in y \circ z} \tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\sup_{c \in (x \circ y) \circ z} \tilde{\lambda}_A(c), \inf_{d \in y \circ z} \tilde{\lambda}_A(d)\right\}$$

(ii) type 6, if there exists $t \in x \circ z$ such that

$$\tilde{\mu}_A(t) \geq \min\left\{\sup_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \sup_{b \in y \circ z} \tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\inf_{c \in (x \circ y) \circ z} \tilde{\lambda}_A(c), \inf_{d \in y \circ z} \tilde{\lambda}_A(d)\right\}$$

(iii) type 7, if there exists $t \in x \circ z$ such that

$$\tilde{\mu}_A(t) \geq \min\left\{\inf_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \sup_{b\in y\circ z} \tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\sup_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \inf_{d\in y\circ z} \tilde{\lambda}_A(d)\right\}$$

(iv) type 8, if there exists $t \in x \circ z$ such that

$$\tilde{\mu}_A(t) \geq \min\left\{\sup_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \inf_{b\in y\circ z} \tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\inf_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \sup_{d\in y\circ z} \tilde{\lambda}_A(d)\right\}.$$

Theorem 3.3

Let $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy subset of H. Then

(i) If $\tilde{A}$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 3, then $\tilde{A}$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 2, 4 and 6.

(ii) If $\tilde{A}$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 2 (or) 4, then $\tilde{A}$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 1.

(iii) If $\tilde{A}$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 2 (type 4), then $\tilde{A}$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 8 (type 7).

(iv) If $\tilde{A}$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 1, then $\tilde{A}$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 5.

Proof: (i) Assume $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 3.

Since $\sup_{b\in y\circ z} \tilde{\mu}_A(b) \geq \inf_{b\in y\circ z} \tilde{\mu}_A(b)$ and $\inf_{b\in y\circ z} \tilde{\lambda}_A(b) \leq \sup_{b\in y\circ z} \tilde{\lambda}_A(b)$

Now for all $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \min\left\{\sup_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \sup_{b\in y\circ z} \tilde{\mu}_A(b)\right\} \geq \min\left\{\sup_{a\in(x o y) o z} \tilde{\mu}_A(a), \inf_{b\in y\circ z} \tilde{\mu}_A(b)\right\}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\inf_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \inf_{d\in y\circ z} \tilde{\lambda}_A(d)\right\} \leq \max\left\{\inf_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \sup_{d\in y\circ z} \tilde{\lambda}_A(d)\right\}$$

Thus $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 2.

Since $\sup_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a) \geq \inf_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a)$ and $\inf_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c) \leq \sup_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c)$

Now for all $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \min\left\{\sup_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \sup_{b\in y\circ z} \tilde{\mu}_A(b)\right\} \geq \min\left\{\inf_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \sup_{b\in y\circ z} \tilde{\mu}_A(b)\right\}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\inf_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \inf_{d\in y\circ z} \tilde{\lambda}_A(d)\right\} \leq \max\left\{\sup_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \inf_{d\in y\circ z} \tilde{\lambda}_A(d)\right\}$$

Thus $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 4.

Clearly, there exists $t \in x \circ z$ such that

$$\tilde{\mu}_A(t) \geq \min\left\{\sup_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \sup_{b\in y\circ z} \tilde{\mu}_A(b)\right\} \text{ and } \tilde{\lambda}_A(t) \leq \max\left\{\inf_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \inf_{d\in y\circ z} \tilde{\lambda}_A(d)\right\}.$$

Thus $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval- valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 6.

(ii) Assume $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 2.

Since $\sup_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a) \geq \inf_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a)$ and $\inf_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c) \leq \sup_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c)$

Now for all $t \in x \circ z$

$$\tilde{\mu}_A(t) \geq \min\left\{\sup_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \inf_{b\in y\circ z} \tilde{\mu}_A(b)\right\} \geq \min\left\{\inf_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \inf_{b\in y\circ z} \tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\inf_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \sup_{d\in y\circ z} \tilde{\lambda}_A(d)\right\} \leq \max\left\{\sup_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \sup_{d\in y\circ z} \tilde{\lambda}_A(d)\right\}$$

Thus $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 1.

Assume $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 4.

Since $\sup_{b\in y\circ z} \tilde{\mu}_A(b) \geq \inf_{b\in y\circ z} \tilde{\mu}_A(b)$ and $\inf_{d\in y\circ z} \tilde{\lambda}_A(d) \leq \sup_{d\in y\circ z} \tilde{\lambda}_A(d)$

Now for all $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \min\left\{\inf_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \sup_{b\in y\circ z} \tilde{\mu}_A(b)\right\} \geq \min\left\{\inf_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \inf_{b\in y\circ z} \tilde{\mu}_A(b)\right\} \text{and}$$

$$\tilde{\lambda}_A(t) \leq \max\left\{\sup_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \inf_{d\in y\circ z} \tilde{\lambda}_A(d)\right\} \leq \max\left\{\sup_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \sup_{d\in y\circ z} \tilde{\lambda}_A(d)\right\}$$

Thus $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 1.

(iii) Assume $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 2. Clearly there exists $t \in x \circ z$

$$\tilde{\mu}_A(t) \geq \min\left\{\sup_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \inf_{b\in y\circ z} \tilde{\mu}_A(b)\right\}, \ \tilde{\lambda}_A(t) \leq \max\left\{\inf_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \sup_{d\in y\circ z} \tilde{\lambda}_A(d)\right\}.$$

Thus $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type-8. Assume that $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 4. Clearly there exists $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \min\left\{\inf_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \sup_{b\in y\circ z} \tilde{\mu}_A(b)\right\}, \ \tilde{\lambda}_A(t) \leq \max\left\{\sup_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \inf_{d\in y\circ z} \tilde{\lambda}_A(d)\right\}.$$

Thus $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 7.

(iv) Assume $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an intuitionistic fuzzy positive implicative hyper BCK-ideal of type 1. Clearly there exists $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \min\left\{\inf_{a\in(x\circ y)\circ z} \tilde{\mu}_A(a), \inf_{b\in y\circ z} \tilde{\mu}_A(b)\right\}, \ \tilde{\lambda}_A(t) \leq \max\left\{\sup_{c\in(x\circ y)\circ z} \tilde{\lambda}_A(c), \sup_{d\in y\circ z} \tilde{\lambda}_A(d)\right\}$$

Thus $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 5.

Example 3.4

Let $H = \{0,1,2,3\}$ be a hyper BCK-algebra with the following Cayley table

∘	**0**	**1**	**2**	**3**
0	{0}	{0}	{0}	{0}
1	{1}	{0}	{0}	{0}
`	{2}	{2}	{0}	{0}
3	{3}	{3}	{2,3}	{0,2,3}

Define an i-v IFS $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ in H by

$\tilde{\mu}_A(0)=\tilde{\mu}_A(1)=[1,1],\ \tilde{\mu}_A(2)=[0.5,0.6],\ \tilde{\mu}_A(3)=[0.2,0.25]$

$\tilde{\lambda}_A(0)=\tilde{\lambda}_A(1)=[0,0],\ \tilde{\lambda}_A(2)=[0.3,0.35],\ \tilde{\lambda}_A(3)=[0.7,0.75]$.

Then $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type1, while it is not type-2, because $3\in 3\circ 0$

$$\tilde{\mu}_A(3)=[0.2,0.25]<[0.5,0.6]=\tilde{\mu}_A(2)=\min\left\{\sup_{a\in(3\circ 2)\circ 0}\tilde{\mu}_A(a),\ \inf_{b\in(2\circ 0)}\tilde{\mu}_A(\mathrm{b})\right\}$$ and

$$\tilde{\lambda}_A(3)=[0.7,0.75]>[0.3,0.35]=\tilde{\lambda}_A(2)=\max\left\{\inf_{c\in(3\circ 2)\circ 0}\tilde{\lambda}_A(c),\ \sup_{d\in(2\circ 0)}\tilde{\lambda}_A(d)\right\}.$$

Example 3.5

Let $H=\{0,1,2\}$ *be a hyper BCK-algebra with hyper operation "ï " the following is the Cayley table*

∘	0	1	2
0	{0}	{0}	{0}
1	{1}	{0,1}	{0,1}
2	{2}	{1,2}	{0,1,2}

Define i-v IFS $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ in *H* by

$\tilde{\mu}_A(0)=\tilde{\mu}_A(2)=[1,1],\ \tilde{\mu}_A(1)=[0.2,0.25]$ and $\tilde{\lambda}_A(0)=\tilde{\lambda}_A(2)=[0,0],\ \tilde{\lambda}_A(1)=[0.6,0.65]$.

Then $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 4, which is not type 3, because $1\in 2\circ 2$

$$\tilde{\mu}_A(1)=[0.2,0.25]<[1,1]=\tilde{\mu}_A(0)=\min\left\{\sup_{a\in(2\circ 0)\circ 2}\tilde{\mu}_A(a),\ \sup_{b\in(0\circ 2)}\tilde{\mu}_A(b)\right\}$$ and

$$\tilde{\lambda}_A(1)=[0.6,0.65]>[0,0]=\max\left\{\inf_{c\in(2\circ 0)\circ 2}\tilde{\lambda}_A(c),\ \inf_{d\in(0\circ 2)}\tilde{\lambda}_A(d)\right\}.$$

Theorem 3.6

Let $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ be an i-v IFS of H. Then the following statements hold:

(i) If *A* is an interval-valued intuitionistic fuzzy positive implicative hyper BCK- ideal of type 1, then *A* is an interval-valued intuitionistic fuzzy weak hyper BCK-ideal of *H*.

(ii) If *A* is an interval-valued intuitionistic fuzzy positive implicative hyper BCK- ideal of type 2, then *A* is an interval-valued intuitionistic fuzzy strong hyper BCK-ideal of *H*.

Proof: (i) Suppose *A* is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 1. Put $z=0$ in type1, for all $t\in x\circ 0=\{x\}$, we get

$$\tilde{\mu}_A(x)\geq\min\left\{\inf_{a\in(x\circ y)\circ 0}\tilde{\mu}_A(a),\ \inf_{b\in y\circ 0}\tilde{\mu}_A(b)\right\}=\min\left\{\inf_{a\in x\circ y}\tilde{\mu}_A(a),\ \tilde{\mu}_A(y)\right\}$$ and

$$\tilde{\lambda}_A(x)\leq\max\left\{\sup_{c\in(x\circ y)\circ 0}\tilde{\lambda}_A(c),\ \sup_{d\in y\circ 0}\tilde{\lambda}_A(d)\right\}=\max\left\{\sup_{c\in x\circ y}\tilde{\lambda}_A(c),\tilde{\lambda}_A(y)\right\}$$

Thus $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy weak hyper BCK-ideal.

Suppose $\tilde{A}$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 2.

Put $z=0$ in type 2, for all $t\in x\circ 0=\{x\}$, we get

$$\tilde{\mu}_A(x)\geq\min\left\{\sup_{a\in(x\circ y)\circ 0}\tilde{\mu}_A(a),\ \inf_{b\in y\circ 0}\tilde{\mu}_A(b)\right\}=\min\left\{\sup_{a\in x\circ y}\tilde{\mu}_A(a),\ \tilde{\mu}_A(y)\right\}$$ and

$$\tilde{\lambda}_A(x)\leq\max\left\{\inf_{c\in(x\circ y)\circ 0}\tilde{\lambda}_A(c),\ \sup_{d\in y\circ 0}\tilde{\lambda}_A(d)\right\}=\max\left\{\inf_{c\in x\circ y}\tilde{\lambda}_A(c),\ \tilde{\lambda}_A(y)\right\}\text{.....(i)}$$

We show that, for $x, y \in H$ if $x \ll y$ then $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$ and $\tilde{\lambda}_A(x) \leq \tilde{\lambda}_A(y)$.

For this, if $x, y \in H$ such that $x \ll y$ that is $0 \in x \circ y$ since $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(a)$ and $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(b)$. This is true for all $a, b \in x \circ y$.

Then $\sup\limits_{a \in x \circ y} \tilde{\mu}_A(a) = \tilde{\mu}_A(0)$ and $\inf\limits_{b \in x \circ y} \tilde{\lambda}_A(b) = \tilde{\lambda}_A(0)$

so $\tilde{\mu}_A(x) \geq \min\left\{\sup\limits_{a \in x \circ y} \tilde{\mu}_A(a),\ \tilde{\mu}_A(y)\right\} = \min\{\tilde{\mu}_A(0), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(y)$ and

$$\tilde{\lambda}_A(x) \leq \max\left\{\inf_{c \in x \circ y} \tilde{\lambda}_A(c),\ \tilde{\lambda}_A(y)\right\} = \max\{\tilde{\lambda}_A(0), \tilde{\lambda}_A(y)\} = \tilde{\lambda}_A(y) \text{.....(ii)}$$

Since by (HK3) then for all $a \in x \circ x$, $a \ll x$ and so $\tilde{\mu}_A(a) \geq \tilde{\mu}_A(x)$, $\tilde{\lambda}_A(a) \leq \tilde{\lambda}_A(x)$

for all $a, b \in x \circ y$. Hence $\inf\limits_{a \in x \circ x} \tilde{\mu}_A(a) \geq \tilde{\mu}_A(x)$ and $\sup\limits_{c \in x \circ x} \tilde{\lambda}_A(c) \leq \tilde{\lambda}_A(x)$ .....(iii)

Combining (i) and (iii), we get

$$\inf_{a \in x \circ x} \tilde{\mu}_A(a) \geq \tilde{\mu}_A(x) \geq \min\left\{\sup_{b \in x \circ y} \tilde{\mu}_A(b), \tilde{\mu}_A(y)\right\} \text{ and}$$

$$\sup_{c \in x \circ x} \tilde{\lambda}_A(c) \leq \tilde{\lambda}_A(x) \leq \max\left\{\inf_{d \in x \circ y} \tilde{\lambda}_A(d), \tilde{\lambda}_A(y)\right\} \text{ for all } x, y \in H.$$

Thus $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy strong hyper BCK-ideal of *H*.

The following examples show that the converse of the Theorem 3.6 is not true in general.

Example 3.7

Let $H = \{0,1,2,3\}$ be a hyper BCK-algebra with the following Cayley table

∘	**0**	**1**	**2**	**3**
0	{0}	{0}	{0}	{0}
1	{1}	{0}	{0}	{0}
2	{2}	{2}	{0}	{0}
3	{3}	{3}	{2}	{0,2}

Define an i-v IFS $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in H by

$\tilde{\mu}_A(0) = \tilde{\mu}_A(1) = [1,1]$, $\tilde{\mu}_A(2) = \tilde{\mu}_A(3) = [0.3, 0.35]$ and

$\tilde{\lambda}_A(0) = \tilde{\lambda}_A(1) = [0,0]$, $\tilde{\lambda}_A(2) = \tilde{\lambda}_A(3) = [0.5, 0.6]$.

Then $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy weak hyper BCK-ideal, but it is not an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type1. Because $2 \in 3 \circ 2 = \{2\}$

$$\tilde{\mu}_A(2) = [0.3, 0.35] < [1,1] = \tilde{\mu}_A(0) = \min\left\{\inf_{a \in (3 \circ 2) \circ 2} \tilde{\mu}_A(a), \inf_{b \in 2 \circ 2} \tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(2) = [0.5, 0.6] > [0,0] = \tilde{\lambda}_A(0) = \max\left\{\sup_{c \in (3 \circ 2) \circ 2} \tilde{\lambda}_A(c), \sup_{d \in 2 \circ 2} \tilde{\lambda}_A(d)\right\}.$$

Example 3.8

Let $H = \{0,1,2\}$ be a hyper BCK-algebra with the following Cayley table

∘	**0**	**1**	**2**
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{2}	{0,2}

Define an i-v IFS $\tilde{A} = (\tilde{\mu}_A, \tilde{\lambda}_A)$ in H by

$\tilde{\mu}_A(0)=[1,1], \tilde{\mu}_A(1)=[0,0], \tilde{\mu}_A(2)=[0.4,0.5]$ and

$\tilde{\lambda}_A(0)=[0,0]$, $\tilde{\lambda}_A(1)=[1,1]$, $\tilde{\lambda}_A(2)=[0.3,0.35]$.

Then $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy strong hyper BCK-ideal of H but it is not an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 2. Because $2 \in 2 \circ 2$

$$\tilde{\mu}_A(2)=[0.4,0.5] < [1,1]=\tilde{\mu}_A(0)=\min\left\{\sup_{a\in(2\circ 0)\circ 2}\tilde{\mu}_A(a), \sup_{b\in 0\circ 2}\tilde{\mu}_A(b)\right\} \text{ and}$$

$$\tilde{\lambda}_A(2)=[0.3,0.35] > [0,0]=\tilde{\lambda}_A(0)=\max\left\{\inf_{c\in(2\circ 0)\circ 2}\tilde{\lambda}_A(c), \inf_{d\in 2\circ 2}\tilde{\lambda}_A(d)\right\}.$$

Corollary 3.9

Let $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 1, if A satisfies the "inf- sup" property, then $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy s-weak hyper BCK-ideal of *H*.

Corollary 3.10

Let $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy Positive implicative hyper BCK-ideal of type 2 and $x, y \in H$. Then

(i) $x \ll y$ imply $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$ and $\tilde{\lambda}_A(x) \leq \tilde{\lambda}_A(y)$

(ii) $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(a),\tilde{\mu}_A(y)\}$ and $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A(b), \tilde{\lambda}_A(y)\}$ for all $a,b \in x \circ y$.

Corollary 3.11

Every interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type-2, is an interval-valued intuitionistic fuzzy s-weak hyper BCK-ideal (hence an interval-valued intuitionistic fuzzy weak-hyper BCK-ideal) and an interval-valued intuitionistic fuzzy hyper BCK-ideal.

Corollary 3.12

If $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy positive implicative hyper BCK-ideal of type 2, then $\tilde{\mu}_A(x) \geq \min\left\{\inf_{a\in x\circ y}\tilde{\mu}_A(a),\tilde{\mu}_A(y)\right\}$ and $\tilde{\lambda}_A(x) \leq \max\left\{\sup_{b\in x\circ y}\tilde{\lambda}_A(b), \tilde{\lambda}_A(y)\right\}$ for all $x, y \in H$

Definition 3.13

An IFS $\tilde{A}=(\tilde{\mu}_A,\tilde{\lambda}_A)$ in H is said to be an interval-valued intuitionistic fuzzy closed if it satisfies $x, y \in H$ such that $x \ll y$, $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$ and $\tilde{\lambda}_A(x) \leq \tilde{\lambda}_A(y)$.

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