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HAM for finding solutions to coupled system of variable coefficient equations arising in fluid dynamics

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ABSTRACT

The present paper applies the homotopy analysis method (HAM) for finding solutions to a coupled system of variable coefficient equations that arise in the problems of fluid dynamics. Flow of fluids particularly non-newtonian fluids through pipes is a problem that has wide range of application. The mathematical formulations of these problems generally give rise to non-linear (and/or) coupled (and/ or) variable coefficient equations. Thus finding exact solutions of these problems is almost impossible. Thus researchers sought to numerical or approximate analytical method for solving them. Here, in the present study, the flow of micropolar fluid in a rigid circular tube is considered and an approximate analytical solution is found. The effect of the fluid flow parameter, micro rotation parameter and the pressure gradient on the velocity and micro rotation of the fluid are studied. The results are presented through graphs.

Keywords: HAM; Micropolar fluid;non-linear coupled equations;

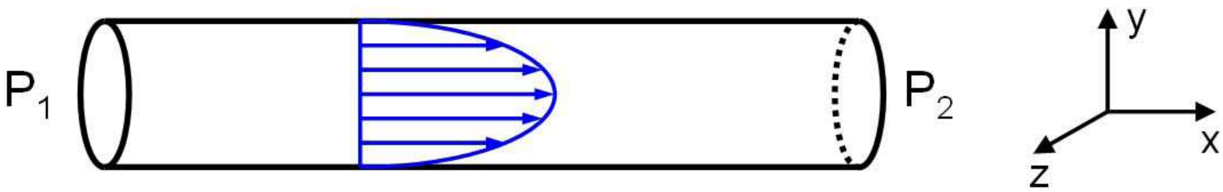
INTRODUCTION

The study of incompressible viscous fluid flow in rigid pipes is a renowned classical problem. An exact steady state solution to this problem, under certain valid assumptions can be obtained by solving the governing equations called Navier Stokes equations [1]. In practice, majority of the fluids found in nature as well as in industrial or medical applications are non- Newtonian. The flow of these fluids through rigid pipes have enormous applications in polymer processing industries, bio medical engineering etc., Thus there is a definite need to undertake the study the flow of these non-Newtonian fluids through pipes. Unlike the steady viscous flow, here, in these problems, it is difficult to relate the instantaneous velocity profiles and the volumetric flow rate to the instantaneous pressure gradient. Further, in some cases the stress – shear rate relation may be highly non- linear and may also be a function of time. These make the problem of pipe flow of non- Newtonian fluids even more complicated and hence mathematically intractable. Inspite of the difficulties in obtaining exact analytical solutions to the above mentioned problems, due to their practical importance, these problems have been attempted by several researchers from diversified fields and they tried finding the solutions by making certain assumptions in order to get an analytical solution or a numerical solution [2-5].

Very recently, some promising approximate analytical methods such as Homotopy analysis method, Homotopy perturbation method, Optimal Homotopy asymptotic method (OHAM) have been proposed [6-14]. These methods can be treated as intermediate methods for the exact analytical methods and the numerical methods and have several advantages over them. These approximate analytical methods, though cannot provide closed form of solutions as the exact analytical methods, can provide good approximate expressions for the solution and hence can solve a wide class of problems whose solutions are almost impossible by the classical methods. Obviously they form the better choice than the numerical methods in many cases, due to the limitations of the numerical methods.

In the present paper, the flow of a non- Newtonian fluid namely Micropolar fluid is considered for study. This fluid model was proposed by Erigen inorder to explain the behavior of real fluids in certain contexts. Eringen has proposed this theory of micropolar fluids [15] in 1966 andLukaszewicz described this fluid model as a well-founded and significant generalization of the classical Navier Stokes model covering both in theory and applications, many more phenomena than the classical one can [16].

Mathematical Formulation:
Consider the flow of incompressible micropolar fluid in a rigid circular pipe under a constant pressure gradient. A schematic diagram of the problem can be found in fig(1). The flow is assumed to be laminar.



Fig(I): Schematic diagram of fluid flow

The equations governing the flow of micropolar fluid are [15]

$$\frac{\partial \rho}{\partial t} + div(\rho\,\vec{q}) = 0 \tag{1}$$

$$\rho \frac{d\vec{q}}{dt} = \rho\,\vec{f} - grad\;\, p + k\; curl\,\vec{v} - (\mu + k)\, curl\;\; curl\,\vec{q} + (\lambda_1 + 2\mu + k) grad\; div\,\vec{q} \tag{2}$$

$$\rho\, j \frac{d\vec{v}}{dt} = \rho\,\vec{l} - 2k\vec{v} + k\; curl\,\vec{q} - \gamma\, curl\; curl\,\vec{v} + (\alpha + \beta + \gamma) grad\; div\,\vec{v} \tag{3}$$

in which $\vec{q}\,,\vec{v}$ are velocity and microrotation vectors, $\bar{f}\,,\bar{l}$ are body force per unit mass, body couple per unit mass respectively and *p* is the fluid pressure at any point. ρ and*j* are density of the fluid and gyration parameters respectively and are assumed to be constants. The material constants (λ_1, μ, k) are viscosity coefficients and (α, β, γ) are gyroviscosity coefficients. These constants confirm to the inequalities

The stress tensor t_{ij} and the couple stress tensor m_{ij} are given by

$$t_{ij} = (-p + \lambda_1 + div\bar{q})\delta_{ij} + (2\mu + k)e_{ij} + k\varepsilon_{ljm}(w_m - v_m) \tag{4}$$

$$m_{ij} = \alpha(div\,\bar{v})\delta_{ij} + \beta v_{i,j} + \gamma v_{j,i} \tag{5}$$

in which the symbols $\delta_{ij}, e_{ij}, 2\,w_m$ and v_m respectively denote Kronecker symbol, components of rate of strain , vorticity vector and microrotation vector. ε_{ijm} denotes the Levi- Civita symbol and comma denotes covariant differentiation.

In the absence of body forces and body couple, the equations governing the steady flow is given by

$$div(\vec{q}) = 0 \tag{6}$$

$$-grad\ p + k\ curl\ \vec{v} - (\mu + k)\ curl\ \ curl\ \vec{q} = 0 \tag{7}$$

$$-2k\vec{v} + k\ curl\ \vec{q} - \gamma\ curl\ curl\ \vec{v} = 0 \tag{8}$$

Assuming $\vec{q} = (0, 0, w(r))$ and $\vec{v} = (0, v(r), 0)$, we get

$$-\frac{\partial p}{\partial r} = 0 \tag{9}$$

$$\frac{k}{r}\frac{d}{dr}(rv) + \frac{\mu + k}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) - \frac{\partial p}{\partial z} = 0 \tag{10}$$

$$-\gamma\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv)\right) + k\frac{dw}{dr} + 2kv = 0 \tag{11}$$

which is a system of coupled ordinary equations with variable coefficients.

The boundary conditions are
$w = 0$ on $r = R$ (no slip condition)
w is finite at $r = 0$
$v = 0$ on $r = R$ (no spin condition)
v is finite at $r = 0$ (12)

Using the following non-dimensionalisation:
$v^* = \frac{v}{U_{micro}}$; $w^* = \frac{w}{U_{average}}$; $r^* = \frac{r}{R}$ and the non- dimensional parameters given by

$pl = \frac{\mu + k}{k}$ (which is the polarity parameter that takes up small values), $fl = \frac{U_{average}}{2RU_{micro}}$ which is referred to as flow parameter, $vis = \frac{\gamma}{2kR^2}$ the viscosity parameter, $Ps = -\frac{R}{2kU_{micro}}\frac{dp}{dz}$ the pressure, equations (9) and (10) after dropping '*' take the form

$$\frac{d}{dr}(rv) + pl * fl\frac{d}{dr}\left(r\frac{dw}{dr}\right) + rPs = 0 \tag{13}$$

$$-vis\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv)\right) + fl\frac{dw}{dr} + v = 0 \tag{14}$$

HAM:
Consider a nonlinear differential equation of the form:

$$N(u(x)) = 0 \tag{15}$$

where N is a nonlinear operator, x is the independent variable and $u(x)$ is the unknown function. Let $u_0(x)$ be the initial approximation of the exact solution $u(x)$ and L be an auxiliary linear operator with the property that

$L(f) = 0$ when $f = 0$. (16)

In this method, we construct the Homotopy which is a continuous mapping $H : u(x) \to \phi(x;q)$ defined as

$$H\left(\phi(x;q);q\right) = (1-q)L\left(\phi(x;q) - u_0(x)\right) - hH(x)qN\left(\phi(x;q)\right) \tag{17}$$

Here $H(x)$ is an auxiliary function and h is an auxiliary parameter called the convergence control parameter, $q \in [0,1]$ is an embedding parameter and $\phi(x;q)$ is the approximate solution to the given problem. We notice from equation (17) that the solution obtained using this method, depends on the four important factors namely the initial approximation $u_0(x)$, the linear operator L, the auxiliary function $H(x)$ and the auxiliary parameter h.

When $q = 0$ and when the Homotopy defined by equation (17) is taken to be zero, we get the zeroth order deformation equation given by

$$L\left(\phi(x;0) - u_0(x)\right) = 0 \tag{18}$$

In view of the linearity of the operator 'L', the zeroth deformation equation is given by

$$\phi(x;0) = u_0(x) \tag{19}$$

Now, when $q = 1$, equation (17) takes the form

$$N\left(\phi(x;1)\right) = 0$$

This equation is same as the given equation provided

$$\phi(x;1) = u(x) \tag{20}$$

This shows that as the embedded parameter q varies from 0 to 1, $\phi(x;q)$ varies from the initial guess $u_0(x)$ (as is seen in equation (19)) to the exact solution $u(x)$ (as seen equation (20)).

Let's now define the m[th] order deformation derivatives as

$$u_0^{[m]}(x) = \frac{\partial^m}{\partial q^m}\left(\phi(x;q)\right)\Bigg|_{q=0} \tag{21}$$

Then, using the Taylor's theorem, $\phi(x;q)$ can be expanded as a power series of q as

$$\phi(x;q) = \phi(x;0) + \sum_{m=1}^{\infty} \frac{u_0^{[m]}(x)}{m!} q^m \tag{22}$$

Writing $u_m(x) = \dfrac{u_0^{[m]}(x)}{m!}$ and using equation (2.19), the above takes the form

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m \tag{23}$$

With suitable choice of the initial guess, the auxiliary linear operator, the convergence control parameter and the auxiliary function, Liao proved that the above power series solution converges for $q = 1$ [10].

Now, to find the solution using equation (2.23), we need to find the functions $u_m(x)$ for m=1,2,3…

Liao [10] has derived that these functions are given by the mth order deformation equation defined as follows:

$$L\left(u_m(x)-\chi_m u_{m-1}(x)\right)=hH(x)R_m\left(\vec{u}_{m-1}(x)\right) \tag{24}$$

where

$$R_m\left(\vec{u}_{m-1}(x)\right)=\frac{1}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial q^{m-1}}\left(N\left(\phi(x;q)\right)\right)\right)_{q=0} \tag{25}$$

and $$\chi_m=\begin{cases}0, m\le 1\\ 1, otherwise\end{cases} \tag{26}$$

After determining $u_m(x)$ for m=1,2…, an approximate solution to the problem given in (14) is

$$u(x)\approx u_0(x)+\sum_{m=1}^{\infty}u_m(x) \tag{27}$$

Solution:

Let $$L_w \equiv w'' \tag{28}$$

$$L_v \equiv v''+x^2 w' \tag{29}$$

Here $$N_w \equiv \frac{d}{dr}(rv)+pl*fl\frac{d}{dr}\left(r\frac{dw}{dr}\right)+rPs=0$$

$$N_v \equiv -vis\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv)\right)+fl\frac{dw}{dr}+v=0 \tag{30}$$

Construct the homotopy for the above nonlinear coupled equations as:

$$H\left(w(r;q);q\right)=(1-q)L\left(w(r;q)-w_0(r)\right)-hN_w\left(w(r;q)\right)$$
$$H\left(v(r;q);q\right)=(1-q)L\left(v(r;q)-v_0(r)\right)-hN_v\left(v(r;q)\right) \tag{31}$$

Using zero initial approximations and assuming the solutions as

$$w(r;q)=w_0(\eta)+\sum_{m=1}^{\infty}w_m(\eta)q^m$$
$$v(r;q)=v_0(\eta)+\sum_{m=1}^{\infty}v_m(\eta)q^m \tag{32}$$

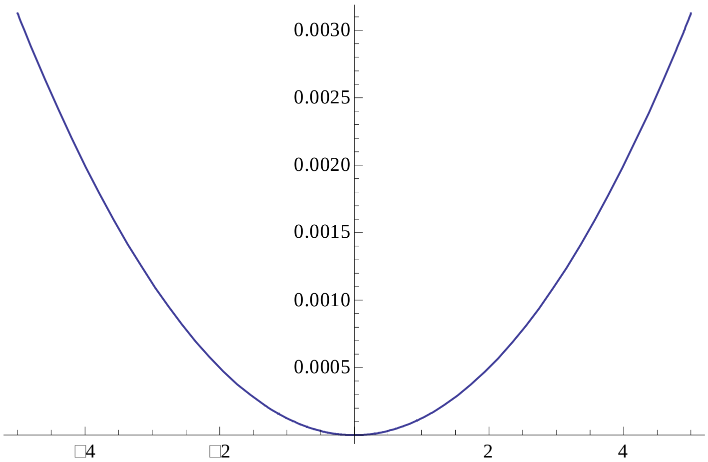
$w_m(r)$ and $v_m(r)$ are obtained using equations (24)- (26).

Using MATHEMATICA, the first four approximations for the fluid velocitycomponent $w(r)$ and the micro rotation component $v(r)$ are calculated and the plots are presented.

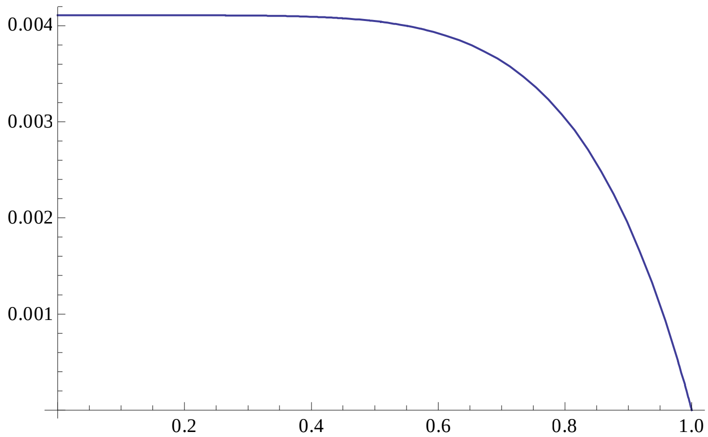
RESULTS AND DISCUSSION

To find the values of the convergence control parameter ‘*h*’ in equations (31), the *h*-graphs for w i.e $w''(h)$ is plotted at $\eta = 0$ an in Fig 1. It can be seen that for for Ps = 0.25; pl = 0.1; fp = 0.01; Vp = 0.01,the control parameter ‘h’ satisfies -0.5<h<0.5

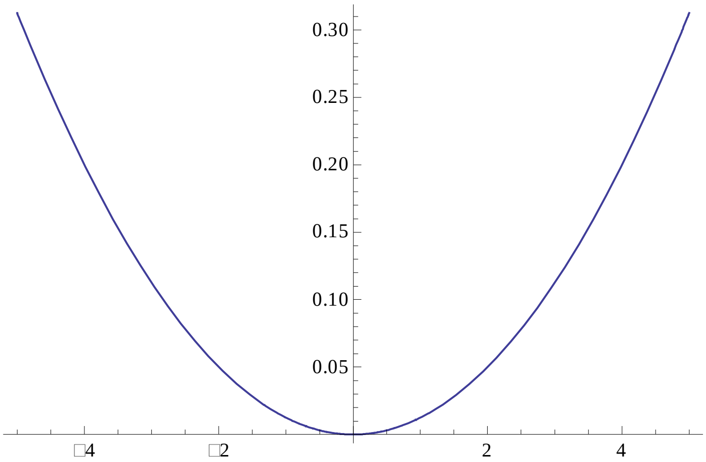
Similarly, to find the convergence control parameter *h* for finding v, the *h*-graph for v is plotted as in Fig3.



Fig(1) Plot of $w''(h)$ **to find the convergence parameter (-0.5<h<0.5)for Ps = 0.25; pl = 0.1; fp = 0.01; Vp = 0.01;**

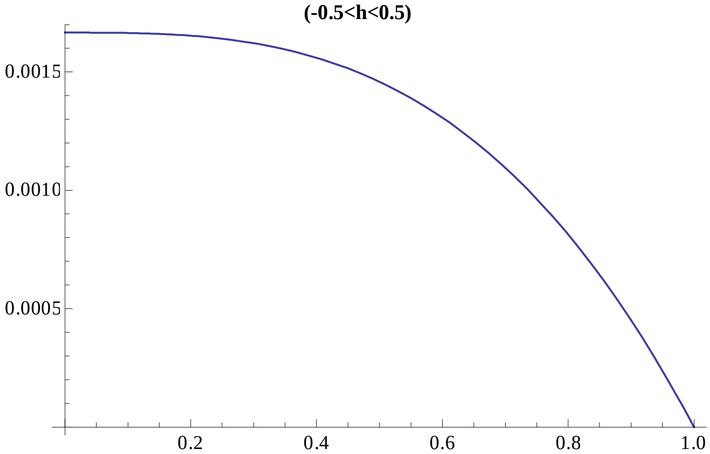


Fig(2) Plot of $w(r)$ **for Ps = 0.25; pl = 0.1; fp = 0.01; Vp = 0.01;**

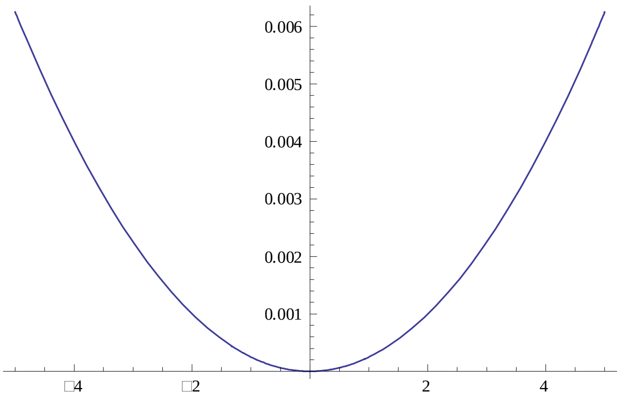


Fig(3) : Plot of $v''(h)$ **for Ps = 0.25; pl = 0.1; fp = 0.01; Vp = 0.01;**

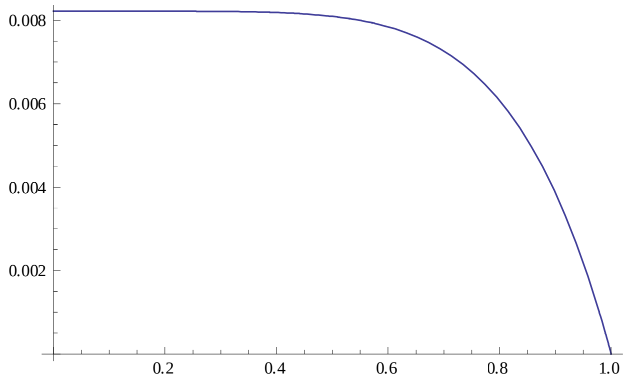
(-0.5<h<0.5)



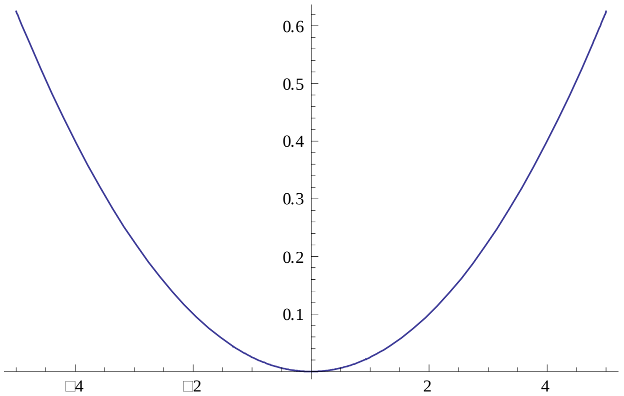
Fig(4) : Plot of $v(r)$ **for Ps = 0.25; pl = 0.1; fp = 0.01; Vp = 0.01;**



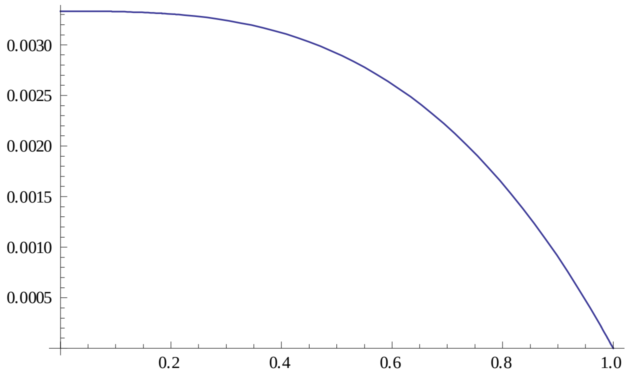
Fig(5) Plot of $w''(h)$ **for Ps = 0.5; pl = 0.1; fp = 0.01; Vp = 0.01;**



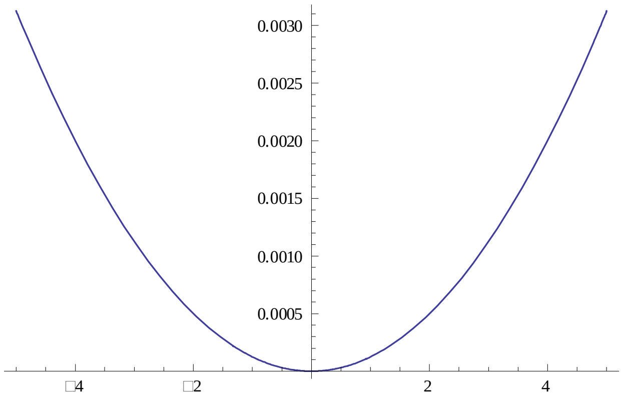
Fig(6): Plot of $w(r)$ $\mathbf{Ps = 0.5; pl = 0.1; fp = 0.01; Vp = 0.01;}$



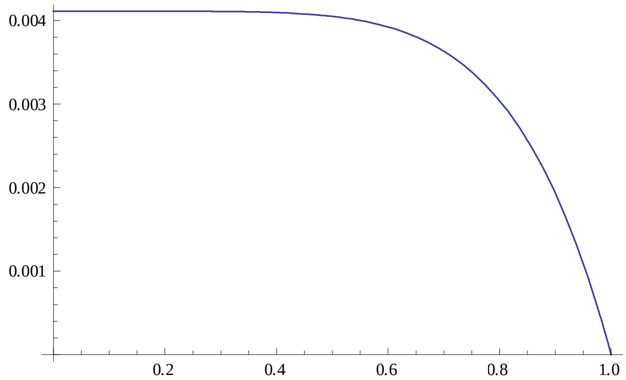
Fig(7) : Plot of $v''(h)$ $\mathbf{Ps = 0.5; pl = 0.1; fp = 0.01; Vp = 0.01;}$



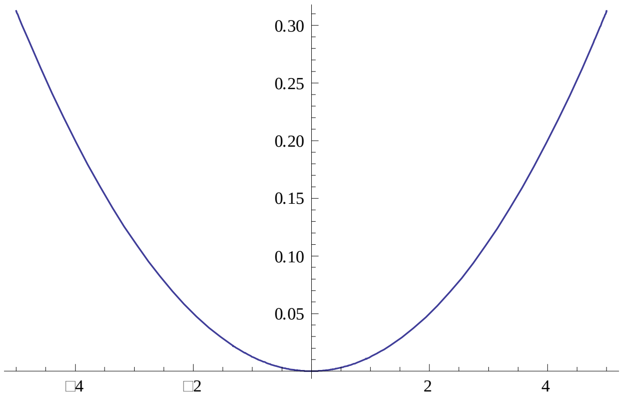
Fig(8): Plot of $v(r)$ $\mathbf{Ps = 0.5; pl = 0.1; fp = 0.01; Vp = 0.01;}$



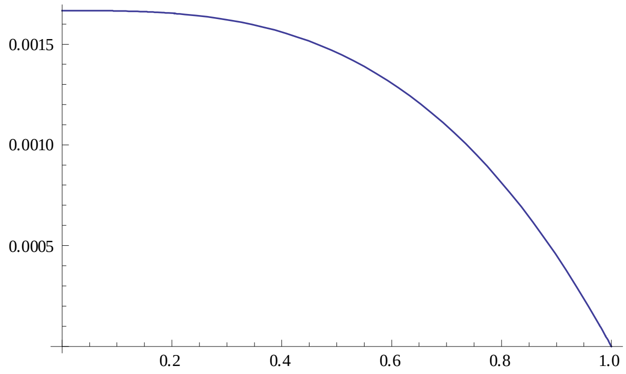
Fig(9) Plot of $w''(h)$ for $Ps = 0.25; pl = 0.4; fp = 0.01; Vp = 0.01$ to find the convergence parameter ($-1<h<0$)



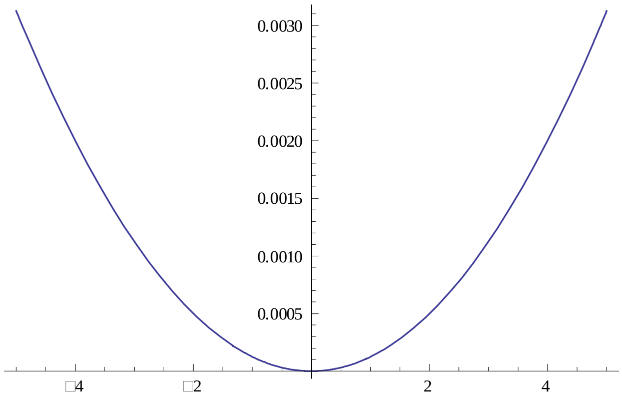
Fig(10): Plot of $w(r)$ $Ps = 0.25; pl = 0.4; fp = 0.01; Vp = 0.01$



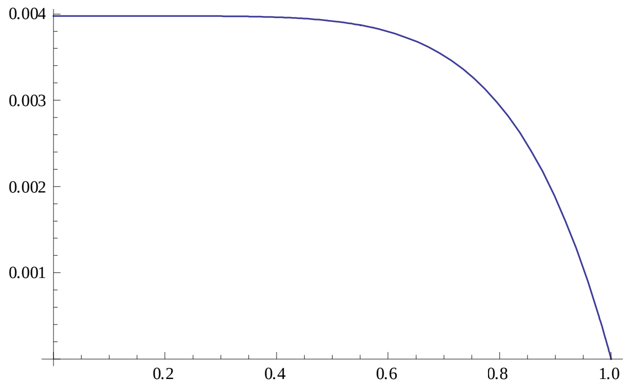
Fig(11) : Plot of $v''(h)$ for $Ps = 0.25; pl = 0.4; fp = 0.01; Vp = 0.01$



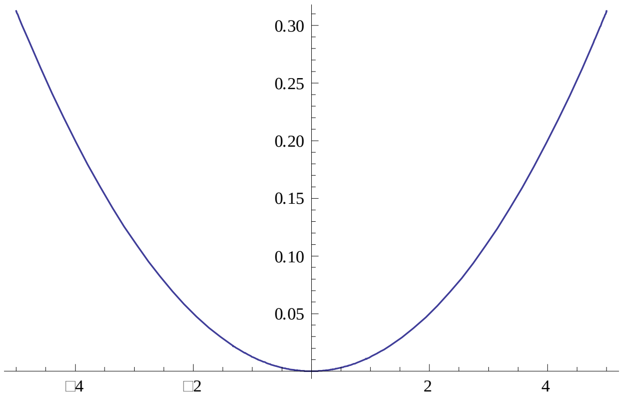
Fig(12): Plot of $v(r)$ $\mathbf{Ps = 0.25; pl = 0.4; fp = 0.01; Vp = 0.01}$



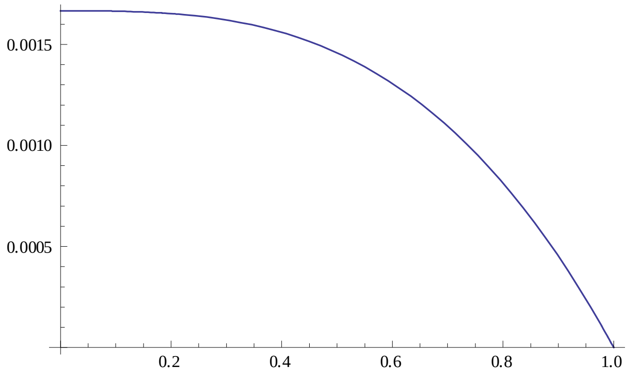
Fig(13) Plot of $w''(h)$ **for for** $\mathbf{Ps = 0.25; pl = 0.1; fp = 0.1; Vp = 0.01}$



Fig(14): Plot of $w(r)$ **for for** $\mathbf{Ps = 0.25; pl = 0.1; fp = 0.1; Vp = 0.01}$



Fig(15) : Plot of $v''(h)$ for Ps = 0.25; pl = 0.1; fp = 0.1; Vp = 0.01



Fig(16): Plot of $v(r)$ for Ps = 0.25; pl = 0.1; fp = 0.1; Vp = 0.01

CONCLUSION

From the above graphs, we draw the following conclusions:

1. As the pressure gradient (Ps) increases, the velocity of fluid at the center of the tube increases as is seen in fig(2) and fig(6).
2. As the reciprocal of the micro polarity (pl) decreases, the velocity of the fluid increases as is seen in fig(2) and (10).
3. Also as the reciprocal of the micro polarity (pl) decreases, the micro rotation of the fluid increases as is seen in fig(4) and (12).
4. As the fluid flow parameter increases, the velocity of the fluid decreases(fig (2) and (14)) and the micro rotation increases (fig (4) and (16).

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