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Fibonacci Polynomial Identities, Binomial coefficients and Pascal's Triangle

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ABSTRACT

In this paper we will prove some generalized identities involving Fibonacci Polynomials and for rapid numerical calculation of identities we present each identity as summation involving binomial coefficients.

Keywords:Fibonacci Polynomial, Binomial coefficient, Pascal's Triangle and Pascal's identity.

INTRODUCTION

The most prominent linear recurrence relation of order two with variable coefficient is one that definesFibonacci polynomial, it is define recursively as:

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \quad , n \geq 1 \text{with} F_0(x) = 0 \text{ and} F_1(x) = 1 \qquad (1)$$

Binet's Formula

The well knownBinet's formula allows us to express the n^{th}Fibonacci Polynomial in function of the roots $r_1 = \alpha$and $r_2 = \beta$of the characteristic equation $r^2 - xr - 1 = 0$ associated to the recurrence relation (1) as:

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad (2)$$

Finding the exact expression of $F_n(x)$from equation (2) requires multiple steps of busy and messy algebraic calculation which is not desirable, so in this paper we present $F_n(x)$ as a summation involving binomial coefficients for quick numerical calculation. Likewise we use this summation to write some fundamental identities concerning Fibonacci polynomial and develop some new identities using Fibonacci Polynomial.

Fibonacci Polynomial, Pascal's triangle and Binomial Coefficient [2]

The well known Pascal's triangle shown in Table 1 is one of the world's most recognized number patterns.

Table 1

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1
…				

Its entries can be presented as binomial coefficient, describe the expansion of $(x+y)^n$ for any integer $n \geq 0$. In particular the k^{th} entry along n^{th} row of Pascal's triangle corresponds to the coefficient of $x^k y^{n-k}$ and is given by the well known factorial formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, (0 \leq k \leq n) \quad (3)$$

The most celebrated property of Pascal's triangle is its triangular recurrence given by:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (4)$$

whichexpress each binomial coefficient as a summation of the two entries immediately above it, moreover this recurrence uniquely define Pascal's triangle if we initialize the boundary value along its outer diagonals to 1 that is:

$$\binom{n}{0} = \binom{n}{n} = 1 \quad (5)$$

all other entries are then generated using (4).

Table 2: Pascal's triangle as Binomial coefficient

$$\binom{0}{0}$$
$$\binom{1}{0}\binom{1}{1}$$
$$\binom{2}{0}\binom{2}{1}\binom{2}{2}$$
$$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$$

…

Table 3: First few Fibonacci Polynomials by (1)

$F_1(x) = 1$
$F_2(x) = x$
$F_3(x) = x^2 + 1$
$F_4(x) = x^3 + 2x$
$F_5(x) = x^4 + 3x^2 + 1$
$F_6(x) = x^5 + 4x^3 + 3x$
…

We may present coefficient of each term of Fibonacci Polynomials as binomial coefficients first few polynomials are presented in Table 4:

Table 4

$$F_1(x) = \binom{0}{0}$$
$$F_2(x) = \binom{1}{0}x$$
$$F_3(x) = \binom{2}{0}x^2 + \binom{1}{1}$$
$$F_4(x) = \binom{3}{0}x^3 + \binom{2}{1}x$$
$$F_5(x) = \binom{4}{0}x^4 + \binom{3}{1}x^2 + \binom{2}{2}$$
$$F_6(x) = \binom{5}{0}x^5 + \binom{4}{1}x^3 + \binom{3}{2}x$$

…

$$F_{n+1}(x) = \binom{n}{0}x^n + \binom{n-1}{1}x^{n-2} + \cdots + \binom{n-\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor}x^{n-2\lfloor\frac{n}{2}\rfloor}$$

So by inspection we can see that for $n \geq 0$

$$F_{n+1}(x) = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-i}{i}x^{n-2i}$$

where$\lfloor n \rfloor$ represent the floor function.

Now using equations (1), (4),(5) and Table 4

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$$

$$= x\left[\binom{n}{0}x^n + \binom{n-1}{1}x^{n-2} + \cdots + \binom{n-\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor}x^{n-2\lfloor\frac{n}{2}\rfloor}\right]$$

$$+\left[\binom{n-1}{0}x^{n-1} + \binom{n-2}{1}x^{n-3} + \cdots + \binom{n-1-\lfloor\frac{n-1}{2}\rfloor}{\lfloor\frac{n-1}{2}\rfloor}x^{n-1-2\lfloor\frac{n-1}{2}\rfloor}\right]$$

$$= \left[\binom{n+1}{0}x^{n+1} + \binom{n}{1}x^{n-1} + \cdots + \binom{n+1-\lfloor\frac{n+1}{2}\rfloor}{\lfloor\frac{n+1}{2}\rfloor}x^{n+1-2\lfloor\frac{n+1}{2}\rfloor}\right]$$

i.e.

$$F_{n+2}(x) = \left[\binom{n+1}{0}x^{n+1} + \binom{n}{1}x^{n-1} + \cdots + \binom{n+1-\lfloor\frac{n+1}{2}\rfloor}{\lfloor\frac{n+1}{2}\rfloor}x^{n+1-2\lfloor\frac{n+1}{2}\rfloor}\right]$$

where$\lfloor n \rfloor$ represent the floor function.

Theorem 1.1: If $F_n(x)$ is any Fibonacci Polynomial then for any integer$n \geq 0$:

$$F_{n+1}(x) = \binom{n}{0}x^n + \binom{n-1}{1}x^{n-2} + \cdots + \binom{n-\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor}x^{n-2\lfloor\frac{n}{2}\rfloor}$$

$$= \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-i}{i}x^{n-2i} \qquad (6)$$

where$\lfloor n \rfloor$ represent the floor function.

Proof: It follows from above discussion, also Theorem 1.1 can be proved using principal of mathematical induction. As direct consequences of Theorem 1.1 and the definition of Fibonacci polynomial we obtained the following theorems

Theorem 2.1[2]: If $F_n(x)$ is any Fibonacci Polynomial then for any integer $n \geq 0$:

(i)

$$1 + x\sum_{i=0}^{n}F_{2i}(x) = \sum_{i=0}^{n}\binom{2n-i}{i}x^{2n-2i}$$

(ii)

$$x\sum_{i=0}^{n}F_{2i+1}(x) = \sum_{i=0}^{\lfloor\frac{2n+1}{2}\rfloor}\binom{2n+1-i}{i}x^{2n+1-2i}$$

(iii)

$$1 + x\sum_{i=0}^{n}F_{4i}(x) = \left[\sum_{i=0}^{n}\binom{2n-i}{i}x^{2n-2i}\right]^2$$

Theorem 2.2[2]: If $F_n(x)$ is any Fibonacci Polynomial then for any integer$n \geq 0$:

(i)

$$x^{n+1} + (x^2+1)\sum_{i=0}^{n}x^{n-i}F_{3i+2}(x) = \sum_{i=0}^{\lfloor\frac{3n+3}{2}\rfloor}\binom{3n+3-i}{i}x^{3n+3-2i}$$

(ii)

$$x^{n+1} + \sum_{i=0}^{n}x^{n-i}F_i(x) = \sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1-i}{i}x^{n+1-2i}$$

(iii)

$$(x^2+1)\sum_{i=0}^{n} x^{n-i}F_{3i+1}(x) = \sum_{i=0}^{\lfloor\frac{3n+2}{2}\rfloor} \binom{3n+2-i}{i} x^{3n+2-2i}$$

(iv)

$$x^{n+1} + (x^2+1)\sum_{i=0}^{n} x^{n-i}F_{3i}(x) = \sum_{i=0}^{\lfloor\frac{3n+1}{2}\rfloor} \binom{3n+1-i}{i} x^{3n+1-2i}$$

Theorem 2.3[1][2]: If $F_n(x)$ is any Fibonacci Polynomial then for any integer$n > 0$:

(i)

$$x\sum_{i=0}^{n} F_i^2(x) = \left[\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1-i}{i} x^{n-1-2i}\right]\left[\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n-i}{i} x^{n-2i}\right]$$

(ii)

$$F_n(x)F_{n+3}^2(x) - F_{n+2}^3(x) = (-1)^{n+1}\left[x^3\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n-i}{i} x^{n-2i} + (x^2-1)\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1-i}{i} x^{n-1-2i}\right]$$

(iii)

$$F_{n+3}(x)F_n^2(x) - F_{n+1}^3(x) = (-1)^{n+1}\left[\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n-i}{i} x^{n-2i} + x\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1-i}{i} x^{n-1-2i}\right]$$

Proof: To prove Theorems 2.1-2.3 we can simply use Theorem 1.1 and the fact that each expression on the left hand side can be written as a single or power of Fibonacci Polynomial, we could use the principal of mathematical induction,Binet's formula, combinatorial arguments or just simple algebra to prove the theorems.

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