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Flow of blood through stenosed artery: A peripheral layer model

Padma Joshi and Garima Gadkari

Department of Mathematics, Mahakal Institute of Technology, Ujjain, India

ABSTRACT

The present paper deals with a mathematical model of blood flow through stenosed artery. The flowing blood has been represented by a two – layered model. Analytical expressions have been obtained for resistance to flow and wall shear stress. The impact of concerned parameters have been examined and depicted through graphs for different values of interest. The outcomes of this paper are compared with previous known results. The model can be useful for approximation of diseased arterial system.

Keywords: Stenosis, Resistance to flow, Arterial wall, Peripheral layer viscosity.

INTRODUCTION

Stenosis is the term which is used in medical science for narrowing of an artery. It is abnormal and unnatural growth in an arterial wall that can be developed at different locations of the cardiovascular systems under diseased condition. The study of blood flow in cardiovascular systems is very important. The basics of fluid mechanics and governing equations of flow were discussed in detail by Fung [1]. The different models of blood flow have been investigated by Kapur [2] and Biswas [3]. There are so many researchers worked on the Newtonian behaviour of blood flow like Saleh and khan [4], Sarifuddin et al. [5] and Siddiqui et al. [6]. On the other side, some researchers worked on the non-Newtonian behaviour of blood like Biswas and Chakraborty [7], Jain et al. [8], Srivastava et al. [9] and Mallik et al. [10]. It is known that the formulation of stenosis is normally symmetric about the wall of the artery. The mathematical formulation for this type of stenosis can be found in literature. The different geometries suggested by various researchers like Shukla et al. [11], Joshi et al. [12], Zuhaila [13] and Joshi and Pathak [14] includes cosine, composite, semi - circular and triangular shaped formation of stenosis. The effects of peripheral layer viscosity in a mildly stenosed tube having cosine, composite and trapezium shaped stenosis have been investigated by Shukla et al. [15], Joshi et al. [16] and Singh et al. [17] respectively. Sankar et al. [18] have discussed the flow of blood through small vessels in the presence of composite stenosis using a two – layered model giving stress on red blood cells concentration. In this paper a triangular geometry of stenosis having a peripheral layer of different viscosity has been discussed and results for resistance to flow and wall shear stress have been obtained.

2. Formulation of the Problem

We assumed that blood is an incompressible fluid which is represented by a two-layered model. The external layer shows peripheral layer of plasma and the internal core layer describes the suspension of red blood cells. The mild axisymmetric triangular stenosis is present in the artery for which the schematic diagram is as follows :

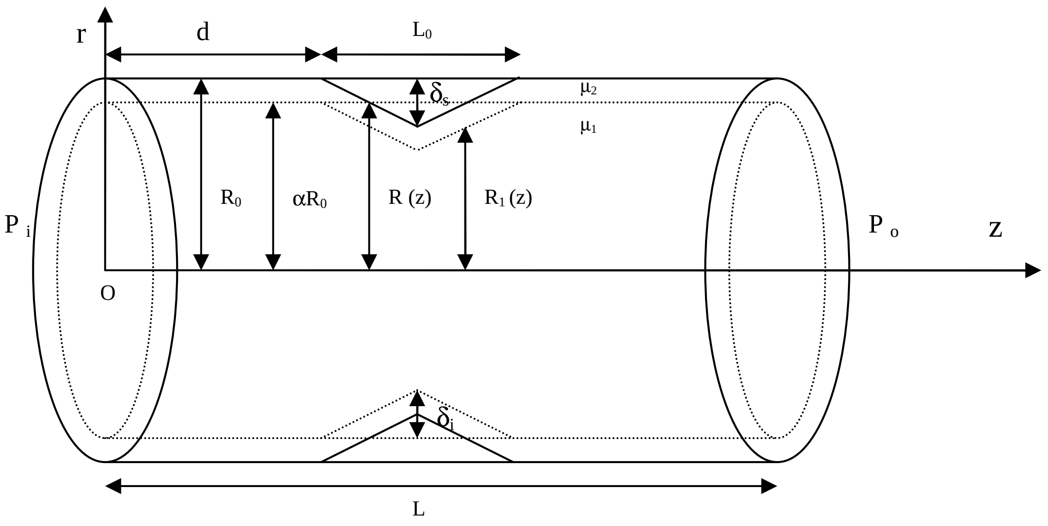

Figure 1: Geometry of stenosed artery

where the symbols stand for

R_0 : Radius of the non-stenotic region

$R(z)$: Radius of the stenotic region

$R_1(z)$: Radius of the central layer in stenotic region

L : The length of the artery

L_0 : The length of the stenosis

d : Location of stenosis

p_i : Inlet fluid pressure

p_0 : Instantaneous outlet fluid pressure

δ_s : Instantaneous maximum height of the stenosis

δ_i : Maximum bulging of interface

μ_1 : Viscosity of fluid in central core layer

μ_2 : Viscosity of fluid in peripheral layer

α : Ratio of central core radius to the tube radius.

The geometry of the stenotic tube without peripheral layer can be expressed as follows,

$$\frac{R(z)}{R_0} = \begin{cases} 1 - \frac{2\delta_S}{R_0 L_0}(z-d) & ; d \le z \le d + \frac{L_0}{2} \\ 1 - \frac{\delta_S}{R_0} + \frac{2\delta_S}{R_0 L_0}\left(z - d - \frac{L_0}{2}\right) & ; d + \frac{L_0}{2} \le z \le d + L_0 \\ 1 & ;\text{otherwise} \end{cases} \tag{1}$$

The governing equation of blood flow is given by Kapur [2],

$$0 = -\frac{dp}{dz} + \frac{1}{r}\frac{\partial}{\partial r}\left\{\mu(r)\, r \frac{\partial w}{\partial r}\right\} \tag{2}$$

where w is axial velocity, p is fluid pressure and $\mu(r)$ is viscosity of fluid.

The boundary conditions are,

$$w = 0 \text{ at } r = R(z) \quad \text{and} \tag{3}$$

$$\frac{\partial w}{\partial r} = 0 \text{ at } r = 0 \tag{4}$$

Solving equation (2) under boundary conditions (3) and (4), we get

$$w = \left(-\frac{1}{2}\frac{dp}{dz}\right)\int_r^R \frac{r}{\mu(r)}dr \tag{5}$$

The volumetric flow rate is given by

$$Q = \int_0^R 2\pi r\, w\, dr \tag{6}$$

which on using equation (5) gives,

$$Q = \left(-\frac{\pi}{2}\frac{dp}{dz}\right)\int_0^R \frac{r^3 dr}{\mu(r)} \tag{7}$$

Thus, the pressure gradient can be obtained as,

$$\frac{dp}{dz} = -\frac{2Q}{\pi I(z)} \tag{8}$$

where, $$I(z) = \int_0^R \frac{r^3 dr}{\mu(r)} \tag{9}$$

Integrating equation (8) using conditions $p = p_i \quad at \quad z = 0$ and

$p = p_0 \quad at \quad z = L$, we have

$$p_i - p_0 = \frac{2Q}{L}\int_0^L \frac{dz}{I(z)} \tag{10}$$

The resistance to flow λ is defined as,

$$\lambda = \frac{p_i - p_0}{Q} \tag{11}$$

From equations (1), (10) and (11), we can find

$$\lambda = \frac{2}{\pi}\left[\frac{L - L_0}{I_0} + \int_d^{d+\frac{L_0}{2}} \frac{dz}{I(z)} + \int_{d+\frac{L_0}{2}}^{d+L_0} \frac{dz}{I(z)}\right] \tag{12}$$

where $I_0 = \int_0^{R_0} \frac{r^3}{\mu(r)} dr$ (13)

Now, the shear stress at wall is given by,

$$\tau_R = \left[-\mu(r) \frac{\partial w}{\partial r} \right]_{r=R(z)} \quad (14)$$

By using equations (5) and (8) in (14), we can find shear stress at maximum height of stenosis i.e. at $z = d + \frac{L_0}{2}$, which is as follows,

$$\tau_s = \left[\frac{R(z)Q}{\pi I(z)} \right]_{z=d+\frac{L_0}{2}} \quad (15)$$

To calculate out the effects of peripheral layer viscosity, the viscosity function $\mu(r)$ can be defined as,

$$\mu(r) = \begin{cases} \mu_1 & ; 0 \le r \le R_1(z) \\ \mu_2 & ; R_1(z) \le r \le R(z) \end{cases} \quad (16)$$

where μ_1 and μ_2 are the viscosities of the central and the peripheral layers respectively. The function $R_1(z)$ represents the shape of the central layer with stenosis. The mathematical representation of this model can be described as,

$$\frac{R_1(z)}{R_0} = \begin{cases} \alpha - \frac{2\delta_i}{R_0 L_0}(z-d) & ; d \le z \le d + \frac{L_0}{2} \\ \alpha - \frac{\delta_i}{R_0} + \frac{2\delta_i}{R_0 L_0}\left(z - d - \frac{L_0}{2} \right) & ; d + \frac{L_0}{2} \le z \le d + L_0 \\ \alpha & ; \text{otherwise} \end{cases} \quad (17)$$

where α is ratio of central core radius to the tube radius in the unobstructed region. By using equation (16) in (5), velocities w_c and w_p can be obtained and then the corresponding volumetric flow rates Q_c and Q_p are obtained as follows,

$$Q_c = \int_0^{R_1} 2\pi r w_c \, dr = \left(-\frac{\pi}{8\mu_2} \frac{dp}{dz} \right) 2R_1^2 \left[R^2 - \left(1 - \frac{\bar{\mu}_2}{2} \right) R_1^2 \right] \quad (18)$$

$$Q_p = \int_{R_1}^{R} 2\pi r w_p \, dr = \left(-\frac{\pi}{8\mu_2} \frac{dp}{dz} \right) \left(R^2 - R_1^2 \right)^2 \quad (19)$$

where $\bar{\mu}_2 = \mu_2 / \mu_1$

Thus, the total volumetric flow rate Q is defined as,

$$Q = Q_c + Q_p = \left(-\frac{\pi}{8\mu_2}\frac{dp}{dz}\right)\left(R^4 - (1-\overline{\mu_2})R_1^4\right) \tag{20}$$

equation (20) can also be obtained by equation (7) using (16) which shows that Q is a constant.
Integrating equation (18), (19) and (20) across the length of artery, assuming that pressure drop is same in each case. We obtain,

$$Q_c = \frac{(p_i - p_0)\pi R_0^4 S_1}{4\mu_2 L\left(1-\frac{L_0}{L}+S_1T_1\right)} \tag{21}$$

where $S_1 = \alpha^2\left\{1-\left(1-\frac{\overline{\mu}_2}{2}\right)\alpha^2\right\}$ (22)

and $T_1 = t_1 + t_2$ (23)

where,

$$t_1 = \frac{1}{L}\int_d^{d+\frac{L_0}{2}} \frac{dz}{\left(\frac{R_1}{R_0}\right)^2\left\{\left(\frac{R}{R_0}\right)^2 - \left(1-\frac{\overline{\mu}_2}{2}\right)\left(\frac{R_1}{R_0}\right)^2\right\}} \tag{24}$$

$$t_2 = \frac{1}{L}\int_{d+\frac{L_0}{2}}^{d+L_0} \frac{dz}{\left(\frac{R_1}{R_0}\right)^2\left\{\left(\frac{R}{R_0}\right)^2 - \left(1-\frac{\overline{\mu}_2}{2}\right)\left(\frac{R_1}{R_0}\right)^2\right\}} \tag{25}$$

and

$$Q_p = \frac{(p_i - p_0)\pi R_0^4 S_2}{8\mu_2 L\left(1-\frac{L_0}{L}+S_2T_2\right)} \tag{26}$$

where $S_2 = \left(1-\alpha^2\right)^2$ (27)

$T_2 = t_3 + t_4$ (28)

where,

$$t_3 = \frac{1}{L}\int_d^{d+\frac{L_0}{2}} \frac{dz}{\left\{\left(\frac{R}{R_0}\right)^2 - \left(\frac{R_1}{R_0}\right)^2\right\}^2} \tag{29}$$

$$t_4 = \frac{1}{L}\int_{d+\frac{L_0}{2}}^{d+L_0} \frac{dz}{\left\{\left(\frac{R}{R_0}\right)^2 - \left(\frac{R_1}{R_0}\right)^2\right\}^2} \tag{30}$$

now,

$$Q = \frac{(p_i - p_0)\pi R_0^4 S}{8\mu_2 L\left(1 - \frac{L_0}{L} + ST\right)} \tag{31}$$

where $S = 1 - (1 - \bar{\mu}_2)\alpha^4$ (32)

$$T = t_5 + t_6 \tag{33}$$

where,

$$t_5 = \frac{1}{L}\int_{d}^{d+\frac{L_0}{2}} \frac{dz}{\left\{\left(\frac{R}{R_0}\right)^4 - (1-\bar{\mu}_2)\left(\frac{R_1}{R_0}\right)^4\right\}} \tag{34}$$

$$t_6 = \frac{1}{L}\int_{d+\frac{L_0}{2}}^{d+L_0} \frac{dz}{\left\{\left(\frac{R}{R_0}\right)^4 - (1-\bar{\mu}_2)\left(\frac{R_1}{R_0}\right)^4\right\}} \tag{35}$$

from equations (21) to (31) and using $Q = Q_c + Q_p$, we can find

$$\frac{S}{\left(1 - \frac{L_0}{L} + ST\right)} = \frac{2S_1}{\left(1 - \frac{L_0}{L} + S_1 T_1\right)} + \frac{S_2}{\left(1 - \frac{L_0}{L} + S_2 T_2\right)} \tag{36}$$

Now using $R_1 = \alpha R$ in equation (17), we get

$$\frac{R(z)}{R_0} = \begin{cases} 1 - \frac{2\delta_i}{\alpha L_0 R_0}(z - d) & , d \le z \le d + \frac{L_0}{2} \\ 1 - \frac{\delta_i}{\alpha R_0} + \frac{2\delta_i}{\alpha L_0 R_0}\left(z - d - \frac{L_0}{2}\right) & , d + \frac{L_0}{2} \le z \le d + L_0 \\ 1 & , \text{otherwise} \end{cases} \tag{37}$$

On comparing equation (1) and (37), we can observe that

$$\delta_i = \alpha \delta_s \tag{38}$$

Now by keeping in mind equation (16), the dimensionless resistance to flow $\bar{\lambda}$ and the dimensionless shear stress $\bar{\tau}_s$ can be obtained by using equation (31) in equations (11) and (15) respectively, as follows

$$\bar{\lambda} = \frac{\lambda}{\lambda_0} = \frac{\overline{\mu_2}}{S}\left(1 - \frac{L_0}{L} + ST\right) \tag{39}$$

where, $\overline{\mu_2} = \frac{\mu_2}{\mu_1}$ and $\lambda_0 = \frac{8\mu_1 L}{\pi R_0^4}$

and

$$\bar{\tau}_s = \frac{\tau_s}{\tau_0} = \frac{\overline{\mu_2}\left(1 - \frac{\delta_s}{R_0}\right)}{\left[\left(1 - \frac{\delta_s}{R_0}\right)^4 - \left(1 - \overline{\mu_2}\right)\left(\alpha - \frac{\delta_i}{R_0}\right)^4\right]} \tag{40}$$

where, $\tau_0 = \frac{4\mu_1 Q}{\pi R_0^3}$ and λ_0 , τ_0 are the resistance to flow and wall shear stress for the case of no stenosis respectively, with $\overline{\mu_2} = 1$.

Evaluating the integrals (34) and (35) after using equation (38) and rewriting the expressions for $\bar{\lambda}$ and $\bar{\tau}_s$ as follows,

$$\bar{\lambda} = \frac{\overline{\mu_2}}{S}\left[1 - \frac{L_0}{L} + \frac{L_0}{L}\left\{1 + 2\left(\frac{\delta_s}{R_0}\right) + \frac{10}{3}\left(\frac{\delta_s}{R_0}\right)^2 + \frac{9}{2}\left(\frac{\delta_s}{R_0}\right)^3 - 3\left(\frac{\delta_s}{R_0}\right)^4 + \frac{5}{3}\left(\frac{\delta_s}{R_0}\right)^5 + \ldots\right\}\right] \tag{41}$$

and

$$\bar{\tau}_s = \frac{\overline{\mu_2}}{\left[\left(1 - \frac{\delta_s}{R_0}\right)^3 S\right]} \tag{42}$$

Here $\bar{\tau}_s$ obtained is same as in Shukla et al. [15].

RESULTS AND DISCUSSION

In the present work, flow of blood through a stenosed artery has been considered. This model consists of a peripheral layer of plasma and a core region of erythrocytes in plasma with different viscosities. The resistance to flow $\bar{\lambda}$ and wall shear stress $\bar{\tau}_s$ have been plotted for different values of parameters. Figs. 2, 3, 4 and 5 represent the variations of $\bar{\lambda}$ and $\bar{\tau}_s$ with δ_s / R_0 for different values of $\overline{\mu_2}$ and L_0 / L respectively. It has been analyzed that resistance to flow $\bar{\lambda}$ and wall shear stress $\bar{\tau}_s$ increases with the increase in the height of stenosis. In the same manner it has been observed that on increase in $\overline{\mu_2}$ there is a finite jump in the values of $\bar{\tau}_s$ and $\bar{\lambda}$. It concludes that peripheral layer thickness is an important factor in blood flow. It can also be seen by the graphs that results obtained in present analysis are in good agreement with the solutions of Joshi et al. [16] and Shukla et al. [15] for different important flow parameters.

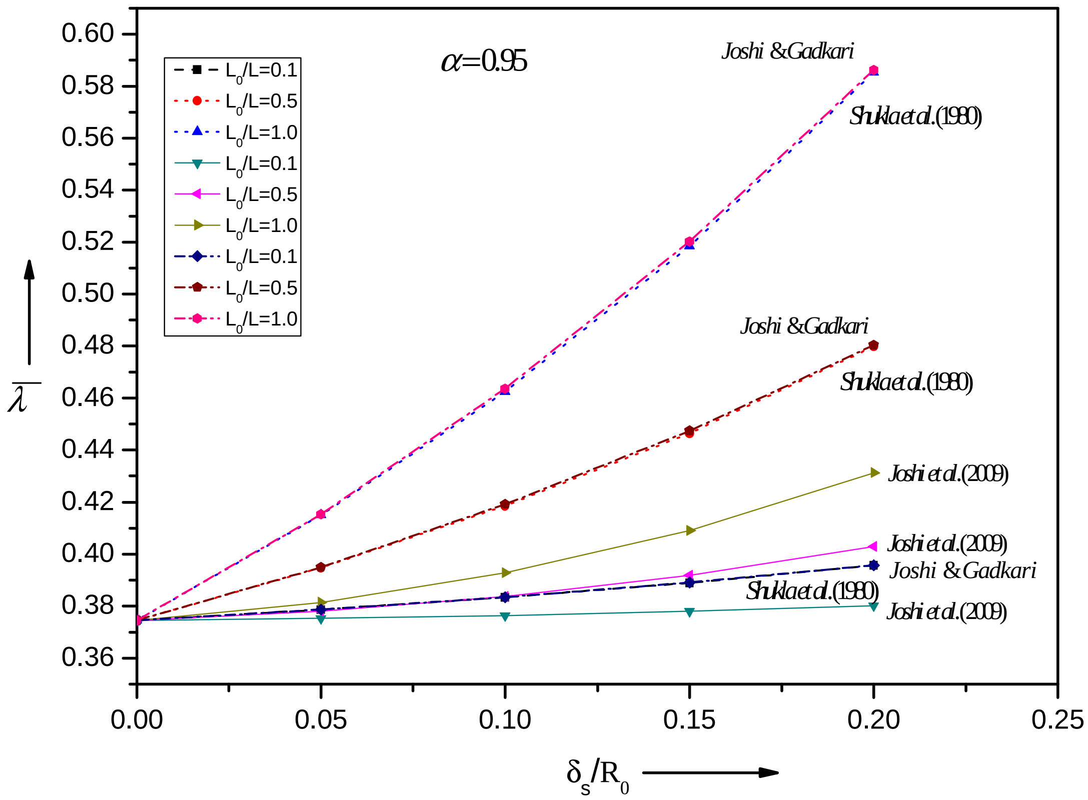

Figure 2. Variation of $\bar{\lambda}$ **with** δ_s / R_0 **for different values of** L_0 / L **and** $\bar{\mu}_2 = 0.1$

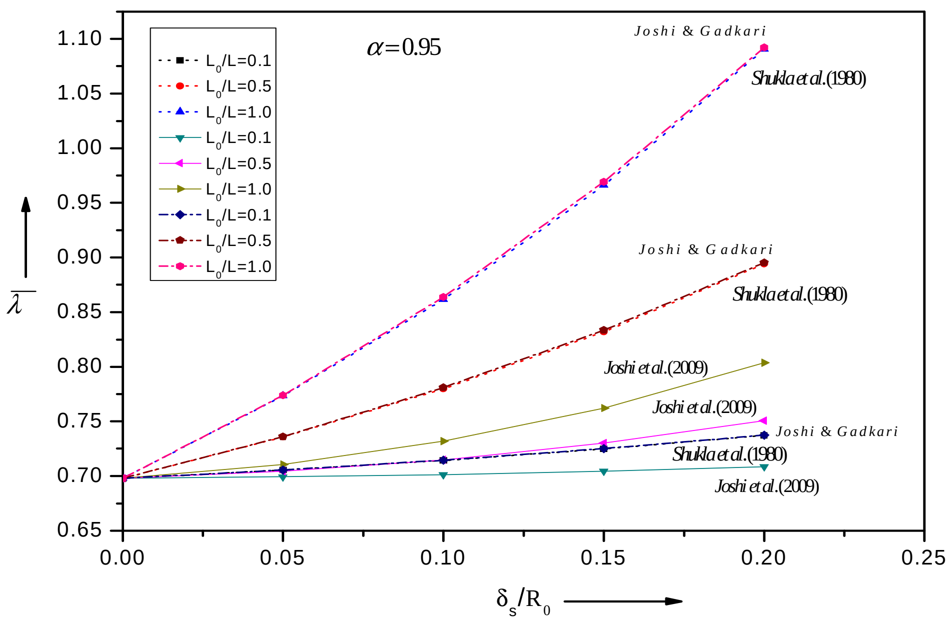

Figure 3. Variation of $\bar{\lambda}$ **with** δ_s / R_0 **for different values of** L_0 / L **and** $\bar{\mu}_2 = 0.3$

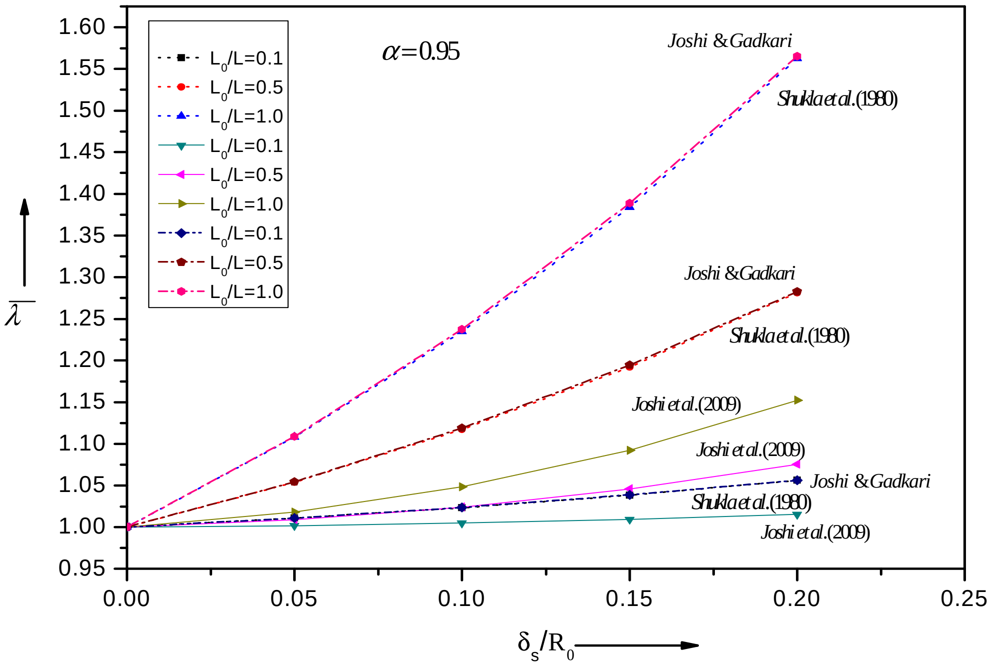

Figure 4. Variation of $\overline{\lambda}$ **with** δ_s / R_0 **for different values of** L_0 / L **and** $\overline{\mu}_2 = 1.0$

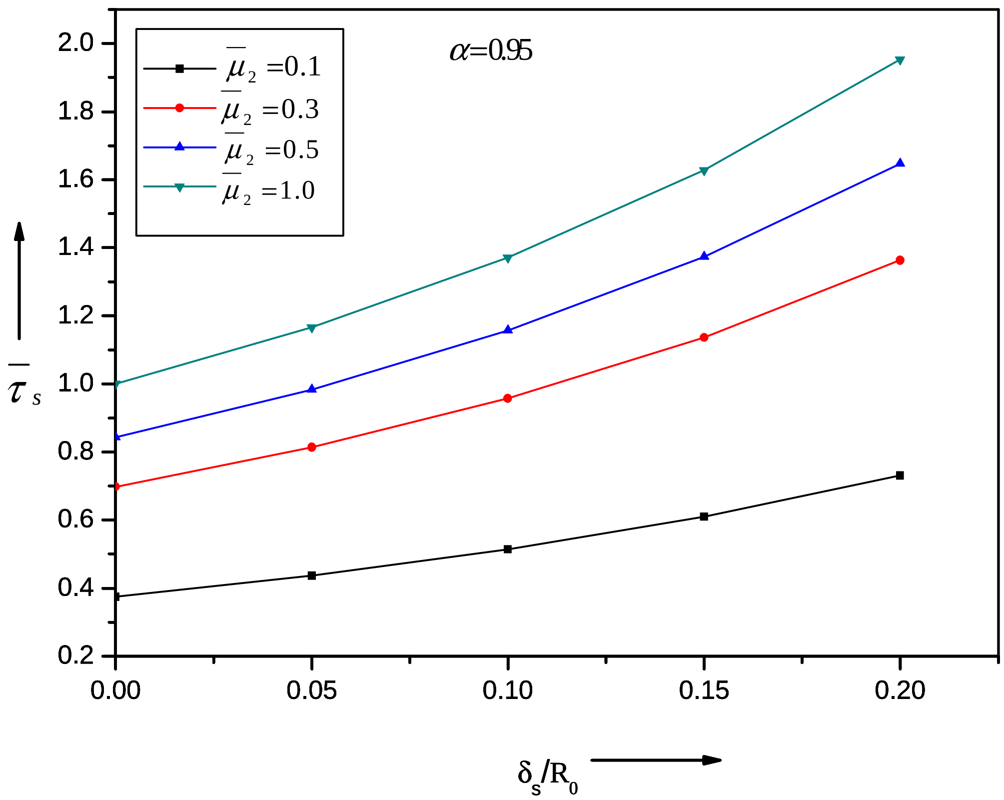

Figure 5. Variation of $\overline{\tau}_s$ **with** δ_s / R_0 **for various values of** $\overline{\mu}_2$.

CONCLUSION

The effects of external peripheral layer of plasma on the flow of blood have been obtained in an artery having mild stenosis. The conclusions are drawn on the basis of resistance to flow and wall shear stress. A comparative analysis is shown with the help of graphs. The numerical computations have been performed by Mathematica software. Thus it can be concluded that a two-layered behaviour of blood is more realistic one and helps in representing the diseased arterial system.

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