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Advances in Applied Science Research, 2012, 3 (3):1266-1270

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ISSN: 0976-8610
CODEN (USA): AASRFC

Fixed Point Theorems for Mappings

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ABSTRACT

Some fixed point theorems for two metric spaces was proved by B.Fisher[1]. Here we also prove fixed point theorems for two pair of mappings in which one of them is coninuous, on two metric space.

Key Words: complete metric space, common fixed point **2010 AMS Subject Classification:** 54H25, 47H10.

INTRODUCTION

Related fixed point theorems on two metric spaces have been studied by R.K. Namdeo et al [2], P.P. Murthy et al [3], B. Fisher [1], R.K. Namdeo et al [4], L. Kikina et al [5]. In this paper, we prove a related fixed point theorem for two mappings, not all of which are necessarily continuous on two metric space. Our theorem is generalization of Theorem (2.1).

Definition 1.1 Let (X,d) be a metric space, a sequence $\{x_n\} \in X$ is said to be Cauchy sequence if $d(x_m, x_n) \to 0$ as $m,n \to \infty$.

Definition 1.2 A metric space (X,d) is said to be complete if and only if every Cauchy sequence in X converges to a point of X.

Definition 1.3 Let (X,d) be a metric space and $A \subseteq X$. Let $F = \{G_{\sim} : \sim \in A\}$ be collection of subsets of X. Then F is called cover of the set A if $A \subseteq \bigcup_{\sim \in A} G_{\sim}$. If each $G_{\sim}$ is an open set in X. Then F is called an open cover of the set A. A cover F is called a finite cover if it has finite members. If it is many members it is called infinite cover.

Definition 1.4 Let (X,d) be a metric space and $A \subseteq X$. A is said to be compact set if every open covering of A is reducible to finite subcovering. In other words, if $F = \{G_{\sim} : \sim \in A\}$ be an open cover for the set A. Then A is compact if $\ni$ a finite many indices $\sim_1, \sim_2, \ldots \sim_n$. If $A \subseteq \bigcup_{i=1}^{n} G_{\sim_i}$.

The following fixed point theorem was proved by B. Fisher [1].

Theorem 1.1 Let (X, d) and $(Y, \circ)$ be complete metric spaces. If $T: X \mapsto Y$ and $S: Y \mapsto X$ satisfying the following inequalities, $d(Sy, STx) \leqslant c \max\{\circ(y, Tx), d(x, Sy), d(x, STx)\}$ and

$\circ(Tx, TSy) \leqslant c \max\{d(x, Sy), \circ(y, Tx), \circ(y, TSy)\}$, $\forall\, x \in X, y \in Y$ where $0 \leqslant c < 1$

then ST has a unique fixed point $z \in T$ and TS has a unique fixed point $w \in Y$. Further $Tz = w$ and $Sw = z$.

RESULTS

We now prove the following related fixed point theorems:

Theorem 2.1 Let (X, d_1) and (Y, d_2) be complete metric spaces. Let $A, B: X \mapsto Y$ and $S, T: Y \mapsto X$ satisfying the inequality.

$$d_1(Sy, TBx) \leqslant c \max\{d_1(x, Sy), d_1(x, TBx), d_2(y, Bx)\} \tag{2.1}$$

$$d_2(Bx, ATy) \leqslant c \max\{d_2(y, Bx), d_2(y, ATy), d_1(x, Ty)\} \tag{2.2}$$

$$\forall\, x \in X \text{ and } y \in Y, \text{ where } 0 \leqslant c < 1.$$

If one of the mappings A, B, S, T is continuous then TB has a unique common fixed point $z \in X$ and AT has unique common fixed point $w \in Y$. Further $Az = Bz = w$ and $Sw = Tw = z$.

Proof. Let x be an arbitrary point in X and

$Ax = y_1, Sy_1 = x_1, Bx_1 = y_2, Ty_2 = x_2, Ax_2 = y_3$

and in general, let $Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}, Ty_{2n} = x_{2n}, Ax_{2n} = y_{2n+1}$ for $n = 1, 2, \ldots$

Using inequality (2.1), we get, $d_1(x_{2n}, x_{2n+1}) = d_1(Sy_{2n}, TBx_{2n})$

$\leqslant c \max\{d_1(x_{2n}, Sy_{2n}), d_1(x_{2n}, TBx_{2n}), \ d_2(y_{2n}, Bx_{2n})\}$

$= c \max\{d_1(x_{2n}, x_{2n}), d_1(x_{2n}, x_{2n+1}), \ d_2(y_{2n}, y_{2n+1})\}$

Which implies, $d_1(x_{2n}, x_{2n+1}) \leqslant c d_2(y_{2n}, y_{2n+1})$ (2.3)

Now using inequality (2.2), we have,

$d_2(y_{2n}, y_{2n+1}) = d_2(Bx_{2n-1}, ATy_{2n})$

$\leqslant c \max\{d_2(y_{2n}, Bx_{2n-1}), d_2(y_{2n}, ATy_{2n}), \ d_1(x_{2n-1}, Ty_{2n})\}$

$= c \max\{d_2(y_{2n}, y_{2n}), d_2(y_{2n}, y_{2n+1}), \ d_1(x_{2n-1}, x_{2n})\}$

Which implies, $d_2(y_{2n}, y_{2n+1}) \leqslant c d_1(x_{2n-1}, x_{2n})$ (2.4)

It follows that using inequalities (2.3) and (2.4), we have

$d_1(x_{2n}, x_{2n+1}) \leqslant c d_2(y_{2n}, y_{2n+1}) \leqslant c^2 d_1(x_{2n-1}, x_{2n}) \leqslant \cdots \leqslant c^{2n} d_1(x, x_1)$ for $n = 1, 2, \ldots$

Since $c < 1$, it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X and w in Y.

Now suppose that A is continuous, then, $\lim_{n\to\infty} Ax_{2n} = Az = \lim_{n\to\infty} y_{2n+1} = w$ and so $Az = w$.

Now using inequality (2.2), we have,

$d_2(Bz, y_{2n}) = d_2(Bz, ATy_{2n-1})$

$\leqslant c \max\{d_2(y_{2n-1}, Bz), d_2(y_{2n-1}, ATy_{2n-1}), \ d_1(z, Ty_{2n-1})\}$

$= c \max\{d_2(y_{2n-1}, Bz), d_2(y_{2n-1}, y_{2n}), \ d_1(z, x_{2n-1})\}$

Letting $n \to \infty$, we get, $d_2(Bz, w) \leqslant c d_2(w, Bz)$

So, $Bz = w = BSw$. Using inequality (2.1), we have,

$d_1(Sw, x_{2n+1}) = d_1(Sw, TBx_{2n})$

$\leqslant c\max\{d_1(x_{2n}, Sw), d_1(x_{2n}, TBx_{2n}), d_2(w, Bx_{2n})\}$

$= c\max\{d_1(x_{2n}, Sw), d_1(x_{2n}, x_{2n+1}), d_2(w, y_{2n+1})\}$

Letting $n \to \infty$, $d_1(Sw, z) \leqslant cd_1(z, Sw)$

So, $Sw = z = SAz$.

Again using inequality (2.1), we have, $d_1(z, Tw) = d_1(Sw, TBz)$

$\leqslant c\max\{d_1(z, Sw), d_1(z, TBz), d_2(w, Bz)\}$

$d_1(z, Tw) \leqslant c\max\{d_1(z, z), d_1(z, Tw), d_2(w, w)\}$

$d_1(z, Tw) \leqslant cd_1(z, Tw)$

Thus, $z = Tw$ or $Tw = z = TBz$.

The same result of course hold if one of the mappings B, S, T is continuous instead of A.

Uniqueness

To prove uniqueness, suppose that TB has a second fixed point z'. Then inequality (2.1) and (2.2), we have, $d_1(z', z) = d_1(SAz', TBz)$

$\leqslant c\max\{d_1(z, SAz'), d_1(z, TBz), d_2(Az', Bz)\}$

$= c\max\{d_1(z, z'), d_1(z, z), d_2(Az', Bz)\}$

$d_1(z', z) \leqslant cd_2(Az', Bz)$

Now, $d_2(Bz, Az') = d_2(Bz, ATAz') \leqslant c\max\{d_2(Bz, Az'), d_2(Az', ATAz'), \ d_1(z, TAz')\}$

Thus, $d_2(Bz, Az') \leqslant cd_1(z, z')$

and so, $d_1(z, z') \leqslant cd_2(Az', Bz) \leqslant c^2 d_1(z, z')$ since $c < 1$, the uniqueness of z follows.

Similarly w is the unique fixed point of AT. This completes the proof of the theorem.

Corollary 2.1 Let (X, d_1) be a complete metric space and let $A, B, S, T : X \mapsto X$ satisfying the inequalities,

$$d_1(Sy, TBx) \leqslant c\max\{d_1(x, Sy), d_1(x, TBx), d_1(y, Bx)\} \qquad (2.5)$$

$$d_1(Bx, ATy) \leqslant c\max\{d_1(y, Bx), d_1(y, ATy), d_1(x, Ty)\} \qquad (2.6)$$

$$\forall\, x, y \in X \text{, where } 0 \leqslant c < 1.$$

If one of the mappings A, B, S, T is continuous then TB has a unique common fixed point z and AT has unique fixed point w. Further $Az = Bz = w$ and $Sw = Tw = z$.

Theorem 2.2 Let (X, d_1) and (Y, d_2) be compact metric spaces. Let $A, B : X \mapsto Y$ and $S, T : Y \mapsto X$ be continuous mappings, satisfying the inequality,

$$d_2(Bx, ATy) < \max\{d_2(y, Bx), d_2(y, ATy), d_1(x, Ty)\} \qquad (2.7)$$

$$\forall\, x \in X \text{ and } y \in Y \text{ with } x \neq Ty \text{ and}$$

$$d_1(Sy, TBx) < \max\{d_1(x, Sy), d_1(x, TBx), d_2(y, Bx)\} \qquad (2.8)$$

$$\forall\, x \in X \text{ and } y \in Y \text{ with } y \neq Bx.$$

Then TB has a unique common fixed point $z \in X$ and AT has unique fixed point $w \in Y$. Further $Az = Bz = w$ and $Sw = Tw = z$.

Proof. Suppose first of all that there exist $z \in X$ and $w \in Y$ such that either,

$$\max\{d_2(w, Bz), d_2(w, ATw), d_1(z, Tw)\} = 0$$

or, $\max\{d_1(z, Sw), d_1(z, TBz), d_2(w, Bz)\} = 0$

Then it follows that,

$$ATw = w, z = TBz, Sw = z$$

$$Tw = z, Bz = w, Az = w$$

Now suppose that it is possible for no such z and w to exist then, $\max\{d_2(y, Bx), d_2(y, ATy), d_1(x, Ty)\} \neq 0$, $\forall\, x \in X$ and $y \in Y$ and so the real valued function $f(x, y)$ defined on $X \times Y$ by $f(x, y) = \dfrac{d_2(Bx, ATy)}{\max\{d_2(y, Bx), d_2(y, ATy), d_1(x, Ty)\}}$ is continuous.

Since $X \times Y$ is compact, f attains its maximum value c_1, because of inequality $c_1 < 1$ and $d_2(Bx, ATy) \leqslant c_1 \max\{d_2(y, Bx), d_2(y, ATy), d_1(x, Ty)\}$ $\forall\, x \in X$ and $y \in Y$.

Similarly there exist $c_2 < 1$ such that $d_1(Sy, TBx) \leqslant c_2 \max\{d_1(x, Sy), d_1(x, TBx), d_2(y, Bx)\}$

$$\forall\, x \in X \text{ and } y \in Y.$$

It follows that the conditions of Theorem (2.1) are satisfied with $c = \max\{c_1, c_2\}$ and so once again there exist $z \in X$ and $w \in Y$ such that inequalities hold.

Uniqueness

Now suppose that TB has a second distinct fixed point z'. Then $Az' \neq Bz$ since $Az' = Bz = w$. This implies $z' = TBz' = Tw = z$

Giving a contradiction. Thus on using inequality (2.8), we have,

$$d_1(z', z) = d_1(SAz', TBz) < \max\{d_1(z, SAz'), d_1(z, TBz), d_2(Az', Bz)\}$$

$$= \max\{d_1(z, z'), d_1(z, z'), d_2(Az', Bz)\}$$

$$d_1(z', z) < d_2(Az', Bz)$$

But on using inequality (2.7), we have since $z' \neq z$.

$$d_2(Az', Bz) = d_2(Bz, Az') = d_2(Bz, ATAz')$$

$$< \max\{d_2(Bz, Az'), d_2(Az', ATAz'), \ d_1(z, TAz')\}$$

$$\Rightarrow d_2(Az', Bz) < d_1(z, z')$$

Thus $d_1(z', z) < d_2(Az', Bz) < d_1(z, z')$ giving a contradiction. The uniqueness of z follows. Similarly, w is the unique fixed point of AT.

This completes the proof of theorem.

Corollary 2.2 Let (X, d_1) be a compact metric space. If $A, B, S, T : X \mapsto X$ are continuous mappings satisfying the inequalities, $d_2(Bx, ATy) < \max\{d_1(y, Bx), d_1(y, ATy), d_1(x, Ty)\}$

$$\forall\, x \in X \text{ and } y \in Y \text{ with } x \neq Ty \text{ and}$$

$d_1(Sy, TBx) < \max\{d_1(x, Sy), d_1(x, TBx), d_1(y, Bx)\}$, $\forall\ x \in X$ and $y \in Y$ with $y \neq Bx$.

Then TB has a unique fixed point z and AT has a unique fixed point w.
Further $Az = Bz = w$ and $Sw = Tw = z$.

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