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Four-vector, Dirac spinor representation and Lorentz Transformations

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ABSTRACT

Author found a simple geometrical representation of the general covariant form of Maxwell's equations and few related relations. We mention here briefly another equivalent way to formulate the geometrization of the four fields, totally need Dirac spin matrices which are used as vectors, do not required tensors knowledge but leading to simplifications of the four vectors, three vector physical quantities.

Key Words: Four-vector, Dirac spinor, Lorentz Transformations, SL (2,C).
PACS: 03.30.+p (Special relativity);03.50.De (Classical electromagnetism, Maxwell equations); 04.20.Gz (Spacetime topology, causal structure, spinor structure)

INTRODUCTION

Symmetries are of fundamental importances in the description of physical phenomena. The discovery of a non-Euclidean geometry posed an extremely complicated problem to physics, particularly explaining real space was Euclidean as have been believed earlier and if it was not, to what type of non-Euclidean space it belonged. Thus, it is necessary to see the validity of the axioms experimentally or extension of Euclidean space to new space, so that the construction of spin-geometries could be justified by the possibility of applying their conclusions to actually existing object and the observation. The fact that these conclusions are expressed in term of geometry is of no real consequence. As to the geometry-structure of real space, comes within the domain of physics[1-3] and cannot be resolve by means pure geometry. Present formulation provides a better description of actual spatial relation than earlier workers. It is well established that the theory of relativity uses the formulas of non-Euclidean geometry but it never says that the Euclidean's geometry must be discarded. In the pseudo Euclidean space, time coordinate have different footing than the space coordinates. Both geometries are the tools for investigating spatial forms but the non-Euclidean enables finer studies to be made in the light of preconceived information. It is the confirmed fact that physical phenomena do not appear same to the other observers in the relative motion, with respect to each other, although the physical law must be same for all observers (inertial frame). Thus the principle of relativity asserts that two observations/observers will describe a physical process by same equation whenever they are stationary or in the uniform linear motion with respect to each other.

A transformation of space-time that maps any observers a reference in four-dimensional spaces into an equivalent one cannot affect the description of the physical processes. Naturally such a transformation obviously forms a group. The principal of relativity does not determine group distinctively because an additional postulate is required for such purpose. For unique identification there are three possibilities.

1. That two observers are equivalent only if they are at least rests with respect to each other, which consist of all

rotation and transformation in three- dimensional space.

2. If in addition, equivalent observers are allowed to be in relative motion and time remains absolute, we have a relativity group of Newtonian mechanics i.e. Galilee group.

3. If instead of above the two possibilities light propagates with same speed for every observer (inertial frame), which is now incompatible with absolute time, we obtain the inhomogeneous Lorenz group also called Poincare group. This is the relativity group of Einstein's theory of special relativity. Relativity group adapts a classical mechanics to the symmetry properties inherent in electromagnetism the principle of relativity can be extended to the observers in gravitational field, but interpretation of general relativity in-group theoretic term is no longer straightforward. Once the relativity group of the theory determined, the principal of relativity must be put in action. But every theory has its particular advantages and drawbacks. The aim of all continuum theories is to derive the atomic nature of the electricity from the property that the differential equations expressing the physical law have only discrete number of solutions which are everywhere regular static and spherically symmetric. In particular one such solution should exist for each of positive and negative kind of electricity. It is clear that the differential equations, which have this property, must have complicated structure. It seems that such complexity of physical law itself speaks against the continuum theory. Thus it is required from a physical point of view that the existence of an atomicity is itself so simple and basic, it should also be interpreted in simple and elementary manner by theory and should not speak, appeals as a trick in analysis. The continuum theory forced to introduce a special forces, which keep the coulomb repulsive forces in the interior of the electrical elementary particle in equilibrium. If such forces are electrical in nature, then we have to assign an absolute meaning to the potential in the domain of four-vector and three-vector which leads to the different types of difficulties.

2. Four Vector Algebra:

Indifference to location of the origin of coordinate system is called homogeneity of space and indifference to direction of axis is called isotropy of space. The requirement of homogeneity of space is expressed by invariance of equations with respect to shift of origin. The equations are said to be invariant when they preserve their forms on transformation to another inertial reference frame. If the value of some physical quantity remains the same after transformation from one frame of reference to an-other, then that quantity is said to be invariant. The requirement of isotropy of space is expressed by requiring covariance of our equations with respect to the rotation of the axis of reference frame. The equations that describe the physical law must be covariant in form i.e. its form is independent of the choice of the inertial frame or both the side have the same tensor character. Thus a scalar cannot be equated to a component of a vector nor can one term in a sum be a tensor of second rank, while another one, a tensor of first rank. If we express the physical law in four-vector/tensor form/equivalent form, then the resulting equations will be automatically be covariant with respect to a given class of transformations.

From the relativity, we have following basic quadratic expression as

$$(P_o/m_oc)^2 - (\mathbf{P}/m_oc)^2 = 1 \qquad (1)$$

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Here we view above equation by defining any physical quantity say A,

$$A_{\pm} = (A_{o,} \pm \mathbf{A}) \qquad (2)$$

and another physical B as,

$$B_{\mp} = (B_o, \mp \mathbf{B}) \qquad (3)$$

Here we postulate that any physical vector quantity must be represented through following mapping given below

$$\mathbf{A} \mapsto \mathbf{A} \cdot \boldsymbol{\sigma} = \mathbf{A_a} \cdot \boldsymbol{\sigma}_\mathbf{a} + \mathbf{A_b} . \boldsymbol{\sigma}_\mathbf{b} + \mathbf{A_c} . \boldsymbol{\sigma}_\mathbf{c} \qquad (4)$$

And

$$\mathbf{B} \mapsto \mathbf{B} \cdot \boldsymbol{\sigma} = \mathbf{B_a} . \boldsymbol{\sigma_a} + \mathbf{B_b} . \boldsymbol{\sigma_b} + \mathbf{B_c} . \boldsymbol{\sigma_c} \quad (5)$$

and any physical scalar quantity through following mapping given below

$$\mathbf{A} \mapsto \mathbf{A} . \boldsymbol{\sigma} = A_0 . \boldsymbol{\sigma_0} \quad (6)$$

Where we have following matrix algebra.

$$\boldsymbol{\sigma_a} . \boldsymbol{\sigma_b} = j \cdot \boldsymbol{\sigma_c},\ j^2 = -1 \quad (7)$$

a,b,c are cyclic and

$$\boldsymbol{\sigma_a} . \boldsymbol{\sigma_a} = \boldsymbol{\sigma_b} . \boldsymbol{\sigma_b} = \boldsymbol{\sigma_c} . \boldsymbol{\sigma_c} = \boldsymbol{\sigma_0} . \boldsymbol{\sigma_0} = \boldsymbol{\sigma_0} \quad (8)$$

So that one can handle simultaneously scalar, vector quantity, scalar product and vector product easily seen form following equation.

$$\mathbf{A} \cdot \mathbf{B} = (\mathbf{A} \cdot \mathbf{B}) + j (\mathbf{A} \otimes \mathbf{B}) \quad (9)$$

Now we define product rule so that new physical quantity such that its existence would depend upon as A and B given by following equation

$$A_{\pm} \cdot B_{\pm} = C_{\pm} \quad (10)$$

$$C_{\pm} = C_{scalar} \mp (\mathbf{C}_{polar} \pm j \cdot \mathbf{C}_{axial}) \quad (11)$$

$$C_{scalar} = (A_o \cdot B_o - \mathbf{A} \cdot \mathbf{B}) \quad (12)$$

$$\mathbf{C}_{polar} = (A_o \cdot \mathbf{B} - \mathbf{A} \cdot B_o) \quad (13)$$

$$\mathbf{C}_{axial} = (\mathbf{A} \otimes \mathbf{B}) \quad (14)$$

This definition show that C contain scalar, polar-vector and axial-vector parts whereas A and B contain scalar and vector part.

Let
$I_o = \gamma$

And

$\mathbf{I} = \gamma \cdot \boldsymbol{\beta}$

So that we can have unitary quantity by following equation.

$$I_{\pm} = (I_o \pm \mathbf{I}) \quad (15)$$

$$I_{\mp} = (I_o \mp \mathbf{I}) \quad (16)$$

so that it yield well known identity given below.

$$I_{\pm} I_{\mp} = (\gamma^2 - (\gamma \beta)^2) \sigma_0 = \sigma_0 \quad (17)$$

Now using above postulate we have following expression for momentum as

$$P_{\pm} = (P_o, \pm P) = (P_o \pm P) = P_o \cdot \boldsymbol{\sigma}_o \pm (\mathbf{P}_a . \boldsymbol{\sigma}_a + \mathbf{P}_b . \boldsymbol{\sigma}_b + \mathbf{P}_c . \boldsymbol{\sigma}_c) \quad (18)$$

and if we use idea of quantum mechanics the we have following mapping

$$(P_o, \pm P) \rightarrow j \cdot \hbar \; (\partial_o, \mp \partial) \quad (19)$$

which can be used to obtain differential of physical quantity. ·

2.1 Invariant Physical Scalar Quantity and Four-vector representation:
Now with the help of above set of postulate, definitions and equations only we can easily construct, different basic invariant physical quantity and its corresponding four-vector representation very easily. The components of four-vector velocity can be obtained from an observed invariant velocity of light c as a transformation relation as

$$\mathbf{U}_{\pm} = \mathbf{I}_{\pm} \cdot \mathbf{c} = (\gamma \pm \gamma \cdot \beta) \cdot c = (\gamma \cdot c, \pm \gamma \cdot \mathbf{V}) \quad (20)$$

Similarly four-vector for momentum can be obtain as

$$\mathbf{P}_{\pm} = \mathbf{I}_{\pm} \cdot m_o \cdot \mathbf{c} = (\gamma \pm \gamma \cdot \boldsymbol{\beta}) \, mo \cdot c = (P_{o,} \pm \mathbf{P}) \quad (21)$$

It is known in the context of special relativity that a charge distribution that is static in one frame will appear to be a current distribution in another inertial frame. It implies that the current and charge densities are not distinct entities and their relationship may be presented with the help of rest charge density as given below.

$$\mathbf{J}_{\pm} = \mathbf{I}_{\pm} \cdot \rho_o \cdot c = (\gamma \pm \gamma \cdot \beta) \cdot \rho_o \cdot c = (J_{o,} \pm \mathbf{J}) \quad (22)$$

Lastly we would construct four-vector potential from scalar potential as given below.

$$\mathbf{A}_{\pm} = \mathbf{I}_{\pm} \cdot \boldsymbol{\varphi} = (\gamma \pm \gamma \cdot \beta) \cdot \varphi = (A_{o,} \pm \mathbf{A}) \quad (23)$$

2.2 Invariant Physical Vector Quantity and Four-vector representation:
If the momentum of a particle

$$\mathbf{P} = m_o \cdot \mathbf{V}$$

then its four-vector would be as

$$\mathbf{P}_{\pm} = \mathbf{I}_{\pm} \cdot m_o \cdot \mathbf{V} = (\gamma \pm \gamma \cdot \beta) \cdot m_o \cdot \mathbf{V} = (\pm \boldsymbol{\beta} \cdot \mathbf{P}, \mathbf{P}) \quad (24)$$

If the force on a particle is F then Minkowski force can be smoothly obtained as

$$\mathbf{K}_{\pm} = \mathbf{I}_{\pm} \cdot \mathbf{F} = (\gamma \pm \gamma \cdot \beta) \cdot F = (\pm \cdot \gamma \cdot \beta \cdot \mathbf{F}, \boldsymbol{\gamma} \cdot \mathbf{F} \pm j \cdot \gamma \cdot (\boldsymbol{\beta} \otimes \mathbf{F})) \quad (25)$$

Above equation shows that

$$(\beta \otimes \mathbf{F}) = 0$$

2.3 Transformation of Four-vector Quantity:
General transformation properties of any four-vector J (charge-current) are given below.

$$I_{\pm} \cdot J_{\mp} = J^{*}_{\mp} \quad (26)$$

$$J^{*}_{\pm} = J^{*}_{scalar} \mp (\mathbf{J}^{*}_{polar} \pm j \cdot \mathbf{J}^{*}_{axial}) \quad (27)$$

$$J^{*} scalar = (I_o \cdot Jo - \mathbf{I} \cdot \mathbf{J}) \quad (28)$$

$$\mathbf{J}^{*}_{polar} = (I_o \cdot \mathbf{J} - \mathbf{I} \cdot J_o) \quad (29)$$

$$\mathbf{J}^{*}_{axial} = (\mathbf{I} \otimes \mathbf{J}) \quad (30)$$

2.4 Electrodynamics:

We can obtain definition of electric field, magnetic field and Lorentz condition simultaneously when we consider following expression for force which can be obtained with the help of four-differentiation of four-vector potential i.e.

$$\mathbf{F}_{\pm} = (\partial_o \pm \partial)\ (A_o \pm \mathbf{A}) = (G_{scalar}, \pm\ (\mathbf{G}_{polar} \pm j\ \ \mathbf{G}_{axial}))\ ; \quad (31)$$

$$G_{scalar} = (\partial_o \cdot A_o + \partial \cdot \mathbf{A}) = 0 \quad (32)$$

$$\mathbf{G}_{polar} = (\partial_o \cdot \mathbf{A} + \partial \cdot \mathbf{A}_o) = \mathbf{E} \quad (33)$$

$$\mathbf{G}_{axial} = (\partial \otimes \mathbf{A}) = \mathbf{B} \quad (34)$$

Hence an easy path to get Lorentz condition along with an regular definition of electric field and magnetic field.

In the next step ,using the four-differential operator we can easily obtain a set of homogeneous/inhomogeneous Maxwell's equations simultaneously from following single equation.

$$(\partial_o \pm \partial) \cdot (\pm \mathbf{E} + j \cdot \mathbf{B}) = (4 \cdot \pi \div c \cdot (J_o, \pm \mathbf{J}) \quad (35)$$

$$(4 \cdot \pi \div c) \cdot J_o = \partial \cdot \mathbf{E} \mp j \cdot \partial \cdot \mathbf{B} \quad (36)$$

$$(4\,\pi \div c) \cdot \mathbf{J} = (\partial \otimes \mathbf{B} - \partial_o \cdot \mathbf{E}) \mp j \cdot (\partial \otimes \mathbf{E} + \partial_o \mathbf{B}) \quad (37)$$

So on separating real and imaginary part we have a statement of Coulomb's Law.

$$(4 \cdot \pi \div c) \cdot J_o = \partial \cdot \mathbf{E} \quad (38)$$

and showing that absence of free magnetic poles.

$$\partial \cdot \mathbf{B} = 0 \quad (39)$$

Similarly we have a statement for Ampere's law.

$$g_{\pm}^{*} = g^{*}_{scalar} \mp (c \cdot \mathbf{g}^{*}_{polar} \mp j \cdot c \cdot \mathbf{g}^{*}_{axial}) \quad (52)$$

$$g^{*}_{scalar} = \gamma\ (g_o - \mathbf{V} \cdot \mathbf{g}) \quad (53)$$

$$c\ \mathbf{g}^{*}_{polar} = \gamma\ (c \cdot \mathbf{g} - \boldsymbol{\beta} \cdot g_o) \tag{54}$$

$$c \cdot \mathbf{g}^{*}_{axial} = \gamma \cdot (\mathbf{V} \otimes \mathbf{g}) \tag{55}$$

But for electrostatic configuration the magnetic field in K is given by

$$\mathbf{B} = \boldsymbol{\beta} \otimes \mathbf{E}$$

so that

$$(g_o - \mathbf{V} \cdot \mathbf{g}) = (\mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B}) \div (8 \cdot \pi) \tag{56}$$

Let the interaction between electromagnetic field and four-vector potential eld a physical quantity be represented as

$$a_{\pm} = A_{\pm} \cdot F_{\pm} = (A_o \pm \mathbf{A}) \cdot (\pm \mathbf{E} + j \cdot \mathbf{B}) = (-\mathbf{A} \cdot \mathbf{E} \pm (A_o \cdot \mathbf{E} + \mathbf{A} \otimes \mathbf{B}) \tag{57}$$

and imaginary part of four-vector is

$$(\pm \mathbf{A} \cdot \mathbf{B} + (A_o \cdot \mathbf{B} - \mathbf{A} \otimes \mathbf{E}) = 0 \tag{58}$$

which is zero. So that we can easily construct following very useful expression in a direction to construction of Lagrangian density for electromagnetic field-particle interaction as

$$L_{\pm} = (\partial_o \pm \partial) \cdot (a_o \pm \mathbf{a}) = (L_{scalar}, \pm (\mathbf{L}_{polar} \pm j \cdot \mathbf{L}_{axial})) \tag{59}$$

Now electromagnetic Lagrangian density for field- field and field-particle could be obtain from single equation

$$L_{scalar} = (\partial_o \cdot a_o + \partial \cdot \mathbf{a}) = 0 \tag{60}$$

along with following conditions.

$$\mathbf{L}_{polar} = (\partial_o \cdot \mathbf{a} + \partial \cdot a_o) = 0 \tag{61}$$

$$\mathbf{L}_{axial} = (\partial \otimes \mathbf{a}) = 0 \tag{62}$$

Invariant interaction between the charged particle and eld is

$$J_{\pm} \cdot A_{\mp} = O_{\mp} \tag{63}$$

$$O_{\pm} = O_{scalar} \pm (O_{polar} \pm j \cdot O_{axial}) \tag{64}$$

Following are the natural condition that the interaction between the charged particle and field to be invariant.

$$\mathbf{O}_{polar} = (J_o \cdot \mathbf{A} - \mathbf{J} \cdot A_o) = 0 \tag{66}$$

$$\mathbf{O}_{axial} = (\mathbf{J} \otimes \mathbf{A}) = 0 \tag{67}$$

We can define generalize momentum using superposition principle as

$$P_{\pm} = (P_o + e \cdot A_o/c) \pm (\mathbf{P} + e \cdot \mathbf{A}/c) \quad (68)$$

Taking four-differentiation of above generalize momentum we have equivalent statement of Newton's law for a particle- field interaction where

$(P_o \cdot c)$
represent a potential energy and the term

$$\partial_o (\mathbf{P} \cdot c) = \mathbf{F} \quad (69)$$

is interpreted as a rate of change of momentum of a particle

$$\mathbf{F}_{\pm} = (\partial_o \pm \partial) \cdot (P \cdot c) = (F_{scalar}, \pm (\mathbf{F}_{polar} \pm j \cdot \mathbf{F}_{axial})) \quad (70)$$

with additional requirements in the form of an equations i.e. equation of continuity etc.

$$F_{scalar} = (\partial_o \cdot (P_o \cdot c + e \cdot A_o) + \partial \cdot (\mathbf{P} \cdot c + e \cdot \mathbf{A})) = 0 \quad (71)$$

and Newton's statement under equilibrium condition as

$$\mathbf{F}_{polar} = (\partial_o \cdot (\mathbf{P} \cdot c + e \cdot \mathbf{A}) + \partial \cdot (P_o \cdot c + e \cdot A_o) = 0 \quad (72)$$

Note that curl of a particle momentum and magnetic field are directly connected.

$$\mathbf{F}_{axial} = (\partial \otimes (\mathbf{P} \cdot c + e \cdot \mathbf{A})) = 0 \quad (73)$$

The Lagrangian treatment of mechanics is based on the principle of least action.in a nonrelativistic mechanics the system is describe by generalized coordinate and velocities i.e.(a displacement, momentum representation). The Lagrangian L is a functional of coordinate and velocities and the action A is de ned as the time integral of Lagragian L along a path of the system i.e. in integral form we have a following definitions.

$$P_{\pm} = ((P_o + Q_o) \pm (\mathbf{P} + \mathbf{Q})) \quad (74)$$

$$d\mathbf{s}_{\pm} = (dx_o \mp \mathbf{dx}) \quad (75)$$

$$P_{\pm} \cdot ds_{\pm} = Ps_{\mp} \quad (76)$$

$$Ps_{\pm} = Ps_{scalar} \mp (\mathbf{Ps}_{polar} \pm j \cdot \mathbf{Ps}_{axial}) \quad (77)$$

$$Ps_{scalar} = ((P_o + Q_o) \cdot dx_o - (\mathbf{P} + \mathbf{Q}) \cdot \mathbf{dx}) \quad (78)$$

$$\mathbf{Ps}_{polar} = ((P_o + Q_o) \cdot \mathbf{dx} - (\mathbf{P} + \mathbf{Q}) \cdot dx_o) = 0 \quad (79)$$

$$\mathbf{Ps}_{axial} = (\mathbf{P} + \mathbf{Q}) \otimes \mathbf{dx} = 0 \quad (80)$$

$$\int L_{\mp} \cdot dt = -\int P_{\pm} \cdot ds_{\mp} \tag{81}$$

So that we have relativistic Lagrangian for a simple particle as given below.

$$L = - m_o \cdot c^2 \cdot \sqrt{1-\beta^2} - (Q_o \cdot c - \mathbf{Q} \cdot \mathbf{v}) \tag{82}$$

$$(Q_o \cdot c - \mathbf{Q} \cdot \mathbf{v}) = \varphi(\mathbf{x}) \cdot (1-\beta^2) \tag{83}$$

On similar line the Lagrangian for a single particle in an electromagnetic field could be defined.

CONCLUSION

Author would like to mention that no references in particular are cited in the paper, and have only referred the well established theoretical development reported in classical text book[1-3]. Hence found another equivalent way, an interesting simple representation technique with simple mapping to obtain covariant form of physical quantity along with additional relations.

REFERENCES

[1] W. Pauli. Theory of Relativity, (Translated from German by G.Field).
[2] J.D.Jackson. Classical Electrodynamics, (Wiley Eastern Limited).
[3]Siegfried Flugge. Practical Quantum Mechanics, (Springer International).