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Hydromagnetic natural convection flow of an incompressible viscoelastic fluid between two infinite vertical moving and oscillating parallel plates

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ABSTRACT

In this paper, we study the thermal and mass diffusion on incompressible viscoelastic fluid flow (Rivlin-Ericksen) between two infinite vertical parallel plates moving in opposite direction, while one plate is oscillating in time and magnitude about a constant non-zero mean under the influence of a uniform transverse magnetic field. The suction velocity at the plate fluctuates with the time harmonically from a constant mean velocity. We have evaluated velocity distribution, temperature distribution, concentration distribution, phase and amplitude of the skin friction. The effects of magnetic parameter M, visco-elastic parameter S, Schmidt number S_c, Grashof number G_r, modified Grashof number G_c, and Prandtl number P_r on the above said physical quantities are discussed.

INTRODUCTION

The mechanical behaviour of a large number of materials of industrial importance such as synthetic fibers, high polymer solutions and many other highly viscous fluids cannot be explained fully by the classical linear stress-strain relation. Attempts in the past especially in the last two decades, have been made to formulate more general non-linear theories which could take fully into account the observed behaviour of the non-Newtonian and elastic viscous fluids. Consequently large number of flow problems has been solved by using such general theories either for some theoretical interest or from the point of view of comparing the experimental results with the prediction of various theories. As the general stress-strain relations are expressed by highly complicated non-linear differential equations, to work out solutions for such a class of fluids even for small flows is not an early task.

Generalizing the stress-strain velocity relations of classical hydrodynamics the rheological behaviour of the non-Newtonian liquids has been studied by Rivlin [12] and Reiner [11].On account of the non-linear nature of the equation of state of even the simplest elastic viscous fluid, it is almost impossible to obtain the exact solution of the equation of motion and one has to resort to the approximate methods. In most of the investigations of elastic viscous fluids, the flow has been considered slow and parameters characterizing the elastic properties of the fluid have been assumed small. In fact the increase emergence of non-Newtonian fluids such as molten plastics, pulps, emulsions, aqueous solutions of polyacrylamid and polyisobutylene etc., as important raw materials and chemical products is a large variety of industrial processes has stimulated a considerable attention in recent years to the study of non- Newtonian fluids and their related transport process.

The two dimensional unsteady flow of an incompressible viscous fluid, when the free stream is oscillating about a non-zero constant mean, has been studied by Karman-pohlausen method by Lighthill [8].Taking the free stream oscillations into account, Stuart [6] has analyzed the forced flow and heat transfer from an infinite porous plate. Soundalgekar [14] has studied the free convection effects on the oscillatory flow of an incompressible viscous fluid past an infinite vertical plate with variable section. Krishnanlal [7] has investigated the application of fluctuating section to free stream laminar flow past a porous vertical wall. Sastri and Bhadram [13] have studied hydromagnetic convective heat transfer in vertical pipes.

Georgantopoules *et al.* [5] have discussed the effects of free convection and mass transfer in a conducting liquid when the fluid is subjected to a transverse magnetic field. Raptis [10] has studied free convection and mass transfer effects on the flow past an infinite moving vertical porous plate with constant suction and heat sources when free stream velocity is an oscillatory function of time. Agarwal *et al.* [3] have discussed the combined buoyancy effects of thermal and mass diffusion on MHD convection flows. Vajravelu [17] has studied the problem of free convection heat transfer between two long vertical plates moving in opposite direction. Agarwal and kishore [2] studied thermal and mass diffusion on MHD natural convection flow between two infinite vertical moving and oscillating porous parallel plates. Johri [6] has considered the flow of visco-elastic fluid induced by elliptic harmonic oscillations of a disc. He also studied the unsteady channel flow of an elastic viscous liquid. Recently, Asghar *et al.* [4] discussed the flow of a non-Newtonian fluid induced due to the oscillations of a porous plate. Ogulu *et al.* [9] studied the Heat transfer to unsteady magneto-hydrodynamic flow past an infinite moving vertical plate with variable suction and Abbas *et al.* [1] studied Hydromagnetic flow in a viscoelastic fluid due to the oscillatory stretching surface

In this paper, we study the thermal and mass diffusion on incompressible visco-elastic fluid flow (Rivlin-Ericksen) between two infinite vertical parallel plates moving in opposite direction, while one plate is oscillating in time and magnitude about a constant non-zero mean under the influence of a uniform transverse magnetic field. The suction velocity at the plate fluctuates with the time harmonically from a constant mean velocity. We have evaluated velocity distribution, temperature distribution, concentration distribution, phase and amplitude of the skin friction. The effects of magnetic parameter M, visco-elastic parameter S, Schmidt number S_c, Grashof number G_r, modified Grashof number G_c, and Prandtl number P_r on the above said physical quantities are discussed in section 3.

2. Rheological equations of the state:
The constitute equation for a Rivlin-Ericksen fluid is

$$T = -PI + \phi_1 d + \phi_2 b + \phi_3 d^2 \quad (2.1)$$

where

$T = \| T_{ij} \|$, T_{ij} is stress tensor
$I = \| \delta_{ij} \|$, δ_{ij} is Kroneckar delta

$d = \|d_{ij}\|$, $d_{ij} = (1/2)(W_{i,j} + W_{j,i})$ = deformation rate tensor
$b = \|b_{ij}\|$, $b_{ij} = a_{i,j} + a_{j,i} + 2W_{m,i} W_{m,j}$ = viscoelastic parameter with
$\overline{a_i} = \frac{\partial W_i}{\partial t} + W_{i,j} W_j$ = acceleration vector

The $ø_1$, $ø_2$, $ø_3$ are coefficients of viscosity, visco-elasticity and cross-viscosity respectively and these are in general functions of temperature, material properties and invariants of d, b, d^2. For many liquids aqueous solutions of polyacralamid and polybutylene, the coefficients $ø_1$, $ø_2$, $ø_3$ may be taken as constant.

3. Formulation and solution of the problem:

We consider the flow of thermal and mass diffusion on viscoelastic (Rivlin-Ericksen) fluid between two infinite parallel plates moving in opposite direction, while one place is oscillating in time and magnitude about a constant non-zero mean, under the influence of a uniform transverse magnetic field. The x-axis is taken along an infinite flat plate moving vertically upwards and a straight line perpendicular to that as y-axis. The magnetic field of small intensity H_o is introduced in the y-direction. Since the fluid is slightly conducting, the magnetic Reynolds number is much less than unity and hence the induced magnetic field is neglected in comparison with the applied magnetic field (Sparrow and Cess [15]). In the absence of an input electric field, the equations governing the motion of the conducting Rivlin-Ericksen fluid are given by

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} = g\beta^*(T - T_s) + g\beta^{**}(C - C_s) + \upsilon\frac{\partial^2 u}{\partial y^2} + \beta\frac{\partial^2}{\partial y^2}\left[\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y}\right] - \frac{\sigma\mu_e^2 H_0^2}{\rho}u \quad (3.1)$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 2(2\beta + \gamma)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \quad (3.2)$$

$$\frac{\partial v}{\partial y} = 0 \quad (3.3)$$

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y} = \frac{K}{\rho C_P}\frac{\partial^2 T}{\partial y^2} \quad (3.4)$$

$$\frac{\partial C}{\partial t} + v\frac{\partial C}{\partial y} = D\frac{d^2 C}{\partial y^2} \quad (3.5)$$

where u, v are the velocity components along the axes of components along the axes of coordinates respectively, g the acceleration due to gravity, β* the coefficient of thermal expansion, β** the volumetric coefficient of expansion with concentration, T the temperature of the plate, T_s the temperature at second plate, υ the kinematic viscosity, β the visco- elasticity, t the time, σ the electrical conductivity of the fluid, μ_e the magnetic permeability ,ρ the density of the fluid, P the fluid pressure , γ the kinematic cross viscosity, C_P the specific heat at constant pressure and K the thermal conductivity of the fluid. The continuity equation (3.3) shows that v is a function of time only. In order to obtain a steady state solution of the boundary layer type it is known that v must be a negative non-zero constant v_0 .In the unsteady case also we shall make this restriction (Stuart [16]).

The boundary conditions are

$$u = U_0,\ T = T_w,\ C = C_w \ \text{ at } \ y = 0 \quad (3.6a)$$

$$u = -U(t) = -U_0\left(1+\in e^{i\omega t}\right),\quad T = T_S,\quad C = C_S \quad \text{at} \quad y = d \qquad (3.6b)$$

From the continuity equation, we get (3.3)

$$v = +v_0 \qquad (3.7)$$

We now introduce the following non-dimensional quantities

$$y^* = \frac{yv_0}{\upsilon},\ m = \frac{dv_0}{\upsilon},\ t^* = \frac{v_0^2\, t}{4\upsilon},\ w^* = \frac{4\upsilon w}{v_0^2}$$

$$u^* = \frac{u}{U_0},\ v^* = \frac{v}{v_0},\ \theta^* = \frac{T-T_s}{T_w-T_s},\ C^* = \frac{C-C_S}{C_W-C_S} \qquad (3.8)$$

In view of the equation (3.8), the equation (3.1) to (3.5) reduce to (dropping the super scripts *)

$$\frac{1}{4}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} = G_r\theta + G_c C + \frac{\partial^2 u}{\partial y^2} - S_0\frac{\partial^2}{\partial y^2}\left(\frac{1}{4}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}\right) - Mu \qquad (3.9)$$

$$\frac{1}{4}P_r\frac{\partial \theta}{\partial t} + P_r\frac{\partial \theta}{\partial y} = \frac{\partial^2 \theta}{\partial y^2} \qquad (3.10)$$

$$\frac{1}{4}S_c\frac{\partial C}{\partial t} + S_c\frac{\partial C}{\partial y} = \frac{\partial^2 C}{\partial y^2} \qquad (3.11)$$

where $P_r = \mu C_p / K$ Prandtl number

$S_c = \upsilon / D$ Schmidt number

$G_r = [v g \beta^* (T_w - T_s)] / (U_0 v_0^2)$ Grashof number

$S = \dfrac{\beta v_0^2}{\upsilon_2^2}$ Visco-elastic parameter

$M = \dfrac{\sigma \mu_e^2 H_0^2 v}{\rho v_0^2}$ Magnetic parameter

$G_c = \left[\upsilon g \beta^{**}(C_\omega - C_S)\right] / \left(U_0 v_0^2\right)$ Modified Grashof number

$S = -S_0$ is the visco-elastic parameters $0 \le S_0 \le \frac{1}{4}$ for Rivlin- Ericksen second order fluid the visco-elasticity parameter S is necessarily negative.

The non dimensional boundary conditions are

$u = 1,\ \theta = 1,\ C = 1$ at $y = 0$ (3.12a)

$u = -\left(1+\in e^{i\omega t}\right),\ \theta = 0,\ C = 0$ at $y = m$ (3.12b)

Substituting equations (3.13) to (3.15) in to (3.9) to (3.11) and comparing the harmonic and non-harmonic term we obtain

$$C = C_0\left(y\right) = \frac{1}{1-e^{(S_c m)}}\left(e^{s_c y} - e^{s_c m}\right) \quad (3.16)$$

$$S_0 u_0''' - u_0'' + u_0' + Mu_0 = G_r\theta_0 + G_c C_0 \quad (3.17)$$

$$S_0 u_1''' + \left[\left(S_0 i\omega/4\right) - 1\right] u_1'' + u_1' + \left[M + \left(i\omega/4\right)\right] u_1 = G_r\theta_1 + G_c C_1 \quad (3.18)$$

$$\theta_0'' - \mathrm{P}_r\,\theta_0' = 0 \quad (3.19)$$

$$\theta_0'' - \mathrm{P}_r\,\theta_1' - \left(i\omega/4\right)\mathrm{P}_r\,\theta_1 = 0 \quad (3.20)$$

In view of (3.13) to (3.15), the boundary conditions (3.12) reduces to

$u_0 = 1$, $u_1 = 0$, $\theta_0 = 0$, $\theta_1 = 0$, $C_0 = 1$, $C_1 = 0$ at $y = 0$ (3.21a)

$u_0 = -1$, $u_1 = -1$, $\theta_0 = 0$, $\theta_1 = 0$, $C_0 = 0$, $C_1 = 0$ at $y = m$ (3.21b)

Following Lighthill [8] we assume the solution in the form

$$u_0 = u_{01} + S_0\, u_{02} + 0\left(S_0^2\right)$$

$$u_1 = u_{11} + S_0\, u_{12} + 0\left(S_0^2\right) \quad (3.22a)$$

$$\theta_0 = \theta_{01} + S_0\theta_{02} + 0\left(S_0^2\right)$$

$$\theta_1 = \theta_{11} + S_0\theta_{12} + 0\left(S_0^2\right) \quad (3.22b)$$

Substituting (3.22) in equations (3.17) to (3.20) and equating the coefficients of S_0, we obtain (neglecting the terms of $0(S_0^2)$ and higher term).

$$-u_{01}'' + u_{01}' + Mu_{01} = G_r\theta_{01} + \frac{G_c}{1-e^{s_c m}}\left(e^{s_c y} - e^{s_c m}\right) \quad (3.23)$$

$$u_{01}''' - u_{02}'' + u_{02}' + Mu_{02} = G_r\theta_{02} \quad (3.24)$$

$$u_{11}'' - u_{11}'' - T_1 u_{11} = -G_r\theta_{11} \quad (3.25)$$

$$u_{11}'' + \frac{iw}{4}u_1'' - u_{12}'' + u_{12}' + T_1 u_{12} = G_r\theta_{12} \quad (3.26)$$

$$\theta_{01}'' - P_r\theta_{01}' = 0 \quad (3.27)$$

$$\theta_{02}'' - P_r\theta_{02}' = 0 \quad (3.28)$$

$$\theta_{11}'' - P_r\theta_{11}' - \frac{iw}{4}P_r\theta_{11} = 0 \quad (3.29)$$

$$\theta_{12}'' - P_r\theta_{12}' - \frac{iw}{4}P_r\theta_{12} = 0 \quad (3.30)$$

where $T_1 = M + \left(i\omega/4\right)$

In view of (3.21) the boundary conditions (3.12) reduce to

$$u_{01} = 1,\ u_{02} = 0,\ \theta_{01} = 1,\ \theta_{02} = 0,\ u_{11} = 0,$$

$$u_{12} = 0,\ \theta_{11} = 0,\ \theta_{12} = 0 \quad at\ \ y = 0 \quad (3.31a)$$

$$u_{01} = -1,\ u_{02} = 0,\ \theta_{01} = 0,\ \theta_{11} = 0,\ u_{11} = -1,$$

$$u_{12} = 0,\ \theta_{02} = 0,\ \theta_{12} = 0 \quad at\ \ y = m \quad (3.31b)$$

Solving equations (3.23) to (3.30) by using the boundary condition (3.31), we obtain the velocity and the temperature distributions as

$$u(y,t) = u_0(y) + \in (M_r Cos\ \omega t - M_i Sin\omega t) \quad (3.32)$$

$$where\ M_r + iM_i = u_1(y)$$

$$\theta = \theta_0(y) = \frac{1}{1-e^{(P_r m)}}\left(e^{p_r y} - e^{p_r m}\right)$$

The equations for u_0 and u_1 are

$$u_0(y) = T_{15}e^{T_4 Y} + T_{16}e^{T_5 Y} + T_{17}e^{P_r Y} + T_{18}e^{S_C Y} + T_8$$

$$u_1(y) = e^{A_1 Y}(K_{23}\ Cos(b/2)\ y - K_{24}\ Sin(b/2)\ y) - e^{A_2 Y}(K_{25}\ Cos(b/2)\ y + K_{26}\ Sin(b/2)\ y$$

$$+ i\,[\ e^{A_1 Y}(K_{24}\ Cos(b/2)\ y + K_{23}\ Sin(b/2)\ y) + e^{A_2 Y}(-K_{26}\ Cos(b/2)\ y + K_{25}\ Sin(b/2)\ y] \quad (3.33)$$

The various constants are given in Appendix before the reference section of the text.

Transient velocity can be deduced from equation (3.32), when $\omega t = \pi/2$,

$$u = u_0(y) - \in M_i \quad (3.34)$$

Skin Friction

The non-dimensional form of the shear stress at the plates is

$$\tau = \left(\frac{\partial u_0}{\partial y} + \in e^{i\omega t}\frac{\partial u_1}{\partial y}\right)\Bigg]_{y=m}^{y=0} \quad (3.35)$$

$$\tau = \left[(\tau_m)\right]_{y=m}^{y=0} + \in e^{i\omega t}\Big[e^{A_1 y}\left(-K_{27}Sin(b/2)\,y + K_{28}Cos(b/2)\,y\right)$$

$$- e^{A_2 y}\left(-K_{29}Sin(b/2)\,y + K_{30}Cos(b/2)\,y + i\left\{e^{A_1 y}\left(K_{28}Sin(b/2)\,y\right.\right.\right. \quad (3.36)$$

$$+K_{27}Cos(b/2)\,y + e^{A_2 y}(K_{30}Sin(b/2)\,y - K_{29}Cos(b/2)\}y)\Big]_{y=m}^{y=0}$$

where mean shear stress is

$$\tau_m = \left[T_4 T_{15}e^{T_4 y} + T_5 T_{16}e^{T_5 y} + T_{17}P_r e^{P_r y} + T_{18}S_c e^{S_c y}\right]_{y=m}^{y=0} \quad (3.37)$$

The amplitude |B| of the skin-friction at the plates is given by

$$B = B_r + iB_i$$

$$= \Big[e^{A_1 y}\left(-K_{27}\ Sin(b/2)\,y + K_{28}Cos(b/2)\,y\right) - e^{A_2 y}(-K_{27}\ Sin(b/2)\,y + K_{28}Cos(b/2)\,y$$

$$+ i\,\{\ e^{A_1 y}(K_{28}\ Sin(b/2)\,y + K_{27}Cos(b/2)\,y + e^{A_2 y}(K_{30}\ Sin(b/2)\,y - K_{29}Cos(b/2)\}y)]_{y=m}^{y=0}$$

(3.38)

where

$$K_{27} = A_1 K_{24} + (K_{23}b/2) \qquad K_{28} = A_1 K_{23} - (K_{24}b/2)$$

$$K_{29} = (K_{23}b/2) - A_2 K_{26} \qquad K_{30} = (K_{26}b/2) + A_2 K_{25}$$

The phase of the skin friction is

$Tan\,\alpha = Bi / Br$

CONCLUSION

In fig.1, the fluctuating part of the velocity profile M_i is drawn against y for different values of magnetic parameter M. We have observed that M_i increases with the increase in M. Further we have also seen that M_i increases with the increase in y. From table 1, we conclude that the fluctuating part of the velocity profile M_r decreases with the increase in M. In fig.2, we have drawn M_i against y for different values of visco-elastic parameter S_0.We have noticed that M_i increases with the increase in S_0. From table 2, we conclude that M_r increases with the increase in S_0. Further we have also seen that M_r first increases as y increases up to y=12 and then the trend gets reversed. In figures (3), (4), (5), (6), (7) and table (3) velocity distribution u is drawn against y for different values of magnetic parameter M, Grashof number G_r, modified Grashof number G_c, Schmidt number S_c, Prandtl number P_r and visco-elastic parameter S_0 respectively. We have noticed that u increases first and then decreases with the increases in y. We have also seen that the velocity increases with the increase in G_r or G_c or S_c or P_r or S_0, whereas the velocity decreases with the increase in M. From fig.8, we conclude that the temperature distribution profile T increases with the increase in P_r. Further we noticed that T decreases with the increase in y. In fig.9, the concentration distribution profile C is drawn against y for different values of S_c .We have observed that C increases with the increase in S_c, where as it decreases with increase in y. In fig.10, the amplitude of the skin friction |B| is drawn against S_0 for different values of M. We have seen that |B| increases with the increase in M, whereas it decreases with the increase in S_0.in fig.11, the phase lead of the skin friction Tan α at the plate is drawn against S_0 for different values of M. We have noticed that Tan α decreases with the increase in M, whereas it increases with the increase in S_0

5. GRAPHS

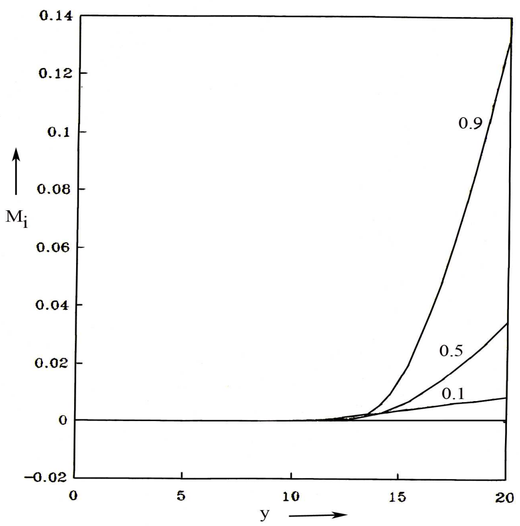

Fig.1. M_i against y for different M

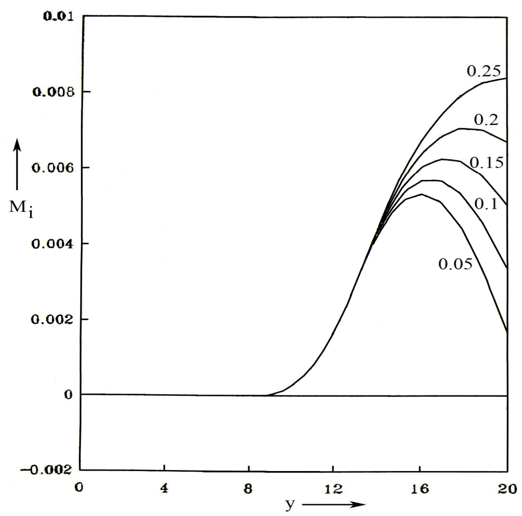

Fig. 2. M_i against y for different S_o

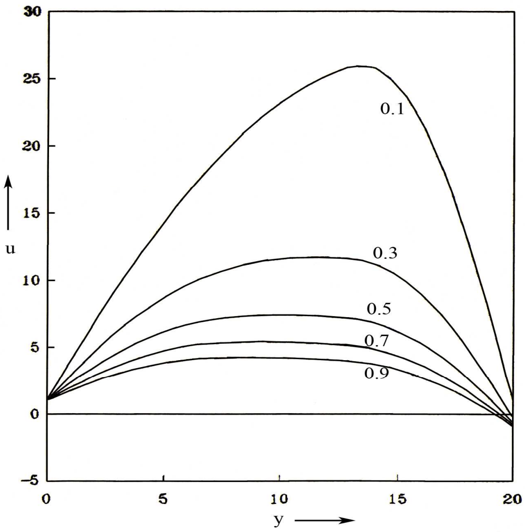

Fig. 3. u against y for different M

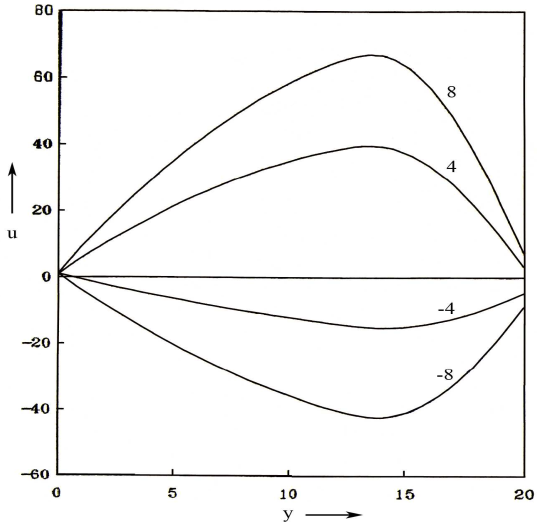

Fig.4. u against y for different G_r

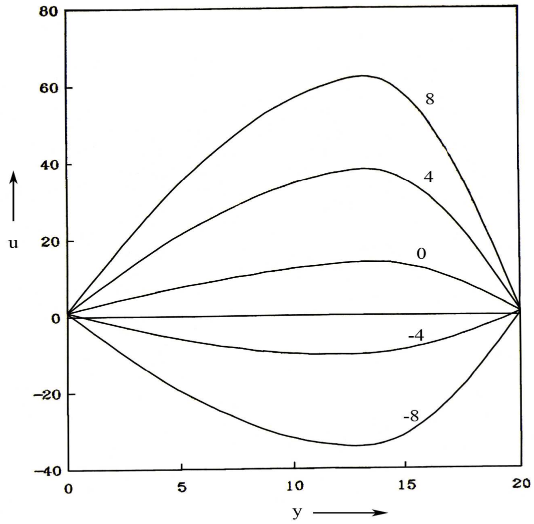

Fig. 5. u against y for different G_c

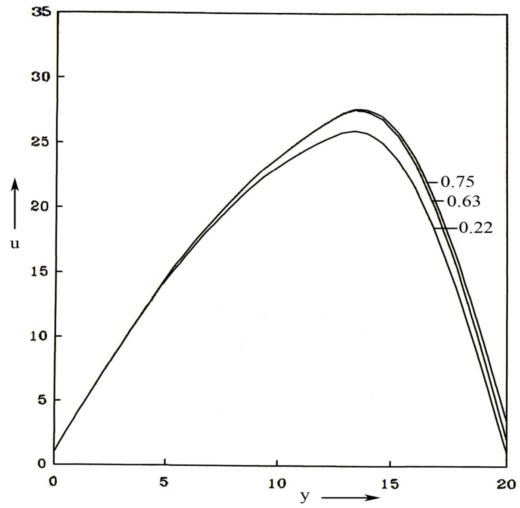

Fig. 6. u against y for different S_c

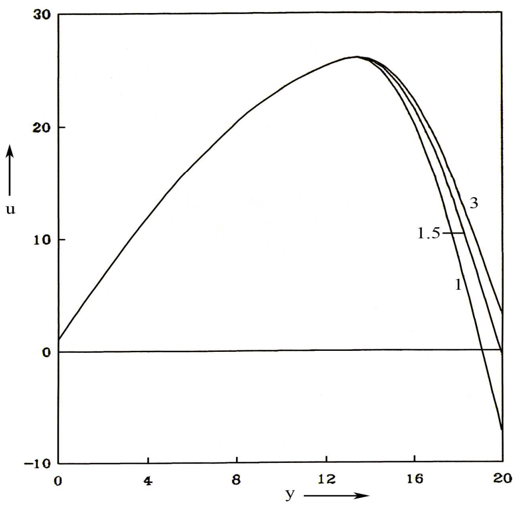

Fig. 7. u against y for different P_r

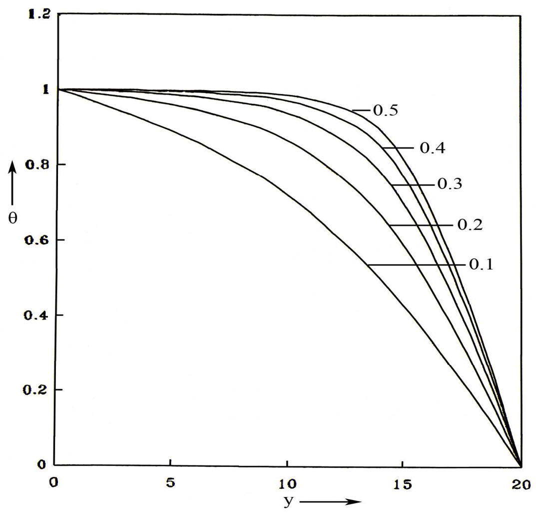

Fig. 8. θ against y for different P_r

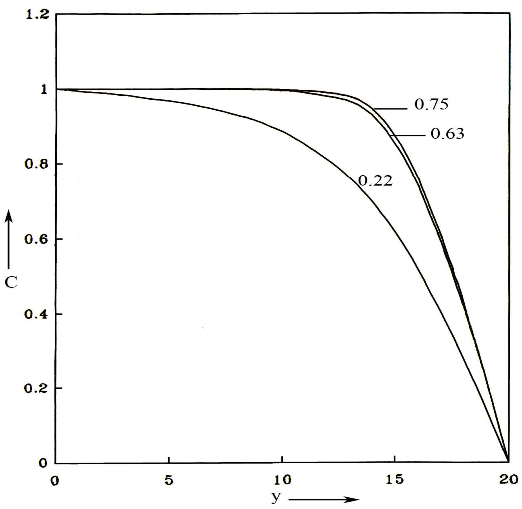

Fig. 9. C against y for different S_c

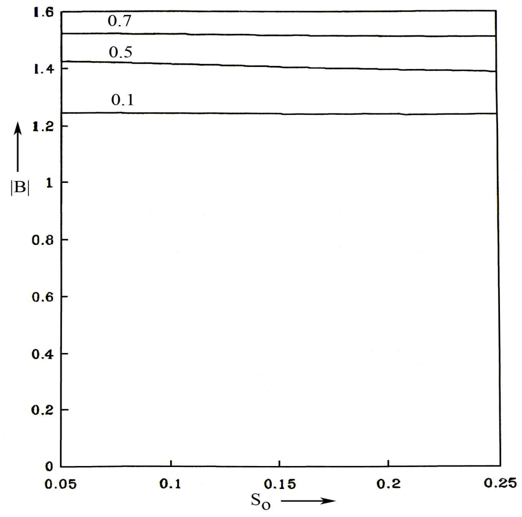

Fig. 10. |B| against S_o for different M

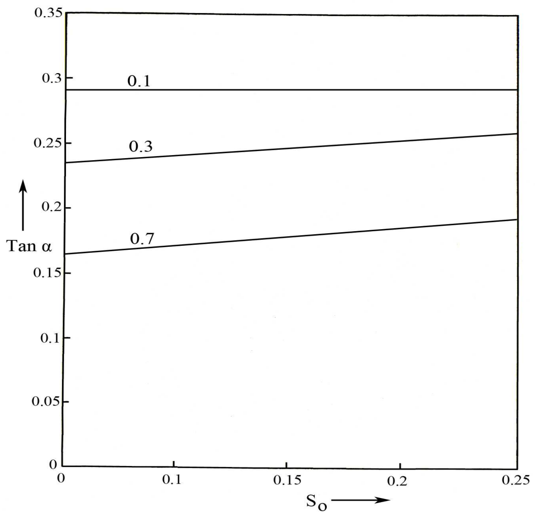

Fig. 11. Tanα against S_o for different M

TABLE 1 M_r against y for different M

M	y=0	y=4	y=8	y=12	y=16	y=20
0.1	0	-4.48829E-09	2.20573E-07	6.906126E-05	-1.000584E-03	-0.9937678
0.3	0	-2.039326E-10	1.351622E-07	2.296075E-05	-1.870739E-03	-1.001581
0.5	0	+4.271583E-11	4.318385E-08	7.617278E-06	-1.452229E-03	-0.9655896
0.7	0	+2.332164E-11	1.382289E-08	2.248725E-06	-1.345178E-03	-0.9893848
0.9	0	+6.628039E-12	4.711978E-09	1.089853E-06	-8.663931E-04	-1.012504

TABLE 2 M_r against y for different S_0

S_0	y=0	y=4	y=8	y=12	y=16	y=20
0.05	0	-4.49374E-09	2.254847E-07	6.928536E-05	-1.062382E-03	-0.9987536
0.1	0	-4.492378E-09	2.242567E-07	6.922934E-05	-1.046932E-03	-0.9975071
0.15	0	-4.491015E-09	2.230288E-07	6.917333E-05	-1.031482E-03	-0.9962606
0.2	0	-4.489653E-09	2.218009E-07	6.91173E-05	-1.016033E-03	-0.9950142
0.25	0	-4.48829E-09	2.20573E-07	6.906126E-05	-1.000584E-03	-0.9937678

TABLE 3 u against y for different S_0

S_0	y=0	y=4	y=8	y=12	y=16	y=20
0.05	1	12.88448	20.90018	25.90065	27.49440	-0.6014626
0.1	1	12.88492	20.90130	25.90446	27.52281	-0.2043802
0.15	1	12.88536	20.90242	25.90828	27.55121	0.1927022
0.2	1	12.88580	20.90354	25.91210	27.57962	0.5897846
0.25	1	12.88623	20.90466	25.91593	27.60803	0.9868669

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Appendix:
The list of the constants appearing in the text.

$$T_2 = \frac{G_r}{e^{P_r m} - 1}, \qquad T_3 = \frac{G_c}{e^{P_r m} - 1}$$

$$T_4 = \frac{1 + \sqrt{1 + 4M}}{2}, \qquad T_5 = \frac{1 - \sqrt{1 + 4M}}{2}$$

$$T_6 = \frac{T_2}{P_r^2 - P_r - M}, \qquad T_7 = \frac{T_3}{S_c^2 - S_c - M}$$

$$T_8 = \frac{1}{M}\left(T_2 e^{P_r m} + T_3 e^{S_c m}\right), \qquad T_9 = \frac{C_1 T_4^3}{T_4^2 - T_4 - M}$$

$$T_{10} = \frac{C_2 T_5^3}{T_5^2 - T_5 - M}, \qquad T_{11} = \frac{T_6 P_r^3}{P_r^2 - P_r - M}$$

$$T_{12} = \frac{T_7 S_c^3}{S_c^2 - S_c - M}, \quad T_{13} = C_1 + T_9, \quad T_{14} = C_4 + T_{10}$$

$$T_{15} = C_1 + S_0 T_{13}, \quad T_{16} = C_2 + S_0 T_{14}, \quad T_{17} = T_6 + S_0 T_{11}$$

$$T_{18} = T_7 + S_0 T_{12}$$

$$C_1 = \frac{1}{\left(e^{T_4 m} - e^{T_5 m}\right)}\left[T_6\left(e^{T_5 m} - e^{P_r m}\right) + T_7\left(e^{T_5 m} - e^{S_c m}\right) + T_8\left(e^{T_5 m} - 1\right) - \left(1 + e^{T_5 m}\right)\right]$$

$$C_2 = \frac{1}{\left(e^{T_4 m} - e^{T_5 m}\right)}\left[T_6\left(e^{P_r m} - e^{T_4 m}\right) + T_7\left(e^{S_c m} - e^{T_4 m}\right) + T_8\left(1 - e^{T_4 m}\right) - \left(1 + e^{T_4 m}\right)\right]$$

$$C_3 = \frac{1}{\left(e^{T_4 m} - e^{T_5 m}\right)}\left[T_9\left(e^{T_5 m} - e^{T_4 m}\right) + T_{11}\left(e^{T_5 m} - e^{P_r m}\right) + T_{12}\left(e^{T_5 m} - e^{S_c m}\right)\right]$$

$$C_4 = \frac{1}{\left(e^{T_4 m} - e^{T_5 m}\right)}\left[T_{10}\left(e^{T_5 m} - e^{T_4 m}\right) + T_{11}\left(e^{P_r m} - e^{T_4 m}\right) + T_{12}\left(e^{S_c m} - e^{T_4 m}\right)\right]$$

$$T_{19} = \left(e^{A_2 m} - e^{A_1 m}\right) Cos\left(b/2\right)m, \qquad T_{20} = \left(e^{A_2 m} - e^{A_1 m}\right) Sin\left(b/2\right)m$$

$$T_{21} = \frac{T_{19}}{T_{19}{}^2 + T_{20}{}^2}, \qquad T_{22} = \frac{T_{20}}{T_{19}{}^2 + T_{20}{}^2}$$

$$L_1 = 1 + 4M$$

$$a = \sqrt{\frac{1}{2}\left[L_1 + \sqrt{L_1{}^2 + W^2}\right]} \qquad b = \sqrt{\frac{1}{2}\left[\sqrt{L_1{}^2 + W^2} - L_1\right]}$$

$$A_1 = (1+a)/2, \quad A_2 = (1-a)/2, \quad A_3 = (3b)/2, \quad A_4 = \left(3b^2\right)/4,$$

$$A_5 = b^3/8, \qquad A_6 = A_1^3 - A_4 A_1 - \left(\omega b A_1/4\right),$$

$$A_7 = A_2^3 - A_4 A_2 + \left(\omega b A_2/4\right), \quad A_8 = A_3 A_1^2 - A_5 + \left(\omega A^2/4\right) - \left(\omega b^2/16\right),$$

$$A_9 = A_3 A_2^2 - A_5 + \left(\omega A_2{}^2/4\right) + \left(\omega b^2/16\right), \ A_{10} = T_{21} A_6 - T_{22} A_8$$

$$A_{11} = T_{21} A_7 + T_{22} A_9, \quad A_{12} = T_{21} A_8 + T_{22} A_6, \quad A_{13} = T_{21} A_9 - T_{22} A_7,$$

$$A_{14} = A_1^2 - A_1 - \left(b^2/4\right), \ A_{15} = \left(A_{14} - M\right)/\left(2A_1 - 1\right), \ A_{16} = \left[\omega/4\left(2A_1 - 1\right)\right]$$

$$A_{17} = \frac{1}{\left(1 - 2A_1\right)}\left[\frac{b^2\left(2A_1 - 1\right)^2}{4} + \left(A_{14}^2 + M^2 - \frac{\omega^2}{16}\right) - 2A_{14} M\right]$$

$$A_{18} = \frac{\omega\left(M - A_{14}\right)}{2\left(2A_1 - 1\right)}, \quad A_{19} = \frac{b}{2\left(2A_{17}^2 + A_{18}^2\right)}$$

$$A_{20} = \frac{A_{15}}{A_{17}^2 + A_{18}^2}, \qquad A_{21} = \frac{A_{16}}{A_{17}^2 + A_{18}^2}, \qquad A_{22} = A_2^2 - A_2 - \frac{b^2}{4}$$

$$A_{23} = \frac{A_{22} - M}{2A_2 - 1}, \qquad A_{24} = \frac{\omega}{4(2A_2 - 1)}$$

$$A_{25}=\frac{1}{(1-2A_2)}\left[\frac{b^2(2A_2-1)^2}{4}+\left(A_{22}^2+M^2-\frac{\omega^2}{16}-2A_{22}M\right)\right]$$

$$A_{26}=\frac{\omega(M-A_{22})}{2(2A_2-1)},\qquad A_{27}=\frac{b}{2\left(A_{25}^2+A_{26}^2\right)}$$

$$A_{28}=\frac{A_{23}}{A_{25}^2+A_{26}^2},\qquad A_{29}=\frac{A_{24}}{A_{25}^2+A_{26}^2}$$

$$B_1=A_{17}A_{19}A_{10}\qquad B_{13}=A_{12}A_{17}A_{19}\qquad K_1=B_1-B_{14}-B_{15}$$

$$B_2=A_{17}A_{20}A_{10}\qquad B_{14}=A_{12}A_{17}A_{20}\qquad K_2=B_2+B_3+B_{13}$$

$$B_3=A_{18}A_{21}A_{10}\qquad B_{15}=A_{12}A_{18}A_{21}\qquad K_3=B_4-B_{17}+B_{18}$$

$$B_4=A_{18}A_{19}A_{10}\qquad B_{16}=A_{18}A_{19}A_{12}\qquad K_4=B_5-B_6+B_{16}$$

$$B_5=A_{18}A_{20}A_{10}\qquad B_{17}=A_{12}A_{18}A_{20}\qquad K_5=B_7-B_{20}-B_{21}$$

$$B_6=A_{17}A_{21}A_{10}\qquad B_{18}=A_{12}A_{17}A_{21}\qquad K_6=B_8+B_9+B_{19}$$

$$B_7=A_{11}A_{25}A_{27}\qquad B_{19}=A_{13}A_{25}A_{27}\qquad K_7=B_{10}-B_{23}+B_{24}$$

$$B_8=A_{11}A_{25}A_{28}\qquad B_{20}=A_{13}A_{25}A_{28}\qquad K_8=B_{11}-B_{12}+B_{22}$$

$$B_9=A_{11}A_{26}A_{29}\qquad B_{21}=A_{13}A_{26}A_{29}\qquad K_9=K_1-K_4$$

$$B_{10}=A_{11}A_{26}A_{27}\qquad B_{22}=A_{13}A_{26}A_{27}\qquad K_{10}=K_2+K_3$$

$$B_{11}=A_{11}A_{26}A_{28}\qquad B_{23}=A_{13}A_{26}A_{28}\qquad K_{11}=K_5+K_8$$

$$B_{12}=A_{11}A_{25}A_{29}\qquad B_{24}=A_{13}A_{26}A_{29}\qquad K_{12}=K_6-K_7$$

$$K_{13}=K_{11}T_{20}+K_{12}T_{19}\qquad K_{19}=K_{18}-K_9$$

$$K_{14}=K_{11}T_{19}-K_{12}T_{20}\qquad K_{20}=K_{17}-K_{10}$$

$$K_{15}=\frac{K_{13}T_{19}-K_{14}T_{20}}{T_{19}^2+T_{20}^2}\qquad K_{21}=K_{11}-K_{16}\qquad K_{22}=K_{12}-K_{15}$$

$$K_{16}=\frac{K_{13}T_{20}+T_{19}K_{14}}{T_{19}^2+T_{20}^2}\qquad K_{23}=T_{21}+S_oK_{20}\qquad K_{24}=T_{22}-S_oK_{19}$$

$$K_{17}=K_{10}-K_{12}+K_{15}\qquad K_{25}=T_{21}-S_oK_{22}$$

$$K_{18}=K_9+K_{11}-K_{16}\qquad K_{26}=T_{22}-S_oK_{21}$$