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Fractional integral operator on Laguerre polynomial of matrix argument

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ABSTRACT

When Fractional integral operators are connected with real-valued scalar functions of matrix argument we can get various results, those results are useful in various problems of mathematics, statistics and natural science. Recently Mathai connected some fractional calculus operators with functions of matrix argument, by using those results we are investigating some new results for the Laguerre function of matrix arguments. When these are used as model for problems in the natural sciences, then these can cover the ideal situations, neighbourhoods, in between stages and paths leading to optimal situations.

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INTRODUCTION

Time to time many authors discussed the theory of special function of matrix argument such as Constantine [14], James [15], [16], [17] Mathai and Saxena [6], Poonam and Sethi [18]. Recently Mathai [1], Nair [2], Mathai [13] also introduced some fractional integral operators with matrix argument case.

1.1 ZONAL POLYNOMIAL

Zonal polynomial associated with a matrix X its description and the development of the theory is available from Mathai, Provost and Hayakawn [3]. Consider a $p\times p$ real positive definite matrix X . Let V_k be the vector space of homogenous polynomial $g(x)$ of degree k in the $p(p+1)/2$ different elements of $p\times p$ symmetric matrix X . It can be shown that V_k decomposes into a direct sum of irreducible subspace V_k corresponding to each partition $K=(k_1,k_2.........k_p)$, $K=(k_1\geq k_2\geq ....k_p\geq 0)$ into not more than p parts. Each subspace contains a unique one dimensional subspace invariant under the orthogonal group of linear transformations. These subspaces are generated by the zonal polynomials $U_K(X)$ which when normalised in a certain fashion give a zonal polynomial $C_K(X)$. Explicit forms of these polynomials are available for small value of k. For large value of k it will be extremely difficult to compute this polynomial. For handling elementary special functions of matrix argument, we need a few properties of these zonal polynomials. These properties will be sufficient to establish the result. One basic result which is an immediate consequence of the definition itself is that when X is 1×1 matrix, namely a scalar quantity x,

$$C_K(X)=x^k \qquad (1.1.1)$$

Hence one can look upon $C_K(X)$ as a generalization of x^k. The exponential function has the following expansion.

$$e^{tr(X)} = \sum_{k=0}^{\infty} \frac{[tr(X)]^k}{k} = \sum_{k=0}^{\infty} \sum_{K} \frac{C_K(X)}{k!} \tag{1.1.2}$$

The binomial expansion is the following for $I - X > 0$ that is, $X = X' > 0$ and all the Eigen values of X are between 0 and 1.

$$[I - X]^{-\alpha} = \sum_{k=0}^{\infty} \sum_{K} \frac{(\alpha)_k C_K(X)}{k!} \tag{1.1.3}$$

$$\int_{U>0} C_K(H' XHT) dU = \frac{C_K(X) C_K(T)}{C_K(I)} \tag{1.1.4}$$

Where I is the identity matrix, the integral is over the orthogonal group of $p \times p$ matrices and dH is the invariant measure.

1.2. HYPER GEOMETRIC FUNCTION:

Generalised Hypergeometric function of matrix argument is given by:

$$ {}_mF_n(a_1,\dots\dots\dots,a_m;b_1,\dots\dots\dots,b_n;X) = \sum_{k=0}^{\infty} \sum_{K} \frac{(a_1)_K \dots\dots\dots (a_m)_K}{(b_1)_K \dots\dots\dots (b_n)_K} \frac{C_K(X)}{k!} \tag{1.2.1}$$

None of the denominator factors is equal to zero.

$$\text{Where} \quad (a)_K = \prod_{i=1}^{p} \left[a - \frac{i-1}{2} \right]_{k_i} \tag{1.2.2}$$

$$(a)_K = \frac{\Gamma_p(a,K)}{\Gamma_p(a)} \tag{1.2.3}$$

$$\Gamma_p(a,K) = \pi^{m(m-1)/4} \prod_{i=1}^{p} \Gamma \left[a + k_i - \frac{1}{2}(i-1) \right] \tag{1.2.4}$$

$$\Gamma_p(a) = \pi^{m(m-1)/4} \prod_{i=1}^{p} \Gamma \left[a - \frac{1}{2}(i-1) \right] \tag{1.2.5}$$

X is a real positive definite matrix of order $p \times p$, $C_K(X)$ are the zonal polynomials

2. RESULT REQUIRED

Let $X = (x_{ij})$ be a $p \times r$, $r \geq p$ matrix of real distinct scalar variables x_{ij}'s. Let A be the $p \times p$ real positive definite constant matrix, that is, $A = A' > 0$, prime denoting the Transpose. Let B is the constant positive definite $r \times r$ matrix, $B > 0$. Let $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ denotes the positive definite square root of matrix A and B. Let X is of full rank p. Then $Z_X = A^{1/2} XBX' A^{1/2}$ is symmetric positive matrix.

In this paper we will consider only real matrices.

The real matrix gamma will be denoted by [11].

$$\Gamma_p(a) = \pi^{m(m-1)/4} \prod_{i=1}^{p} \Gamma \left[a - \frac{1}{2}(i-1) \right] \tag{2.1}$$

For $\mathrm{Re}(\alpha) > \left(\frac{p-1}{2} \right)$, where Re(.) means that real part of (.). It can be shown that $\Gamma_p(a)$ is the integral representation, [11]

$$\Gamma_p(a) = \int_{S>O} |S|^{\alpha-(p+1)/2} e^{-tr(S)} dS, \ \mathrm{Re}(\alpha) > \left(\frac{p-1}{2} \right), \tag{2.2}$$

Where $|S|$ means the determinant of $p\times p$ positive definite matrix S and tr (S) denotes the trace of S.
The type – 1 beta integral is given by [9]

$$\beta_p(a,b)=\frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}=\int_{O<S<I}|S|^{a-(p+1)/2}\,|I-S|^{b-(p+1)/2}\,dS \tag{2.3}$$

Where $\mathrm{Re}(a)>(p-1)/2$, $\mathrm{Re}(b)>(p-1)/2$, where S is $p\times p$ positive definite and $O<S<I$ means $S>0$, $I-S>0$. In general $\int_X$ means the integral over X. By making suitable matrix transform above, one get [9], the representation

$$\beta_p(a,b)=\int_{O<S<I}|S|^{a-(p+1)/2}\,|I+S|^{-(a+b)}\,dS\,,\ \mathrm{Re}(a)>(p-1)/2\ ,\mathrm{Re}(b)>(p-1)/2 \tag{2.4}$$

Above result is known as beta type – 2 integral.
Let $Z_X=A^{1/2}XBX'A^{1/2}$, $Z_Y=A^{1/2}YBY'A^{1/2}$

X and Y are $p\times r$, $r\geq p$ matrices of real elements, and of rank p, consider the evaluation of the integral

$$I_1=\int_{O<S<I}|Z_X|^{a}\,|I-Z_X|^{c-a-\frac{p+1}{2}}\,|I-Z_XZ_Y|^{-b}\,dX\,, \tag{2.5}$$

Where $O<z_X<I$, $O<z_Y<I$.

This integral corresponds to the Euler's integral for the Gauss hypergeometric function .

Let $U=A^{1/2}XB^{1/2}$ then $dU=|A|^{r/2}|B|^{p/2}dX$ (2.6)

See Mathai [7] , then after integration over the Stiefel manifold,

$$Z_X=UU'=V\ \text{Then}\ dU=\frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\frac{r}{2})}|V|^{\frac{r}{2}-\frac{p+1}{2}}dV \tag{2.7}$$

Now the integral I_1becomes,

$$I_1=|A|^{-r/2}|B|^{-P/2}\frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\frac{r}{2})}\int_{O<V<I}|V|^{a+\frac{r}{2}-\frac{p+1}{2}}\,|I-V|^{c-a-\frac{p+1}{2}}\,\left|I-Z_Y^{\frac{1}{2}}VZ_X^{\frac{1}{2}}\right|^{-b}dV \tag{2.8}$$

By using Zonal Polynomials as Mathai [03]

$$\left|I-Z_Y^{1\backslash 2}\,V\,Z_X^{1\backslash 2}\right|^{-b}=\sum_{k=0}^{\infty}\sum_{K}\frac{(b)_K}{k!}\,C_K(Z_YV) \tag{2.9}$$

We can evaluate the integral

$$\int_{O<V<I}|V|^{a+\frac{r}{2}-\frac{p+1}{2}}\,|I-V|^{c-a-\frac{p+1}{2}}\,C_K(Z_YV)dV=\frac{\Gamma_p(a+\frac{r}{2})\Gamma_p(c-a)}{\Gamma_p(c+\frac{r}{2},K)}C_K(Z_Y) \tag{2.10}$$

By using above relations I_1 becomes

$$I_1=|A|^{-r/2}|B|^{-p/2}\frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\frac{r}{2})}\frac{\Gamma_p(a+\frac{r}{2})\Gamma_p(c-a)}{\Gamma_p(c+\frac{r}{2})}\sum_{k=0}^{\infty}\sum_{K}\frac{(b)_K(a+\frac{r}{2})_K}{(c+\frac{r}{2})_K}\frac{C_K(Z_Y)}{k!} \tag{2.11}$$

2.1. FRACTIONAL INTEGRAL OPERATORS

Let Z_X and Z_Y be as defined above. Let the generalised fractional integral operator of matrix argument be defined as

$$\left({}_0D_X^{-\alpha}f\right)X=\frac{1}{\Gamma_p(\alpha)}\int_{Z_X>Z_Y>O}|Z_X-Z_Y|^{\alpha-\frac{P+1}{2}}f(Z_Y)dY$$

$$=\frac{|Z_X|^{\alpha-\frac{p+1}{2}}}{\Gamma_p(\alpha)}\int_{Z_X>Z_Y>O}\left|I-Z_X^{-1/2}Z_YZ_X^{-1/2}\right|^{\alpha-\frac{P+1}{2}}f(Z_Y)dY$$

(2.1.1)

By the transform $U = A^{1/2}YB^{1/2}$, $V = UU'$, $W = Z_X^{-1/2}VZ_X^{-1/2}$. Then after integration over the Stiefel manifold, we have

$$\left({}_0D_X^{-\alpha}f\right)X = \frac{|Z_X|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}}{\Gamma_p(\alpha)|A|^{r/2}|B|^{p/2}}\frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\frac{r}{2})}\int_{O<W<I}|I-W|^{\alpha-\frac{p+1}{2}}|W|^{\frac{r}{2}-\frac{p+1}{2}}f(Z_X^{1/2}WZ_X^{1/2})dW \quad (2.1.2)$$

See Mathai [13]

Mathai [13] derived following result by taking $f(Z_X) = C_K(Z_X)$ in above equation

$${}_0D_X^{-\alpha}C_K(Z_X) = \frac{|Z_X|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}}{|A|^{r/2}|B|^{p/2}}\frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\alpha+\frac{r}{2})}\frac{(\frac{r}{2})_K}{(\alpha+\frac{r}{2})_K}C_K(Z_X) \quad (2.1.3)$$

Where $\text{Re}(\alpha+\frac{r}{2}) > \left(\frac{p-1}{2}\right)$

Easily we can obtain the result by taking $f(Z_X) = |Z_X|^{\beta}C_K(Z_X)$ in (2.1.2),

$${}_0D_X^{-\alpha}\left[|Z_X|^{\beta}C_K(Z_X)\right] = \frac{|Z_X|^{\alpha+\beta+\frac{r}{2}-\frac{p+1}{2}}\pi^{\frac{rp}{2}}}{|A|^{r/2}|B|^{p/2}\Gamma_p(\frac{r}{2})}\frac{\Gamma_p(\beta+\frac{r}{2})(\beta+\frac{r}{2})_K}{\Gamma_p(\alpha+\beta+\frac{r}{2})(\alpha+\beta+\frac{r}{2})_K}C_K(Z_X) \quad (2.1.4)$$

Where $\text{Re}(\alpha+\beta+\frac{r}{2}) > \left(\frac{p-1}{2}\right)$

3. MAIN RESULT:

On the basis of above discussion, we can establish the following theorems.

Theorem 1. If $L_b^{(a)}(Z_X)$ is a Laguerre polynomials of matrix arguments then:

$${}_0D_X^{-\alpha}\left[L_b^{(a)}(Z_X)\right] = \frac{|Z_X|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}\pi^{\frac{rp}{2}}}{|A|^{r/2}|B|^{p/2}\Gamma_p(\alpha+\frac{r}{2})}\frac{\prod_p(a+b)}{\prod_p(a)}{}_2F_2(-b,\tfrac{r}{2};a+\tfrac{p+1}{2},\alpha+\tfrac{r}{2};Z_X) \quad (3.1)$$

Where $\text{Re}(\alpha+\frac{r}{2}) > \left(\frac{p-1}{2}\right)$

Proof: Laguerre polynomial of matrix argument is given as

$$L_b^{(a)}(Z_X) = \frac{\prod_p(a+b)}{\prod_p(a)}\,{}_1F_1(-b;a+\tfrac{p+1}{2};Z_X) \quad (3.2)$$

Let

$${}_0D_X^{-\alpha}\left[{}_1F_1(-b;a+\tfrac{p+1}{2};Z_X)\right]$$

$$= {}_0D_X^{-\alpha}\left[\sum_{k=0}^{\infty}\sum_K\frac{(-b)_K}{(a+\frac{p+1}{2})_K}\frac{C_K(Z_X)}{k!}\right]$$

$$= \sum_{k=0}^{\infty}\sum_K\frac{(-b)_K}{(a+\frac{p+1}{2})_K\,k!}\left[{}_0D_X^{-\alpha}C_K(Z_X)\right]$$

Now by using equation (2.1.3) , we get ,

$${}_0D_X^{-\alpha}\left[{}_1F_1(-b;a+\tfrac{p+1}{2};Z_X)\right]$$

$$= \sum_{k=0}^{\infty}\sum_K\frac{(-b)_K}{(a+\frac{p+1}{2})_K\,k!}\frac{|Z_X|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}}{|A|^{r/2}|B|^{p/2}}\frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\alpha+\frac{r}{2})}\frac{(\frac{r}{2})_K}{(\alpha+\frac{r}{2})_K}C_K(Z_X)$$

$$= \frac{|Z_X|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}}{|A|^{r/2}|B|^{p/2}} \frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\alpha+\frac{r}{2})} \sum_{k=0}^{\infty}\sum_{K} \frac{(-b)_K}{(a+\frac{p+1}{2})_K\, k!} \frac{(\frac{r}{2})_K}{(\alpha+\frac{r}{2})_K} C_K(Z_X)$$

$$= \frac{|Z_X|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}}{|A|^{r/2}|B|^{p/2}} \frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\alpha+\frac{r}{2})}\ {}_2F_2(-b,\tfrac{r}{2};a+\tfrac{p+1}{2},\alpha+\tfrac{r}{2};Z_X)$$

Now by multiply $\dfrac{\prod_p(a+b)}{\prod_p(a)}$ both sides we get,

$${}_0D_X^{-\alpha}\left[\frac{\prod_p(a+b)}{\prod_p(a)}\,{}_1F_1(-b;a+\tfrac{p+1}{2};Z_X)\right]$$

$$= \frac{|Z_X|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}}{|A|^{r/2}|B|^{p/2}} \frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\alpha+\frac{r}{2})} \frac{\prod_p(a+b)}{\prod_p(a)}\ {}_2F_2(-b,\tfrac{r}{2};a+\tfrac{p+1}{2},\alpha+\tfrac{r}{2};Z_X)$$

By the definition of Laguerre polynomial of matrix argument, above equation becomes

$${}_0D_X^{-\alpha}\left[L_b^{(a)}(Z_X)\right]$$

$$= \frac{|Z_X|^{\alpha+\frac{r}{2}-\frac{p+1}{2}}}{|A|^{r/2}|B|^{p/2}} \frac{\pi^{\frac{rp}{2}}}{\Gamma_p(\alpha+\frac{r}{2})} \frac{\prod_p(a+b)}{\prod_p(a)}\ {}_2F_2(-b,\tfrac{r}{2};a+\tfrac{p+1}{2},\alpha+\tfrac{r}{2};Z_X)$$

Where $\mathrm{Re}(\alpha+\frac{r}{2}) > \left(\dfrac{p-1}{2}\right)$

Hence the proof is complete.

Theorem 2 If $L_b^{(a)}(Z_X)$ is a Laguerre polynomials of matrix arguments then :

$${}_0D_X^{-\alpha}\left[|Z_X|^{\beta} L_b^{(a)}(Z_X)\right]$$

$$= \frac{|Z_X|^{\alpha+\beta+\frac{r}{2}-\frac{p+1}{2}}\pi^{\frac{rp}{2}}\Gamma_p(\beta+\frac{r}{2})}{|A|^{r/2}|B|^{p/2}\Gamma_p(\frac{r}{2})\Gamma_p(\alpha+\beta+\frac{r}{2})} \frac{\prod_P(a+b)}{\prod_P(a)}\,{}_2F_2(-b,\beta+\tfrac{r}{2};a+\tfrac{p+1}{2},\alpha+\beta+\tfrac{r}{2};Z_X)$$

(3.2)

Proof: LHS of above theorem is

LHS

$$= {}_0D_X^{-\alpha}\left[|Z_X|^{\beta} L_b^{(a)}(Z_X)\right]$$

$$= {}_0D_X^{-\alpha}\left[|Z_X|^{\beta}\frac{\prod_P(a+b)}{\prod_P(a)}\,{}_1F_1(-b;a+\tfrac{p+1}{2};Z_X)\right]$$

$$= {}_0D_X^{-\alpha}\left[|Z_X|^{\beta}\frac{\prod_P(a+b)}{\prod_P(a)}\sum_{k=0}^{\infty}\sum_{K}\frac{(-b)_K}{(a+\frac{p+1}{2})_K\, k!}C_K(Z_X)\right]$$

$$= \frac{\prod_P(a+b)}{\prod_P(a)}\sum_{k=0}^{\infty}\sum_{K}\frac{(-b)_K}{(a+\frac{p+1}{2})_K\, k!}\,{}_0D_X^{-\alpha}\left[|Z_X|^{\beta}C_K(Z_X)\right]$$

By (2.1.4),

LHS

$$= \frac{\prod_P (a+b)}{\prod_P (a)} \sum_{k=0}^{\infty}\sum_K \frac{(-b)_K}{(a+\frac{p+1}{2})_K\, k!} \frac{|Z_X|^{\alpha+\beta+\frac{r}{2}-\frac{p+1}{2}} \pi^{\frac{rp}{2}}}{|A|^{r/2}|B|^{p/2}\Gamma_p(\frac{r}{2})} \frac{\Gamma_p(\beta+\frac{r}{2})(\beta+\frac{r}{2})_K}{\Gamma_p(\alpha+\beta+\frac{r}{2})(\alpha+\beta+\frac{r}{2})_K} C_K(Z_X)$$

$$= \frac{\prod_P (a+b)}{\prod_P (a)} \frac{|Z_X|^{\alpha+\beta+\frac{r}{2}-\frac{p+1}{2}} \pi^{\frac{rp}{2}}\Gamma_p(\beta+\frac{r}{2})}{|A|^{r/2}|B|^{p/2}\Gamma_p(\frac{r}{2})\Gamma_p(\alpha+\beta+\frac{r}{2})} \sum_{k=0}^{\infty}\sum_K \frac{(-b)_K}{(a+\frac{p+1}{2})_K\, k!} \frac{(\beta+\frac{r}{2})_K}{(\alpha+\beta+\frac{r}{2})_K} C_K(Z_X)$$

$$= \frac{|Z_X|^{\alpha+\beta+\frac{r}{2}-\frac{p+1}{2}} \pi^{\frac{rp}{2}}\Gamma_p(\beta+\frac{r}{2})}{|A|^{r/2}|B|^{p/2}\Gamma_p(\frac{r}{2})\Gamma_p(\alpha+\beta+\frac{r}{2})} \frac{\prod_P (a+b)}{\prod_P (a)} \sum_{k=0}^{\infty}\sum_K \frac{(-b)_K}{(a+\frac{p+1}{2})_K\, k!} \frac{(\beta+\frac{r}{2})_K}{(\alpha+\beta+\frac{r}{2})_K} C_K(Z_X)$$

$$= \frac{|Z_X|^{\alpha+\beta+\frac{r}{2}-\frac{p+1}{2}} \pi^{\frac{rp}{2}}\Gamma_p(\beta+\frac{r}{2})}{|A|^{r/2}|B|^{p/2}\Gamma_p(\frac{r}{2})\Gamma_p(\alpha+\beta+\frac{r}{2})} \frac{\prod_P (a+b)}{\prod_P (a)} {}_2F_2(-b,\beta+\tfrac{r}{2};a+\tfrac{p+1}{2},\alpha+\beta+\tfrac{r}{2};Z_X)$$

= RHS

OR,

$${}_0D_X^{-\alpha}\left[|Z_X|^{\beta} L_b^{(a)}(Z_X)\right]$$

$$= \frac{|Z_X|^{\alpha+\beta+\frac{r}{2}-\frac{p+1}{2}} \pi^{\frac{rp}{2}}\Gamma_p(\beta+\frac{r}{2})}{|A|^{r/2}|B|^{p/2}\Gamma_p(\frac{r}{2})\Gamma_p(\alpha+\beta+\frac{r}{2})} \frac{\prod_P (a+b)}{\prod_P (a)} {}_2F_2(-b,\beta+\tfrac{r}{2};a+\tfrac{p+1}{2},\alpha+\beta+\tfrac{r}{2};Z_X)$$

Where $\mathrm{Re}(\alpha+\beta+\frac{r}{2}) > \left(\frac{p-1}{2}\right)$

Hence the proof is complete.

CONCLUSION

Above results are very useful for various problems of mathematics and statistics and in natural sciences.

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