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Fixed point theorems in PM-spaces satisfying strict contractive condition for weakly compatible maps

G. P. S. Rathore[1], Bijendra Singh[2] and Kirty Chauhan[3]

[1]College of Horticulture, Mandsaur (M. P.), India
[2]Department of Mathematics, Vikram University, Ujjain (M. P.), India
[3]Department of Mathematics, A T C, Indore (M. P.), India

ABSTRACT

The purpose of this paper is to generalize weak compatible mappings with sharing the property (E. A.) and derive some fixed point theorems in the framework of Menger spaces, which demonstrate the utility of strict contractive condition. This work extends the results contained in available research work over Menger spaces as well as metric spaces.

Keywords: Fixed point, compatible maps, weakly compatible maps, property (E. A.).

INTRODDUCTION

One important generalization of metric space that is probabilistic metric space was introduces by Menger [2] and expanded by Schweizer and Sklar [6]. This has fundamental and paramount importance in Probabilistic functional analysis, where contraction is one of the main tools to prove existence and uniqueness of fixed point. Notion of compatible mappings in metric spaces is introduced by Jungck [3], who give specific direction to many researchers. However, non-compatible mappings are also equally interested and initiated by Pant [8, 9].The study of common fixed points of weak-compatible mapping be an interesting aspect for further investigation and extent by well known obtained results of [1, 7].

It is possible to prove fixed point theorem beyond compact metric spaces, on strict contraction of non-compatible mappings. Sometimes the strict conditions are replaced by some stronger conditions as [3, 8] because in the setting of metric space, the strict contractive condition do not ensure the existence of common fixed point. This unique concept was generally used to promote existing theorems. Research along this direction has been initiated by many mathematicians.

Aamri and Mountawakil [10] give a property (E.A.) which is generalization of compatible and non-compatible mappings. Several researchers extended this in various spaces. It has been noticed by Imdad and Ali [4] that property (E.A.) can be realized without following any pattern of containment of range of one map into the range of other. In view of their observations two fixed point theorems are slightly formed and we prove them in pattern of 2-menger spaces.

MATERIALS AND METHODS

We begin with some known definitions.
Definition 2.1[1]: A Probabilistic metric space (PM space) is a pair (X, F), where X is a non empty set and F is a mapping from X × X into Δ^+ (set of all distribution functions). For (u, v) ∈ X × X, the distribution function F(u, v) is denoted by $F_{u,v}$. The function F(u, v) assumed to satisfy the following conditions:

(PM1) $F_{u,v}(0) = 0$ for $\exists u, v \in X$,
(PM2) $F_{u,v}(x) = 1$ for $\exists\, x > 0 \Leftrightarrow u = v$,
(PM3) $F_{u,v}(x) = F_{v,u}(x)$ for $\exists\, u, v \in X$,
(PM4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$.
Then $F_{u,w}(x + y) = 1$ for $\exists\, u, v, w \in X$

Definition 2.2[1]: A menger space is a triple (X, F, t) where (X, F) is a PM space and t is T-norm with the following condition:

(PM5) $F_{u,w}(x+y) \geq t(F_{u,v}(x), F_{v,w}(y))$ for $\exists u, v, w \in X$ and $x, y \in R^+$

Definition 2.3[14]: Let x be any nonempty set and Δ^+ the set of all left continuous distribution functions. A triplet (X, F, t) is said to be a 2-menger space if F is a mapping from X^3 into Δ^+ satisfying the following conditions where the value of F at $u, v, w \in X^3$ is denoted by $F_{u,v,w}$ or F(u, v, w) for $\exists u, v, w \in X$ such that

(2MS1) $F_{u,v,w}(0) = 0$,
(2MS2) $F_{u,v,w}(x) = 1$ for $\exists\, x > 0 \Leftrightarrow$ at least two of $u, v, w \in X$ are equal,
(2MS3) $F_{u,v,w}(x) = F_{u,w,v}(x) = F_{w,v,u}(x)$ for $\exists\, x > 0$ and $u, v, w \in X$,
(2MS4) $F_{u,v,w}(x) \geq t(F_{u,v,s}(x), F_{u,s,w}(y), F_{s,v,w}(z))$.
Where x, y, z > 0, $u, v, w, s \in X$ and t is the 3rd order t norm.

Definition 2.4[1]: Let (X^3, F, t) be a 2-menger space such that the T-norm t is continuous and S, T be mapping from X into itself. Then S and T are said to be compatible if $\lim_{n\to\infty} F(STx_n, TSx_n(x)) = 1$ for all x > 0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z ∈ X.

Definition 2.5[4]: A pair (S,T) of self mappings of a 2-menger space (X^3, F, t) is said to be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} F(STx_n, TSx_n(x)) =$ either less than 1 or nonexistent, for some x > 0.

Definition 2.6[1]: Two self mappings S and T are said to be weakly compatible f they commute at their coincidence points, i. e. if Tu = Su for some u ∈ X, then TSu = STu.

Note: Every pair of weakly compatible mappings need not be compatible.

Definition 2.7[1]: Let (S,T) be a pair of self mappings of a 2-menger space (X^3, F, t). we say that S and T satisfy property (E. A.) iff there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z ∈ X
Clearly, a pair of compatible as well as non-compatible mappings satisfies property (E. A.).

Remark: By taking the reference of Sharma and Deshpande [12], Sharma and Choubey [13] and Jungck [5] it is clear that the pair of self mappings (S, T) of 2-menger space (X^3, F, t), is non compatible if there exists any sequence $\{x_n\} \in X$ such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z ∈ X
But $\lim_{n\to\infty} F(STx_n, TSx_n)$ is either non-existent or not equal to 1. In this way every pair of non compatible self mappings of 2-menger spaces satisfy the property (E. A.).

Definition 2.8[11]: A self mapping S of a 2-menger space (X^3, F, t) is said to be strict contraction on X, if for $u \neq v \neq w \in$ X, F(u, v, w) > F(Su, Sv, w).

Definition 2.9: Let X be a set , S and T be self maps of X. A point x ∈ X is called coincidence point of S and T iff Sx = Tx. We shall call w = Sx = Tx a point of coincidence of S and T.

Lemma[5]: Let S and T be weakly compatible self mappings of a set X. If S and T have a unique point of coincidence, that is, w = Sx = Tx, then w is the unique common fixed point of S and T.

RESULTS AND DISCUSSION

In this section we utilize results of [1] and [4] to derive corresponding common fixed point theorem in area of 2-menger space.

Theorem 3.1: Let (X^3, F, t) be a 2-menger space with two weakly compatible mappings S and T of X into itself satisfying the following inequality

(i) T(X) ⊂ S(X),
(ii) S and T satisfy the property (E.A.),
(iii) S(X) or T(X) be a closed subset of X,
(iv) $F(y_{2n}, y_{2n+1}(kx), y_{2n+2}(kx)) \leq F(y_{2n}, y_{2n+1}(x), y_{2n+2}(x))$
(v) F(Tu, Tv(kx), Tw(ky)) ≥ min{F(Su, Sv(x), Sw(y)), F(Su, Tu(x), Tw(y)), F(Sv, Tv(x), Tw(y)), F(Sv, Tu(x), Tw(y)), F(Su, Tv(x), Tw(y))}

where $u, v, w \in$ X and k ∈ (0, 1).

Then S and T have a unique common fixed point.

Proof: As S and T satisfies property (E. A.), so that there exists a sequence $\{x_n\}$ in X such that
$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n$ = z for some z ∈ X

Because S(X) is closed, then we have $\lim_{n\to\infty} Sx_n$ = Sa and also $\lim_{n\to\infty} Tx_n$ = Sa for a ∈ X.
To show that Sa = Ta, we starts from Sa ≠ Ta.
Now By (v) we have

F(Tx_n, Ta(kx), Ta(ky)) ≥ min{F(Sx_n, Sa(x), Sa(y)), F(Sx_n, Tx_n(x), Ta(y)), F(Sa, Ta(x), Ta(y)), F(Sa, Tx_n(x), Ta(y)), F(Sx_n, Ta(x), Ta(y))}
Letting $n \to \infty$, yield

F(Sa, Ta(kx), Ta(ky)) ≥ min{F(Sa, Sa(x), Sa(y)), F(Sa, Sa(x), Ta(y)), F(Sa, Ta(x), Ta(y)), F(Sa, Sa(x), Ta(y)), F(Sa, Ta(x), Ta(y))}
By (i)

≥ min{F(Sa, Ta(x), Ta(y)), F(Sa, Ta(x), Ta(y)), F(Sa, Ta(x), Ta(y)), F(Sa, Ta(x), Ta(y)), F(Sa, Ta(x), Ta(y))}
≥ F(Sa, Ta(x), Ta(y))

Which is contradiction, so that Sa = Ta.

As S and T are weakly compatible mappings i. e. TSa = STa and therefore TSa = STa = SSa = TTa. To represent Ta is a common fixed point of S and T, we initiate with Ta ≠ TTa. By (v)

F(Ta, TTa(kx), TTa(ky)) ≥ min{F(Sa, STa(x), STa(y)), F(Sa, Ta(x), TTa(y)), F(STa, TTa(x), TTa(y)), F(STa, Ta(x), TTa(y)), F(Sa, TTa(x), TTa(y))}

≥ min{F(Ta, TTa(x), TTa(y)), F(Ta, Ta(x), TTa(y)), F(TTa, TTa(x), TTa(y)), F(TTa, Ta(x), TTa(y)), F(Ta, TTa(x), TTa(y))}

≥ F(Ta, TTa(x), TTa(y))

Which is contradiction, so that Ta = TTa and thus, Ta = TTa = STa.

Above calculation shows that Ta is common fixed point of S and T. Uniqueness can be follows easily.

Theorem 3.2: Let (X^3, F, t) be a 2-menger space with three weakly compatible mappings A, B and S of X into itself satisfying the following inequality
(i) A(X) ⊂ S(X), B(X) ⊂ S(X),
(ii) (A, S) and (B, S) satisfy the property (E.A.),
(iii) One of A(X), B(X) or S(X) is a closed subset of X,
(iv) F(y_{2n}, y_{2n+1}(kx), y_{2n+2}(kx)) ≤ F(y_{2n}, y_{2n+1}(x), y_{2n+2}(x))
(v) F(Au, Bv(kx), Sw(ky)) ≥ min{F(Su, Sv(x), Sw(y)), F(Su, Bv(x), Aw(y)), F(Sv, Bv(x), Av(y)), F(Au, Su(x), Bu(y)), F(Au, Sv(x), Bw(y))}

where u, v, w ∈ X and k ∈ (0, 1).

Then A, B and S have a unique fixed point.

Proof: As (A, S) and (B, S) satisfies property (E. A.), so that there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ax_n$ = $\lim_{n\to\infty} Bx_n$ = $\lim_{n\to\infty} Sx_n$ = z for some z ∈ X
By (i), there exist a sequence $\{y_n\}$ in X such that Ax_n= Bx_n= Sy_n.

Hence $\lim_{n\to\infty} Sy_n$ = z.

Let us show that $\lim_{n\to\infty} Ay_n$ = $\lim_{n\to\infty} By_n$ = z
In view of (v)

F(Ay_n, Bx_n(kx), Sx_n(ky)) ≥ min{F(Sy_n, Sx_n(x), Sx_n(y)), F(Sy_n, Bx_n(x), Ax_n(y)), F(Sx_n, Bx_n(x), Ax_n(y)), F(Ay_n, Sy_n(x), By_n(y)), F(Ay_n, Sx_n(x), Bx_n(y))}

≥ min{F(Ay_n, Bx_n(x), Sx_n(y)), F(Ay_n, Bx_n(x), Sx_n(y)), F(Ax_n, Bx_n(x), Sx_n(y)), F(Ay_n, Bx_n(x), Sy_n(y)), F(Ay_n, Bx_n(x), Sx_n(y))}

≥ min{F(Ay_n, Bx_n(x), Sx_n(y)), F(Ay_n, Bx_n(x), Sx_n(y)), F(Sy_n, Bx_n(x), Sx_n(y)), F(Ay_n, Bx_n(x), Ax_n(y)), F(Ay_n, Bx_n(x), Sx_n(y))}

≥ min{F(Ay_n, Bx_n(x), Sx_n(y)), F(Ay_n, Bx_n(x), Sx_n(y)), F(Ay_n, Bx_n(x), Sx_n(y)), F(Ay_n, Bx_n(x), Sx_n(y)), F(Ay_n, Bx_n(x), Sx_n(y))}

F(Ay_n, Bx_n(kx), Sx_n(ky)) ≥ F(Ay_n, Bx_n(x), Sx_n(y)

Which is contradiction, therefore we deduce that $\lim_{n\to\infty} Ay_n$ = $\lim_{n\to\infty} By_n$ = z.

As S(X) is a closed subset of X, then for some u ∈ X we have Su = z. Also

$\lim_{n\to\infty} Ay_n$ = $\lim_{n\to\infty} Bx_n$ =$\lim_{n\to\infty} Sy_n$ = $\lim_{n\to\infty} Sx_n$ = Su.
In view of (v)

F(Au, Bx_n(kx), Sx_n(ky)) ≥ min{F(Su, Sx_n(x), Sx_n(y)), F(Su, Bx_n(x), Ax_n(y)), F(Sx_n, Bx_n(x), Ax_n(y)), F(Au, Su(x), Bu(y)), F(Au, Sx_n(x), Bx_n(y))}

F(Au, Su(kx), Su(ky)) ≥ min{F(Au, Su(x), Su(y)), F(Au, Su(x), Sx_n(y)), F(Au, Su(x), Sx_n(y)), F(Au, Su(x), Su(y)), F(Au, Su(x), Su(y))}

≥ min{F(Au, Su(x), Su(y)), F(Au, Su(x), Su(y)), F(Au, Su(x), Su(y)), F(Au, Su(x), Su(y)), F(Au, Su(x), Su(y))}
F(Au, Su(kx), Su(ky)) ≥ F(Au, Su(x), Su(y))

Which is contradiction, so that Au = Su.

By the property of weak compatibility of A and S we can say that
ASu = SAu and therefore AAu = SSu = ASu = SAu.

Similarly, because A(X) ⊂ S(X) then for v ∈ X we have Au = Sv. Now we claim for Bv = Sv.
In view of (v)
F(Au, Bv(kx), Sw(ky)) ≥ min{F(Su, Sv(x), Sw(y)), F(Su, Bv(x), Aw(y)), F(Sv, Bv(x), Av(y)), F(Au, Su(x), Bu(y)), F(Au, Sv(x), Bw(y))}
≥ min{F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Au(x), Su(y)), F(Au, Bv(x), Sw(y))}
≥ min{F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Sv(x), Au(y)), F(Au, Bv(x), Sw(y))}
≥ min{F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sv(y)), F(Au, Bv(x), Sw(y))}
≥ min{F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Av(y)), F(Au, Bv(x), Sw(y))}
≥ min{F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y)), F(Au, Bv(x), Sw(y))}
F(Au, Bv(kx), Sw(ky)) ≥ F(Au, Bv(x), Sw(y))

Which is contradiction, therefore we have Au = Bv.

Thus it is confirm that Au = Bv = Sv = Su.

Same as by the property of weak compatibility of B and S -

BSv = SBv and therefore BBv = SSv = BSv = SBv.

Now we show that Au is a common fixed point of A, B and S.
In view of (v)
F(AAu, Bv(kx), Sw(ky)) ≥ min{F(SAu, Sv(x), Sw(y)), F(SAu, Bv(x), Aw(y)), F(Sv, Bv(x), Av(y)), F(AAu, SAu (x), BAu(y)), F(AAu, Sv(x), Bw(y))}
F(AAu, Au(kx), Sw(ky)) ≥ min{F(AAu, Au(x), Sw(y)), F(AAu, Au(x), Sw(y)), F(Au, Au(x), Sw(y)), F(AAu, AAu (x), Bu(y)), F(AAu, Au(x), Sw(y))}
≥ min{F(AAu, Au(x), Sw(y)), F(AAu, Au(x), Sw(y)), F(Au, Au(x), Sw(y)), F(AAu, AAu (x), Au(y)), F(AAu, Au(x), Sw(y))}
≥ min{F(AAu, Au(x), Sw(y)), F(AAu, Au(x), Sw(y)), F(Au, Au(x), Sw(y)), F(AAu, AAu (x), Sw(y)), F(AAu, Au(x), Sw(y))}

F(AAu, Au(kx), Sw(ky)) ≥ F(AAu, Au(x), Sw(y))
Which is contradiction, therefore AAu = SAu = Au.
It means Au is a common fixed point of A and S. In similar manner we can prove that Bv is a common fixed point of B and S. As Au = Bv already proved, thus it is conclude that Au is a common fixed point of A, B and S.
At last to show uniqueness, let u ≠ v and Au = Bu = Su = u and Av = Bv = Sv = v.
In view of (v)

F(Au, Bv(kx), Sw(ky)) ≥ min{F(Su, Sv(x), Sw(y)), F(Su, Bv(x), Aw(y)), F(Sv, Bv(x), Av(y)), F(Au, Su(x), Bu(y)), F(Au, Sv(x), Bw(y))}

F(u, v(kx), Sw(ky)) ≥ min{F(u, v(x), Sw(y)), F(u, v(x), Sw(y)), F(Bv, v(x), Sw(y)), F(u, Au(x), Su(y)), F(u, v(x), Sw(y))}

≥ min{F(u, v(x), Sw(y)), F(u, v(x), Sw(y)), F(Au, v(x), Sw(y)), F(u, Bv(x), Au(y)), F(u, v(x), Sw(y))}

≥ min{F(u, v(x), Sw(y)), F(u, v(x), Sw(y)), F(u, v(x), Sw(y)), F(u, v(x), Sw(y)), F(u, v(x), Sw(y))}
F(u, v(kx), Sw(ky)) ≥ F(u, v(x), Sw(y))

Which is contradiction, thus u = v means common fixed point is unique.
This completes the proof.

CONCLUSION

In the present paper we have proved the existence and uniqueness of fixed point through the property (E.A.) defined over 2-menger space.

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