

## Fixed point theorems in fuzzy metric spaces using (CLRG) property

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### ABSTRACT

The present paper deals with the common fixed point theorem for a pair of occasionally weakly compatible mappings by using the (CLRg) property in fuzzy metric space. We also cited an example in support of our result. Our result improves the result of Alamgir, M., and Sumitra[3].

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### INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [20], in 1965, as a new way to represent the vagueness in everyday life. In mathematical programming problems are expressed as optimizing some goal function given certain constraints, and there are real life problems that consider multiple objectives. Generally it is very difficult to get a feasible solution that brings us to the optimum of all objective functions. A possible method of resolution that is quite useful is the one using fuzzy sets. It was developed extensively by many authors and Used in various fields to use this concept in topology and analysis. Abbas [1], Balasubramaniam[4], Chauhan S. and Kumar S.[5], Chauhan S.[4], Kumar S., Fisher B.[13],Sharma S.[17] have defined fuzzy metric space in various ways. George and Veeramani[9] modified the concept of fuzzy metric space introduced by Kramosil and Michalek[12] in order to get the Hausdorff topology. Jungck[11] introduced the notion of compatible maps for a pair of self mapping. The importance of CLRg property ensures that one does not require the closeness of range subspaces.

In 2008 Altun I. [2] proved common fixed point theorem on fuzzy metric space with an implicit relation. Sintunavarat [20] introduced a new concept of property (CLRg). Chauhan et al [6] utilize the notion of common limit range property to prove unified fixed point theorems for weakly compatible mapping in fuzzy metric spaces. Implicit relation and (CLRg) property are used as a tool for finding common fixed point of contraction maps. The intent of this paper is to establish the concept of E.A. property and (CLRg) property for coupled mappings and an affirmative answer of question raised by Rhoades [15]. The importance of (CLRg) property ensures that one does not Require the closeness of range subspaces., First, we give some definitions.

### MATERIALS AND METHODS

#### 2. Preliminaries:-

**Definitions 2.1.** [10] A binary operation  $*$ :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is called a t –norm if  $([0, 1], *)$  is an abelian topological monoid with unit 1 such that

$a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$  for  $a, b, c, d \in [0, 1]$ .

Examples of t-norms are  $a * b = ab$  and  $a * b = \min \{a, b\}$

**Definition 2.2.**[10] The 3-tuple  $(X, M, *)$  is said to be a fuzzy metric space, if  $X$  is an arbitrary set,  $*$  is a continuous  $t$ -norm and  $M$  is a fuzzy set in  $X^2 \times [0, \infty)$  satisfying the following conditions:

for all  $x, y, z \in X$  and  $s, t > 0$ .

- (FM-1)  $M(x, y, 0) = 0$ ,
- (FM-2)  $M(x, y, t) = 1$  for all  $t > 0$  if and only if  $x = y$ ,
- (FM-3)  $M(x, y, t) = M(y, x, t)$ ,
- (FM-4)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ ,
- (FM-5)  $M(x, y, \cdot): [0, \infty) \rightarrow [0, 1]$  is left continuous,
- (FM-6)  $\lim_{n \rightarrow \infty} M(x, y, t) = 1$ .

Note that  $M(x, y, t)$  can be considered as the degree of nearness between  $x$  and  $y$  with respect to  $t$ . we identify  $x=y$  with  $M(x, y, t)=1$  for all  $t>0$ .The following example shows that every metric space induces a Fuzzy metric space.

**Example 2.3** [15] Let  $(X, d)$  be a metric space. Define a  $* b = \min \{a, b\}$  and  $M(x, y, t) = \frac{t}{t+d(x,y)}$  for all  $x, y \in X$  and all  $t > 0$ .Then  $(X, M, *)$  is a fuzzy metric space. It is called the fuzzy metric space induced by the metric  $d$ .

**Lemma 2.4**[5] Let  $(X, M, *)$  be a fuzzy metric space. If there exist  $k \in (0, 1)$  such that  $M(x, y, kt) \geq M(x, y, t)$  for all  $x, y \in X$  and  $t > 0$  then  $x = y$ .

**Definition 2.5** [5] A sequence  $\{x_n\}$  in  $X$  is said to be a Cauchy sequence if and only if for each  $\epsilon > 0, t > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $M(x_n, x_m, t) > 1-\epsilon$  for all  $n, m \geq n_0$ .

A sequence  $\{x_n\}$  is said to be a converge to a point  $x$  in  $X$  if and only if for each  $\epsilon > 0, t > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $M(x_n, x, t) > 1-\epsilon$  for all  $n \geq n_0$ .

A Fuzzy metric space  $(X, M, *)$  is said to be complete if every Cauchy sequence in it converges to a point in it.

**Definition 2.6**[14] Two maps  $A$  and  $B$  from a fuzzy metric space  $(X, M, *)$  into itself are said to be compatible if  $\lim_{n \rightarrow \infty} M(ABx_n, BAx_n, t) = 1$  for all  $t > 0$ , whenever  $\{x_n\}$  is a sequence such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = x \text{ for some } x \in X.$$

**Definition 2.7**[14] Two maps  $A$  and  $B$  from a fuzzy metric space  $(X, M, *)$  into itself are said to be weak - compatible if they commute at their coincidence points, i.e.,  $Ax = Bx$  implies  $ABx = BAx$ .

**Definition 2.8**[14] Self mappings  $A$  and  $S$  of a fuzzy metric space  $(X, M, *)$  are said to be occasionally weakly compatible (owc) if and only if there is a point  $x$  in  $X$  which is coincidence point of  $A$  and  $S$  at which  $A$  and  $S$  commute.

**Definition 2.9**[14] A pair of self mappings  $A$  and  $S$  of a fuzzy metric space  $(X, M, *)$  is said to satisfy the (CLRg) property if there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = Bu$ .for some  $u \in X$ .

**Proposition 3.** [16] In a fuzzy metric space  $(X, M, *)$  limit of a sequence is unique.

**Proposition 3.1**[16] Let  $S$  and  $T$  be compatible self maps of a fuzzy metric space  $(X, M, *)$  and let  $\{x_n\}$  be a sequence in  $X$  such that  $Sx_n, Tx_n \rightarrow u$  for some  $u$  in  $X$ . Then  $STx_n \rightarrow Tu$  provided  $T$  is continuous.

**Proposition 3.2**[16] Let  $S$  and  $T$  be compatible self maps of a fuzzy metric space  $(X, M, *)$  and  $Su = Tu$  for some  $u$  in  $X$  then  $STu = TSu = SSu = TTu$ .

**Lemma 3.3**[8] Let  $(X, M, *)$  be a fuzzy metric space. Then for all  $x, y \in X$ ,  $M(x, y, \cdot)$  is a non-decreasing function.

**Lemma 3.4**[2] Let  $(X, M, *)$  be a fuzzy metric space. If there exists  $k \in (0, 1)$  such that for all  $x, y \in X$

$$M(x, y, kt) \geq M(x, y, t) \quad \forall t > 0 \text{ then } x=y.$$

**Lemma 3.5**[19] Let  $\{x_n\}$  be a sequence in a fuzzy metric space  $(X, M, *)$ . If there exists a number  $k \in (0, 1)$  such that  $M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t) \quad \forall t > 0$  and  $n \in \mathbb{N}$ .

Then  $\{x_n\}$  is a Cauchy sequence in  $X$ .

**Lemma 3.6**[12] The only t-norm  $*$  satisfying  $r * r \geq r$  for all  $r \in [0, 1]$  is the minimum t-norm, that is  $a * b = \min\{a, b\}$  for all  $a, b \in [0, 1]$ .

**3. Main Result**

**Theorem 3.7.** Let  $(X, M, *)$  be a Fuzzy Metric Space,  $*$  being continuous

t-norm with  $a * b \geq ab, \forall a, b \in [0, 1]$ . Let  $P, Q: X \times X \rightarrow X$  and  $R, S: X \times X \rightarrow X$  be four mappings satisfying following conditions:

- 1) The pairs  $(P, R)$  and  $(Q, S)$  satisfy CLRg property
  - 2)  $M(P(x, y), Q(u, v), kt) \geq \phi\{M(Rx, Su, t) \times M(P(x, y), Rx, t) \times M(Q(u, v), Su, t)\}$
- $$\forall x, y, u, v \in X, k \in (0, 1) \text{ and } \phi: [0, 1] \rightarrow [0, 1]$$

Such that  $\phi(t) > t$  for  $0 < t < 1$ . Then  $(P, R)$  and  $(Q, S)$  have point of coincidence. Moreover if the pairs  $(P, R)$  and  $(Q, S)$  are occasionally weakly compatible, then there exists unique  $x$  in  $X$ .

$$\text{Such that } P(x, x) = S(x) = Q(x, x) = R(x) = x.$$

**Proof:** - Since the pairs  $(P, R)$  and  $(Q, S)$  satisfy CLRg property,

there exist sequences  $\{x_n\}, \{y_n\}, \{x'_n\}$  and  $\{y'_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} P(x_n, y_n) = \lim_{n \rightarrow \infty} R(x_n) = Ra,$

$$\lim_{n \rightarrow \infty} P(y_n, x_n) = \lim_{n \rightarrow \infty} R(y_n) = Rb \text{ and } \lim_{n \rightarrow \infty} Q(x'_n, y'_n) = \lim_{n \rightarrow \infty} S(x'_n) = Sa', \lim_{n \rightarrow \infty} Q(y'_n, x'_n) = \lim_{n \rightarrow \infty} S(y'_n) = Sb,$$

for some  $a, b, a', b'$  in  $X$ .

**Step 1:** We now show that the pairs  $(P, R)$  and  $(Q, S)$  have common coupled coincidence point. We first show that  $Ra = Sa'$ . Using (3.2), we have,

$$M(P(x_n, y_n), Q(x'_n, y'_n), kt) \geq \phi\{M(Rx_n, Sx'_n, t) \times M(P(x_n, y_n), Rx_n, t) \times M(Q(x'_n, y'_n), Sx'_n, t)\}$$

Taking  $n \rightarrow \infty$ , we get

$$\begin{aligned} M(Ra, Sa', kt) &\geq \phi\{M(Ra, Sa', kt) * 1 * 1\} \\ &\geq \phi\{M(Ra, Sa', t)\} \\ &\geq \{M(Ra, Sa', t)\} \\ \text{i.e } M(Ra, Sa', kt) &\geq M(Ra, Sa', t) \\ \Rightarrow Ra &= Sa'. \end{aligned}$$

Similarly we can have  $Rb = Sb'$ .

Also,

$$\begin{aligned} &M(P(y_n, x_n), Q(x_n', y_n'), kt) \\ &\geq \emptyset \{M(Ry_n, Sx_n', t) \times M(P(y_n, x_n), Ry_n, t) \times M(Q(x_n', y_n'), Sx_n', t)\} \\ \text{i.e } &M(Rb, Sa', kt) \geq M(Rb, Sa', t) \\ \Rightarrow &Rb = Sa' \end{aligned}$$

Hence  $Rb = Sa' = Ra = Sa'$ . Now, for all  $t > 0$ , using condition (3.2)

We have

$$\begin{aligned} &M(P(x_n, y_n), Q(a', b'), kt) \\ &\geq \emptyset \{M(Rx_n, Sa', t) \times M(P(x_n, y_n), Rx_n, t) \times M(Q(a', b'), Sa', t)\} \\ \text{i.e } &M(Ra, Q(a', b'), kt) \geq M(Ra, Q(a', b'), t) \\ \Rightarrow &Ra = Q(a', b'). \end{aligned}$$

Similarly, we can get that  $Rb = Q(b', a')$

In a similar fashion, we can have  $Sa' = P(a, b)$  and  $Sb' = P(b, a)$ .

Thus,  $Q(a', b') = Ra = Sa' = P(a, b)$  and  $Q(b', a') = Rb = Sb' = P(b, a)$ .

Thus the pairs (P, R) and (Q, S) have Coincidence points.

Let  $Ra = P(a, b) = Q(a', b') = Sa' = x$  and  $Rb = P(b, a) = Q(b', a') = Sb' = y$ . Since (P, R) and (Q, S) are occasionally weakly compatible, so

$$Rx = RP(a, b) = P(Ra, Rb) = P(x, y)$$

and

$$\begin{aligned} Ry &= RP(b, a) = P(Rb, Ra) = P(y, x) . \\ Sx &= SQ(a', b') = Q(Sa', Sb') = Q(x, y) \end{aligned}$$

and

$$Sy = SQ(b', a') = Q(Sb', Sa') = Q(y, x).$$

**Step 2:** We next show that  $x = y$ . From (3.2),

$$\begin{aligned} &M(x, y, kt) = M(P(a, b), Q(a', b'), kt) \\ &\geq \emptyset \{M(Ra, Sa', t) \times M(P(a, b), Sa, t) \times M(Q(a', b'), Sa', t)\} = 1 \\ \text{Thus, } &x = y. \end{aligned}$$

**Step 3:** Now, we prove that  $Rx = Sx$ , using (3.2) again

$$\begin{aligned} &M(Rx, Sx, kt) = M(P(x, y), Q(y, x), kt) \\ &\geq \emptyset \{M(Rx, Sy, t) \times M(P(x, y), Rx, t) \times M(Q(y, x), Sy, t)\} \\ &\geq \emptyset \{M(Rx, Sx, t) * M(P(x, y), Rx, t) * M(Q(y, x), Sy, t)\} \\ \text{i.e } &M(Rx, Sx, kt) \geq M(Rx, Sy, t) \\ \Rightarrow &Rx = Sx = Sy. \end{aligned}$$

**Step 4:** Lastly, we prove that  $Rx = x$

$$\begin{aligned} &M(Rx, x, kt) = M(Rx, y, kt) = M(P(x, y), Q(x, y), kt) \\ &\geq \emptyset \{M(Rx, Sx, t) * M(P(x, y), Rx, t) * M(Q(x, y), Sx, t)\} \end{aligned}$$

Hence  $x = Rx = Sx = P(x, x) = Q(x, x)$ .

This shows that P, Q, R, S have a common fixed point and uniqueness of x follows easily from (3.2).

**Example 3.8** Let  $X = [0, \infty)$  be the usual metric space. Define  $f, g: X \rightarrow X$  by  $fx = x + 3$  and  $gx = 4x$  for all  $x \in X$ . we consider the sequence  $\{x_n\} = \{1 + \frac{1}{n}\}$ . Since

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 4 = g(1) \in X.$$

Therefore  $f$  and  $g$  satisfy the (CLRg) property.

**Example 3.9** The conclusion of Example 3.8 remains true if the self mappings  $f$  and  $g$  is defined on  $X$  by  $f(x) = \frac{x}{5}$  and  $g(x) = \frac{2x}{4}$  for all  $x \in X$ . Let a sequence

$\{x_n\} = \{\frac{1}{n}\}$  in  $X$ . Since

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 0 = g(0) \in X.$$

Therefore  $f$  and  $g$  satisfy the (CLRg) property.

**Example 4.** Let  $(X, M, *)$  be a fuzzy metric space,  $*$  being a continuous norm with  $X = [0, \infty)$ . Define  $M(x, y, t) = \frac{t}{t + |x - y|}$  for all  $x, y$  in  $X$  and  $t > 0$ . Define mappings  $f: X \times X \rightarrow X$  and  $g: X \rightarrow X$  as follows.

$$f(x, y) = \begin{cases} x + y, & x \in [0, 1), y \in X \\ \frac{x+y}{2}, & x \in [1, \infty), y \in X \end{cases} \text{ and } g(x) = \begin{cases} 1 + x, & x \in [0, 1) \\ \frac{x}{2}, & x \in [1, \infty) \end{cases}$$

We consider the sequence  $x_n = \{\frac{1}{n}\}$  and  $y_n = \{1 + \frac{1}{n}\}$  then,

$$f(x_n, y_n) = f(\frac{1}{n}, 1 + \frac{1}{n}) = 1 + \frac{2}{n}, \quad f(y_n, x_n) = f(1 + \frac{1}{n}, \frac{1}{n}) = \frac{1}{2} + \frac{1}{n}$$

$$g(x_n) = g(\frac{1}{n}) = 1 + \frac{1}{n}, \quad g(y_n) = g(1 + \frac{1}{n}) = \frac{1}{2} + \frac{1}{2n}$$

$$\lim_{n \rightarrow \infty} M(f(x_n, y_n), g(x_n), t) \rightarrow 1 = g(0)$$

$$\text{and } \lim_{n \rightarrow \infty} M(f(y_n, x_n), g(y_n), t) \rightarrow 1 = g(0)$$

therefore, the maps  $f$  and  $g$  satisfy (CLRg) property but the maps are not continuous.

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