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Fixed Point Theorems for Pairs of Mappings on Three Metric Spaces

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ABSTRACT

In this paper we obtain fixed points for three pairs of mappings, not all of which need to be continuous, on three metric spaces.This result is an extension of many results already proved for one, two or three metric spaces.

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INTRODUCTION

Related fixed point theorems on three complete metric spaces have been studied by B. Fisher et al [1], R.K. Jain et al [3], S. Jain et al [5], N.P. Nung et al [2], R.K. Namdeo et al [4]. Recently, Vishal Gupta et al proved a Fixed point theorem for two pair of mappings on two metric spaces. Here, a generalization is given for three pairs of mappings on three complete metric space. Our theorem improves the result of Vishal Gupta et al [6] and of R.K. Jain et al [3].

2 Preliminaries

The following fixed point theorem was proved by R.K. Jain et al[3].

Theorem 2.1 Let (X,d), $(Y,\circ)$ and $Z,\pm)$ be complete metric spaces. If T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a mapping of Z into X.

$$d(RSTx,RSTx')\leqslant c\max\{d(x,x'),d(x,RSTx),\ d(x',RSTx'),\circ(Tx,Tx'),\pm(STX,STx')\}$$

$$\circ(TRSy,TRSy')\leqslant c\max\{\circ(y,y'),\circ(y,TRSy),\ \circ(y',TRSy'),\pm(Sy,Sy'),d(RSy,RSy')\}$$

$$\pm(STRz,STRz')\leqslant c\max\{\pm(z,z'),\pm(z,STRz),\ \pm(z',STRz'),d(Rz,Rz'),\circ(TRz,TRz')\}$$

$\forall\ x,x'\in X, y,y'\in Y$ and $z,z'\in Z$ where $0\leqslant c<1$ then RST has a unique fixed point $u\in X$, TRS has a unique fixed point $v\in Y$ and STR has a unique fixed point $w\in Z$. Further $Tu=v$, $Sv=w$ and $Rw=u$.

3 Main Result

We now prove the following related fixed point theorem.

Theorem 3.1 Let (X, d_1), (Y, d_2) and (Z, d_3) be complete metric spaces. If $A, B: X \to Y$, $E, F: Y \mapsto Z$ and $U, V: Z \mapsto X$ satisfying inequalities.

$$d_1(VEBx, UFAx') \leqslant c\max\{d_1(x, x'), d_1(x, VEBx), d_1(x', UFAx'), d_2(Bx, Ax'), d_3(EBx, FAx')\} \tag{3.1}$$

$$d_2(BVEy, AUFy') \leqslant c\max\{d_2(y, y'), d_2(y, BVEy), d_2(y', AUFy'), d_3(Ey, Fy'), d_1(VEy, UFy')\} \tag{3.2}$$

$$d_3(FAUz, EBVz') \leqslant c\max\{d_3(z, z'), d_3(z, FAUz), d_3(z', EBVz'), d_1(Uz, Vz'), d_2(AUz, BVz')\} \tag{3.3}$$

$\forall\, x, x' \in X$, $y, y' \in Y$ and $z, z' \in Z$ where $0 \leqslant c < 1$. If A, F and V are continuous, then VEB and UFA a unique common fixed point $\sim \in X$. BVE and AUF have a unique commn fixed point $\in Y$ and FAU and EBV have a unique common fixed point ¥ $\in Z$. Further $A\sim = B\sim =$, $E = F =$ ¥ and U¥ $= V$¥ $= \sim$.

Proof. Let x_0 be an arbitrary point in X. Define sequence $\{x_n\}$ in X, $\{y_n\}$ in Y and $\{z_n\}$ in Z.

$$Ax_{2n} = y_{2n+1}, Ey_{2n-1} = z_{2n-1}, Uz_{2n-1} = x_{2n-1}$$
$$Bx_{2n-1} = y_{2n}, Fy_{2n} = z_{2n}, Vz_{2n} = x_{2n} \text{ for } n = 1, 2, \ldots$$

Applying inequality (3.2), we have,

$$d_2(y_{2n}, y_{2n+1}) = d_2(BVEy_{2n-1}, AUFy_{2n})$$
$$\leqslant c\max\{d_2(y_{2n-1}, y_{2n}), d_2(y_{2n-1}, BVEy_{2n-1}), d_2(y_{2n}, AUFy_{2n}), d_3(Ey_{2n-1}, Fy_{2n}), d_1(VEy_{2n-1}, UFy_{2n})\}$$
$$= c\max\{d_2(y_{2n-1}, y_{2n}), d_2(y_{2n-1}, y_{2n}), d_2(y_{2n}, y_{2n+1}), d_3(z_{2n-1}, z_n), d_1(x_{2n-1}, x_{2n})\}$$

This implies,

$$d_2(y_{2n}, y_{2n+1}) \leqslant c\max\{d_2(y_{2n-1}, y_{2n}), d_3(z_{2n-1}, z_{2n}), d_1(x_{2n-1}, x_{2n})\} \tag{3.4}$$

Using inequality (3.3), we have

$$d_3(z_{2n}, z_{2n+1}) = d_3(FAUz_{2n-1}, EBVz_{2n}) \leqslant c\max\{d_3(z_{2n-1}, z_{2n}), d_3(z_{2n-1}, FAUz_{n-1}),$$
$$d_3(z_{2n}, EBVz_{2n}), d_1(Uz_{2n-1}, Vz_{2n}), d_2(AUz_{2n-1}, BVz_{2n})\}$$
$$= c\max\{d_3(z_{2n-1}, z_{2n}), d_3(z_{2n-1}, z_{2n}), d_3(z_{2n}, z_{2n+1}), d_1(x_{2n-1}, x_{2n}), d_2(y_{2n}, y_{2n+1})\}$$

This implies,

$$d_3(z_{2n}, z_{2n+1}) \leqslant c\max\{d_3(z_{2n-1}, z_{2n}), d_1(x_{2n-1}, x_{2n}), d_2(y_{2n}, y_{2n+1})\} \tag{3.5}$$

By using (3.4), we have,

$$d_3(z_{2n}, z_{2n+1}) \leqslant c\max\{d_1(x_{2n-1}, x_{2n}), d_2(y_{2n-1}, y_{2n}), d_3(z_{2n-1}, z_{2n})\}$$

Using inequality (3.1), we have,

$$d_1(x_{2n}, x_{2n+1}) = d_1(VEBx_{2n-1}, UFAx_{2n})$$

$$\leqslant c\max\left\{d_1\left(x_{2n-1},x_{2n}\right),d_1\left(x_{2n-1},x_{2n}\right),\ d_1\left(x_{2n},x_{2n+1}\right),d_2\left(y_{2n},y_{2n+1}\right),\ d_3\left(z_{2n},z_{2n+1}\right)\right\}$$

This implies,

$$d_1\left(x_{2n},x_{2n+1}\right)\leqslant c\max\left\{d_1\left(x_{2n-1},x_{2n}\right),d_2\left(y_{2n},y_{2n+1}\right),\ d_3\left(z_{2n},z_{2n+1}\right)\right\}$$

$$d_1\left(x_{2n},x_{2n+1}\right)\leqslant\ c\max\left\{d_1\left(x_{2n-1},x_{2n}\right),d_2\left(y_{2n-1},y_{2n}\right),\ d_3\left(z_{2n-1},z_{2n}\right)\right\} \tag{3.6}$$

using inequalities (3.4) and (3.5).

It now follows easily by induction on using inequalties (3.4), (3.5) and (3.6),

$$d_1\left(x_{2n},x_{2n+1}\right)\leqslant c^{n-1}\max\left\{d_1\left(x_1,x_2\right),d_2\left(y_1,y_2\right),d_3\left(z_1,z_2\right)\right\}$$

$$d_2\left(y_{2n},y_{2n+1}\right)\leqslant c^{n-1}\max\left\{d_1\left(x_1,x_2\right),d_2\left(y_1,y_2\right),d_3\left(z_1,z_2\right)\right\}$$

$$d_3\left(z_{2n},z_{2n+1}\right)\leqslant c^{n-1}\max\left\{d_1\left(x_1,x_2\right),d_2\left(y_1,y_2\right),d_3\left(z_1,z_2\right)\right\}$$

Since $c<1$, it follows that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences with limits $\sim\in X$, $\ \in Y$ and $\text{¥}\in Z$ respectively. Since A, F and V are continuous, we have,

$$\lim_{n\to\infty} y_{2n+1}=\lim_{n\to\infty} Ax_{2n}=A\sim=\ \ ,$$

$$\lim_{n\to\infty} z_{2n-1}=\lim_{n\to\infty} Fy_{2n-1}=F\ \ =\text{¥},$$

$$\lim_{n\to\infty} x_{2n-1}=\lim_{n\to\infty} Vz_{2n-1}=V\text{¥}=\sim$$

Using inequality (3.1), we have,

$$d_1\left(U\text{¥},x_{2n}\right)=d_1\left(x_{2n},U\text{¥}\right)=d_1\left(VEBx_{2n-1},UFA\sim\right)$$

$$\leqslant c\max\left\{d_1\left(x_{2n-1},\sim\right),d_1\left(x_{2n-1},VEBx_{2n-1}\right),\ d_1\left(\sim,UFA\sim\right),d_2\left(Bx_{2n-1},A\sim\right),\ d_3\left(EBx_{2n-1},FA\sim\right)\right\}$$

$$=c\max\left\{d_1\left(x_{2n-1},\sim\right),d_1\left(x_{2n-1},x_{2n}\right),\ d_1\left(\sim,U\text{¥}\right),d_2\left(y_{2n},\ \ \right),d_3\left(z_{2n},\text{¥}\right)\right\}$$

Letting $n\to\infty$, we have,

$$d_1\left(\sim,U\text{¥}\right)\leqslant c\max d_1\left(\sim,U\text{¥}\right)$$

Thus, $U\text{¥}=\sim\ \Rightarrow UF\ \ =\sim$. Also $UFA\sim=a$.

Using inequality (3.2), we have, $d_2\left(B\sim,y_{2n}\right)=d_2\left(BVE\ \ ,AUFy_{2n-1}\right)$

$$\leqslant c\max\left\{d_2\left(\ \ ,y_{2n-1}\right),d_2\left(\ \ ,B\sim\right),d_2\left(y_{2n-1},y_{2n}\right),\ d_3\left(\text{¥},z_{2n-1}\right),d_1\left(\sim,x_{2n-1}\right)\right\}$$

Letting $n\to\infty$, we have, $d_2\left(B\sim,\ \ \right)\leqslant cd_2\left(\ \ ,B\sim\right)$

Thus, $B\sim=\ \ ,BV\text{¥}=\ \ $. Also $BVE\ \ =\ \ $.

Now using inequality (3.3), we have,

$$d_3\left(E\ \ ,z_{2n}\right)=d_3\left(z_{2n},E\ \ \right)=d_3\left(FAUz_{2n-1},EBV\text{¥}\right)$$

$$\leqslant c\max\left\{d_3\left(z_{2n-1},\text{¥}\right),d_3\left(z_{2n-1},FAUz_{2n-1}\right),\ d_3\left(\text{¥},EBV\text{¥}\right),d_1\left(Uz_{2n-1},V\text{¥}\right),\ d_2\left(AUz_{2n-1},BV\text{¥}\right)\right\}$$

$$=c\max\left\{d_3\left(z_{2n-1},\text{¥}\right),d_3\left(z_{2n-1},z_{2n}\right),d_3\left(\text{¥},E\ \ \right),\ d_1\left(x_{2n-1},\sim\right),d_2\left(y_{2n},\ \ \right)\right\}$$

Letting $n \to \infty$, we have $d_3\left(E\quad ,¥\right) \leqslant c d_3\left(¥, E\quad\right)$

Thus, $E\quad = ¥, EB\sim = ¥$. Also $EBV¥ = ¥$. As $c < 1$ and thus $\sim$ is a fixed point of UFA, is a fixed point of BVE and ¥ is a fixed point of EBV. Thus we have,

$$AUF\quad = AUFA\sim = A\sim = \quad, FAU\quad = FAUF\quad = F\quad = ¥, \text{ and so on.}$$
$$VEB\sim = VEBV¥ = V¥ = \sim.$$

Thus $\sim$ is a fixed point of UFA and VEB, is a fixed point of AUF and BVE and ¥ is a fixed point of FAU and EBV.

Uniqueness: We know prove the uniqueness of the fixed point $\sim$. Suppose UFA has a second fixed point $\sim'$. Then using inquality (3.1), we have,

$$d_1\left(\sim,\sim'\right) = d_1\left(VEB\sim, UFA\sim'\right)$$
$$\leqslant c\max\left\{d_1\left(\sim,\sim'\right), d_1\left(\sim, VEB\sim\right), d_1\left(\sim', UFA\sim'\right),\ d_2\left(B\sim, A\sim'\right), d_3\left(EB\sim, FA\sim'\right)\right\}$$
$$= c\max\left\{d_2\left(B\sim, A\sim'\right), d_3\left(EB\sim, FA\sim'\right)\right\}$$

Further using inequality (3.2), we have,

$$d_2\left(B\sim, A\sim'\right) = d_2\left(BVEB\sim, AUFA\sim'\right)$$
$$\leqslant c\max\left\{d_2\left(B\sim, A\sim'\right), d_2\left(B\sim, BVEB\sim\right), d_2\left(A\sim', AUFA\sim'\right), d_3\left(EB\sim, FA\sim'\right), d_1\left(VEB\sim, UFA\sim'\right)\right\}$$
$$= c\max\left\{d_1\left(\sim,\sim'\right), d_3\left(EB\sim, FA\sim'\right)\right\}$$

Hence, we have, $d_1\left(\sim,\sim'\right) \leqslant c d_3\left(EB\sim', FA\sim'\right)$

$$= c d_3\left(FA\sim', EB\sim\right)$$

Using inequality (3.3), we have,

$$d_1\left(\sim,\sim'\right) \leqslant c d_3\left(FA\sim', EB\sim\right)$$
$$= c d_3\left(FAUFA\sim', EBVEB\sim\right)$$
$$\leqslant c^2 \max\left\{d_3\left(FA\sim', EB\sim\right), d_3\left(FA\sim', FAUFA\sim'\right), d_3\left(EB\sim, EBVEB\sim\right), d_1\left(UFA\sim', VEB\sim\right), d_2\left(AUFA\sim', BVEB\sim\right)\right\}$$
$$= c^2 \max\left\{d_3\left(FA\sim', EB\sim\right), d_1\left(\sim',\sim\right), d_2\left(A\sim', B\sim\right)\right\}$$
$$= c^2 d_1\left(\sim',\sim\right)$$

Since $c < 1$, it follows that $\sim = \sim'$ and the uniqueness of $\sim$ follows.

Similarly it can be proved $\sim$ is fixed point of VEB, is fixed point of AUF and BVE and ¥ is a fixed point of FAU and EBV. This completes the proof of the theorem.

Theorem 3.2 Let (X, d_1), (Y, d_2) and (Z, d_3) be compact metric spaces. If $A, B : X \mapsto Y$, $E, F : Y \mapsto Z$ and $U, V : Z \mapsto X$ be continuous mappings, satisfying the inequalities,

$$d_1\left(VEBx,UFAx'\right) < c\max\left\{d_1\left(x,x'\right),d_1\left(x,VEBx\right),\, d_1\left(x',UFAx'\right),d_2\left(Bx,Ax'\right),\;\; d_3\left(EBx,FAx'\right)\right\} \tag{3.7}$$

$$d_2\left(BVEy,AUFy'\right) < c\max\left\{d_2\left(y,y'\right),d_2\left(y,BVEy\right),\, d_2\left(y',AUFy'\right),d_3\left(Ey,Fy'\right),\, d_1\left(VEy,UFy'\right)\right\} \tag{3.8}$$

$$d_3\left(FAUz,EBVz'\right) < c\max\left\{d_3\left(z,z'\right),d_3\left(z,FAUz\right),\, d_3\left(z',EBVz'\right),d_1\left(Uz,Vz'\right),\, d_2\left(AUz,BVz'\right)\right\} \tag{3.9}$$

$\forall\, x,x'\in X$, $y,y'\in Y$ and $z,z'\in Z$. Then VEB and UFA have a unique common fixed point $\sim\in X$, BVE and AUF have a unique common fixed point $\;\in Y$ and FAU and EBV have a unique common fixed point ¥ $\in Z$. Further $A\!\sim\; = B\!\sim\; = \;$, $E\; = F\; =$ ¥ and U¥ $= V$¥ $=\sim$.

Proof. Suppose first of all that there exists $\sim,\sim'\in X$ such that,

$$\max\left\{d_1\left(\sim,\sim'\right),d_1\left(\sim,VEB\sim\right),d_1\left(\sim',UFA\sim'\right),\; d_2\left(B\sim,A\sim'\right),d_3\left(EB\sim,FA\sim'\right)\right\}=0$$

Then it follows immediately that $\sim\;=\;\sim'$, $VEB\sim\;=\;\sim$, $A\sim\;=B\sim$, $UFA\sim\;=\;\sim$ and on putting $A\sim\;=\;$, $E\;\;=$ ¥ , U¥ $=\sim$.

We have, $VE\;\;=\sim\;\; BVE\;\;=B\sim\;=\;$ and $EBVE\;\;=EBV$¥ $=E\;\;=$ ¥ $\Rightarrow\; VE\;\;=\sim\;\Rightarrow V$¥ $=\sim$

The results of the theorem therefore hold in this case. Similarly if there exists $\;,\;'\in Y$ such that,

$$\max\left\{d_2(\;,\;'),d_2(\;,BVE\;\;),d_2(\;',AUF\;\;'),\; d_3(E\;\;,F\;\;'),d_1(VE\;\;,UF\;\;')\right\}=0$$

or if there exist ¥ ,¥ $'\in Z$ such that,

$$\max\left\{d_3(¥,¥'),d_3(¥,FAU¥),d_3(¥',EBV¥'),\; d_1(U¥,V¥'),d_2\left(AU¥,BU¥\right)\right\}=0$$

then the result of the theorem again hold.

Now suppose that it is possible for n_0 such $\sim,\sim'$; $\;,\;'$ or ¥ ,¥ $'$ to exist. Then,

$$\max\left\{d_1\left(x,x'\right),d_1\left(x,VEBx\right),d_1\left(x',UFAx'\right),\; d_2\left(Bx,Ax'\right),d_3\left(EBx,FAx'\right)\right\}\neq 0 \quad \forall\, x,x'\in X$$

and so the real valued function $f(x,x')$ defined on $X\times X$ by

$$f\left(x,x'\right)=\frac{d_1\left(VEBx,UFAx'\right)}{\max\left\{d_1\left(x,x'\right),d_1\left(x,VEBx\right),d_1\left(x',UFAx'\right),d_2\left(Bx,Ax'\right),d_3\left(EBx,FAx'\right)\right\}}$$

is continuous. Since $X\times X$ is compact, f attain its maximum value c_1 becuase of inequality, $c_1<1$ and so,

$$d_1\left(VEBx,UFAx'\right)\leqslant c_1\max\left\{d_1\left(x,x'\right),d_1\left(x,VEBx\right),\;\; d_1\left(x',UFAX'\right),d_2\left(Bx,Ax'\right),\;\; d_3\left(EBx,FAx'\right)\right\}$$
$$\forall\, x,x'\in X\,.$$

Similarly, there exist $c_2,c_3<1$ such that,

$$d_2\left(BVEy,AUFy'\right)\leqslant c_2\max\left\{d_2\left(y,y'\right),d_2\left(y,BVEy\right),\, d_2\left(y',AUFy'\right),d_3\left(Ey,Fy'\right),d_1\left(VEy,UFy'\right)\right\}$$
$$\forall\, y,y'\in Y\,.$$

$$d_3\left(FAUz,EBVz'\right)\leqslant c_3\max\left\{d_3\left(z,z'\right),d_3\left(z,FAUz\right),\;\; d_3\left(z',EBVz'\right),d_1\left(Uz,Vz'\right),\;\; d_2\left(AUz,BVz'\right)\right\}$$
$$\forall\, z,z'\in Z\,.$$

It follows that the conditions of theorem are satisfied with $c = \max\{c_1, c_2, c_3\}$ and so the results of the theorem are again satisfied. The uniqueness of U, V and W follows easily.

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