Available online at www.pelagiaresearchlibrary.com

Pelagia Research Library

Advances in Applied Science Research, 2011, 2 (6):145-154

Pelagia Research Library

ISSN: 0976-8610
CODEN (USA): AASRFC

Encircled Energy Factor as a Point-Image Quality-Assessment Parameter

Alety. Srisailam*, Vendavetla. Dharmaiah, Ramanamurthy M.V and Pranab Kumar Mondal.

** Department of Mathematics, Osmania University, Hyderabad, A.P, India*
Mondal Institute of Optics, Gandhi agar, Hyderabad, A.P., India

---

ABSTRACT

Encircled Energy Factor (EEF) is an important corollary of the Point Spread Function (PSF) of an optical system. In the present paper, we have studied the Encircled Energy Factor and other associated corollaries of the PSF,viz. Displaced Energy in order to understand their role as point-image quality assessment parameters.

Key-words: Fourier Optics, Apodisation, Encircled Energy Factor Displaced Energy factor etc.

INTRODUCTION

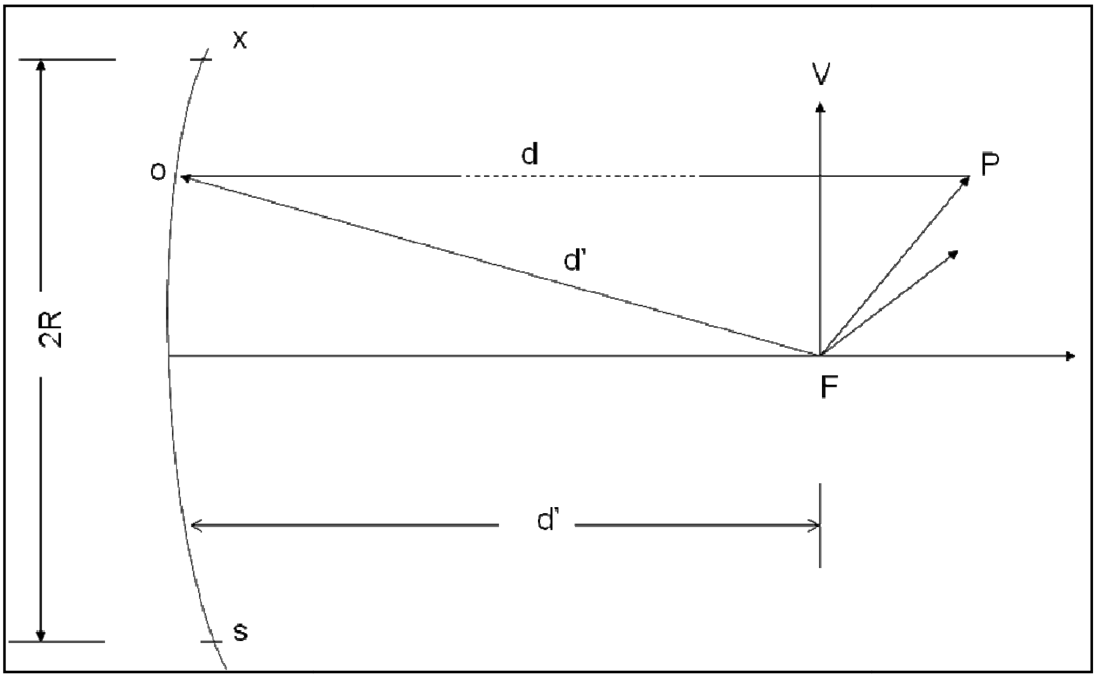
The most important corollary of the Point Spread Function (PSF) is the **"Encircled Energy Factor"** or the **"Encircled Power".** It measures the fraction of the total energy in the PSF, which lies within a specified radius '$\boldsymbol{\delta}$' in the plane of observation or detection. It is one of the significant parameters which serve as an index of the performance of an optical system. The radial distribution of energy within the image, called the encircled power, is a classical measure of the quality of the optical system producing that particular image. We will designate this important parameter by the symbol $EEF(\boldsymbol{\delta})$. $EEF(\boldsymbol{\delta})$, obviously vanishes when $\boldsymbol{\delta}$ is zero and approaches unity when $\boldsymbol{\delta}$ becomes infinity. Lord Rayleigh (1). Was the first to point out the importance of the encircled energy factor to find the illuminations in the various rings of the diffraction pattern and presented a formula for calculating the same.

When a converging spherical wave is diffracted by a circular aperture the classical theory of focusing predicts that light energy is highly concentrated in the geometrical focal plane. It means that there is a maximum amount of energy within a receiving circle of a given radius centered at the aperture axis and placed in the geometrical focal plane which contains more energy per unit area than any other plane parallel to it. Thus, it comes out that the $EEF(\boldsymbol{\delta})$ is the primary corollary of the PSF and is the factor, which describes the integrated behavior of the point source diffraction image. It is a sensible image quality evaluation parameter of an optical system, which may be diffraction–limited, defocused, aberrated, apodised or even a combined form of all these phenomena.

Previous studies on encircled energy factor: LORD RAYLEIGH (1) gave the mathematical formula for a circular aperture without apodisation i.e., in diffraction limited Point Spread Function. STOKES (2) and RAYLEIGH (1) also proved a number of general theorems on encircled energy. The idea of making the encircled energy maximum for a certain radius and thereby optimize the diffraction pattern has been utilized and studied by STRAUBEL (3) WOLF and LINFOOT (4) carried out a detailed study of Encircled Energy for a circular aperture. LANSRAUX (5) gave an alternate interpretation of Encircled Energy as the contrast at the centre of the image with respect to the background. LANSRAUX and BOIVIN (6) found that there is a maximum limit to the concentration of energy in the central core of the diffraction pattern. SHANNON and NEWMAN (7) developed an apparatus for measuring Encircled Energy. BARAKAT and HOUSTON(8) have shown that the encircled energy can be obtained directly by measuring the transfer function, An improved method for measuring encircled or enclosed energy for imaging optical systems, makes use of previously micro- machined detections which are positioned with a great accuracy at the centre of an image, by LEVI (9).

SURENDAR, SESHAGIRI RAO and MONDAL (10) have studied the encircled energy and its complimentary quantity, excluded energy using Lanczos apodisation filters.

2. MATHEMATICAL EXPRESSION FOR *EEF*



The figure.1 shows a schematic representation of diffraction at a circular aperture of diameter $2R$. Let us consider a spherical wave-front S having the radius $d^{'}$ and which momentarily fills the aperture, is emerging from the optical system and converging towards the axial focal point F .

Let $P(u,v,w)$ be a typical point not far away from F. Also, let $FP = \bar{\rho}$ be the position vector by which the point P is specified and let $d^{'} = PF$. Our intention is to study the diffracted field $A_p(u,v,w)$ i.e., the amplitude of the light diffracted at the point P. Let d be the distance of the point P from an arbitrary point $O(\xi,\eta,\zeta)$ on the wave-front just at the moment where it is incident on the aperture. Let $\frac{A}{d^{'}}$ be the amplitude of the incident wave at the point O. We shall assume that the incident light is quasi-monochromatic light and that the wave length is very much small compared to the radius of the aperture i.e., $\lambda << R$.

A general expression for the complex amplitude at the point P can now be obtained by applying the Huygens's Fresnel principle. Thus, following BORN and WOLF [1], the complex amplitude of light diffracted at the point P can be written as

(1) $$G(u,v,w) = -\frac{i}{\lambda}\left(\frac{A}{d}\right)\exp(-ikd')\iint_S f(r)[\exp(ikd)]\frac{dS}{d}$$

In the above expression, k stands for propagation constant $2\frac{\pi}{\lambda}$ and $f(r)$ known is the pupil function which defines the nature of transmission over the pupil of the aperture of the optical system under consideration. The double integral has to be carried out over the entire surface area of the wave-front incident on the aperture. The usual inclination factor has been omitted here since only small angles are involved. If $\bar{0}$ denotes a unit vector in the direction OF, we can write, with good approximation,

(2) $$(d'-d) = \bar{o}.\,\bar{\rho}$$

The surface element dS can be expressed as

(3 $$dS = (d)^2\, d\alpha$$

Where dS is the element which subtends a solid angle $d\alpha$ at the point F.

We can write (4) $$d\alpha = \frac{dS}{(d)^2} = \frac{(R^2 r dr d\theta)}{(d^{'})^2}$$

Where (r,θ) are the polar co-ordinates of the point O. Without introducing an appreciable error, d can be replaced by d' in the denominator of the integrand in the expression (II-1). Thus, after simplification, we obtain

(5) $$G(u,v,w) = -\frac{iA}{\lambda}\iint_\alpha f(r)\exp(-ik\,\bar{o}.\bar{\rho})\,d\alpha$$

The integration extending over the solid angle $d\alpha$ subtended by the aperture at the point P. For a clear aperture $f(r)=1$ and the expression (5) reduces to the well known Debye integral viz.,

(6) $$G(u,v,w) = -\frac{iA}{\lambda}\iint_{\alpha} \exp\left(-ik\,\bar{o}.\,\bar{\rho}\right) d\alpha$$

which can be written, after substituting for $d\alpha$ from equation (4), as

(7) $$G(u,v,w) = -\frac{iA}{\lambda}\int_0^1\int_0^{2\pi} f(r)\exp\left(-ik\,\bar{o}.\bar{\rho}\right) R^2 r dr d\theta \Big/ d'^2$$

Let us use the polar co-ordinates (σ,ϕ) for P and (r,θ) for O. We can express the Cartesian co-ordinates of the points O and P as.

(8) $$u = \sigma\sin\phi \qquad v = \sigma\cos\phi$$

and

(9) $$\xi = Rr\sin\theta \qquad \eta = Rr\cos\theta$$

and

(10) $$\zeta = \left[(d')^2 - R^2r^2\right]^{1/2} = d'\left[1 - \frac{1}{2}\frac{R^2r^2}{[d']} + ...\right]$$

We therefore, get (11) $$\bar{o}.\,\bar{\rho} = \frac{u\xi + v\eta + w\zeta}{d'}$$

Substituting the values of u, v, ξ, η and ζ from (8), (9) and (10) in, (11).

We obtain

$$\bar{o}.\,\bar{\rho} = \left[\frac{\sigma\sin\varphi Rr\sin\theta + \sigma\cos\varphi Rr\cos\theta}{d'}\right] + \frac{wd'}{d'}\left[1 - \frac{1}{2}\frac{R^2r^2}{(d')^2} + ...\right]$$

(12) $$= \frac{Rr\sigma\cos(\theta - \varphi)}{d'} + w\left[1 - \frac{1}{2}\frac{R^2r^2}{(d')^2}\right]$$

where the higher powers $\left(\frac{R^2r^2}{(d')^2}\right)$ have been neglected. Let us now introduce the two dimensionless variables Y and Z to specify the position of the point P.

(13) $$Y = \frac{2\pi}{\lambda}\left(\frac{R}{d'}\right)^2 w$$

And $$Z = \frac{2\pi}{\lambda}\left(\frac{R}{d'}\right)\sigma$$

Where $\sigma = (u^2 + v^2)^{1/2}$,

It has to be noted that the point P lies on the direct beam of light or in the geometrical shadow according as $\frac{Z}{Y} < 1$. Using $\frac{2\pi}{\lambda} = k$, we can write

(14) $$k\,\bar{o}.\,\bar{\rho} = Zr\cos(\theta - \phi) - \left(\frac{d'}{R}\right)^2 y + \frac{1}{2} yr^2$$

Substituting the value of $k\,\bar{o}.\,\bar{\rho}$ value from (14) in equation (7), we obtain

(15) $$G(y,Z) = -\frac{iA}{\lambda}\int_0^1\int_0^{2\pi} f(r)\exp\left[-iZr\cos(\theta-\phi) + i\left(\frac{d'}{R}\right)^2 y - \frac{1}{2} iyr^2\right]\frac{R^2 r dr d\theta}{d'^2}$$

$$= \frac{iA}{\lambda}\int_0^1\int_0^{2\pi} f(r)\exp\left[(-irZ)\cos(\theta-\phi) - \frac{1}{2}(iyr^2) + i\left(\frac{d'}{R}\right)^2 y\right]\frac{R^2 r\, dr d\theta}{d'^2}$$

(16)

The integral w.r.t θ is a well known standard integral being equal to $2\pi J_0(Zr)$ where $J_0(Zr)$ is the Bessel function of the first kind and zero order for the argument (Zr). Thus,

$$G(y,Z) = -\left(\frac{i}{\lambda}\right)\left(\frac{AR^2}{d'^2}\right)\exp\left[i\left(\frac{d'}{R}\right)^2 y\right] 2\pi\int_0^1 f(r)\exp\left(\frac{-iyr^2}{2}\right) J_0(Zr)\, r dr \qquad (17)$$

Putting $-\frac{A}{\lambda d'^2}\exp\left[i\left(\frac{d'}{R}\right) y\right] = \phi,$ we get

(18) $$G(y,Z) = 2\pi i\phi R^2 \int_0^1 f(r)\exp\left(\frac{-iyr^2}{2}\right) J_0(Zr)\, r dr$$

The term $\pi i R^2 \phi$ outside the sign of integration does not have any effect on the diffraction pattern. Neglecting, therefore, this term we obtain

(19) $$G(y,Z) = 2\int_0^1 f(r)\exp\left(\frac{-iyr^2}{2}\right) J_0(Zr)\, r dr$$

as the expression for the amplitude of light diffracted at the point P. The factor of 2 has been retained in the above diffraction integral as a normalizing factor. point spread function of the optical system can be evaluated by knowing the explicit expression of the pupil function $f(r)$ and then taking the squared modulus of the equation (19), at the focused plane of observation corresponding to $y = 0$, the above expression reduces to,

(20) $$G(0,Z) = 2\int_0^1 f(r) J_0(Zr)\, r\, dr$$

The Encircled Energy Factor (EEF) is defined as the ratio of the flux inside a circle of radius '**δ**' centered on the diffraction head to the total flux in the image of a point object. Thus,

(21) $$EEF(\boldsymbol{\delta}) = \frac{\int_0^{2\pi}\int_0^{\delta} |G\ (0,z)|^2\, zdzd\phi}{\int_0^{2\pi}\int_0^{\infty} |G\ (0,z)|^2\, zdzd\phi}$$

where ϕ is the azimuthally angle; $G\ (0,z)$ is the amplitude in the image plane at a point z units away from the diffraction head due to the aperture function $f(r)$. Since, the integration over ϕ introduces the same constant 2π in the numerator and the denominator, the above expression reduces to

(22) $$EEF(\delta) = \frac{\int_0^{\delta} |G\ (0,z)|^2\, zdz}{\int_0^{\infty} |G\ (0,z)|^2\, zdz}$$

The denominator in the expression [2] represents the total flux in the entire image plane. This implies an impossible task of evaluating the denominator by integrating the *PSF* over the image plane, i.e., for the limits of z in the range $0 \le z \le \infty$. However, in actual practice, $G\ (0,z)$ is rapidly convergent and drops to zero value at a finite distance from $z \ge 0$ to $z \le 15.0$. This happens due to the fact that G contains Bessel functions of the first kind, which oscillate from positive to negative values very rapidly and become zero at a finite distance from the centre of the diffraction image $(z = 0)$. Thus, it will be sufficient for all practical purposes if the upper limit of integration in the denominator of [2] is fixed around 15.0. Therefore,

(23) $$EEF(\delta) = \frac{\int_0^{\delta} |G\ (0,z)|^2\, zdz}{\int_0^{15} |G\ (0,z)|^2\, zdz}.$$

It may be mentioned here that analytically too, a closed form solution of the integral in the denominator of [2] can be obtained in terms of another corollary of the PSF, **"Passing – Flux ratio"** denoted by τ.

RESULTS AND DISCUSSION

For calculation we have using Mathematica4.1 software we given program as input then we got results. This parameter is also known as the encircled power, i.e., the amount of power contained in a circle of radius δ in the Gaussian plane of observation $(y = 0)$, which is centered on the Gaussian image point $(z = 0)$. Fig.2 shows how the encircled power on the encircled energy factor varies withδ. It is observed from the table-1 that the central Airy disc $(\,for\ \beta = 0)$ contains 83.8% of the total power. The first bright ring contains 7.2%, the second bright ring 2.8% and the third bright ring 1.4% of the total power.

Figure.2

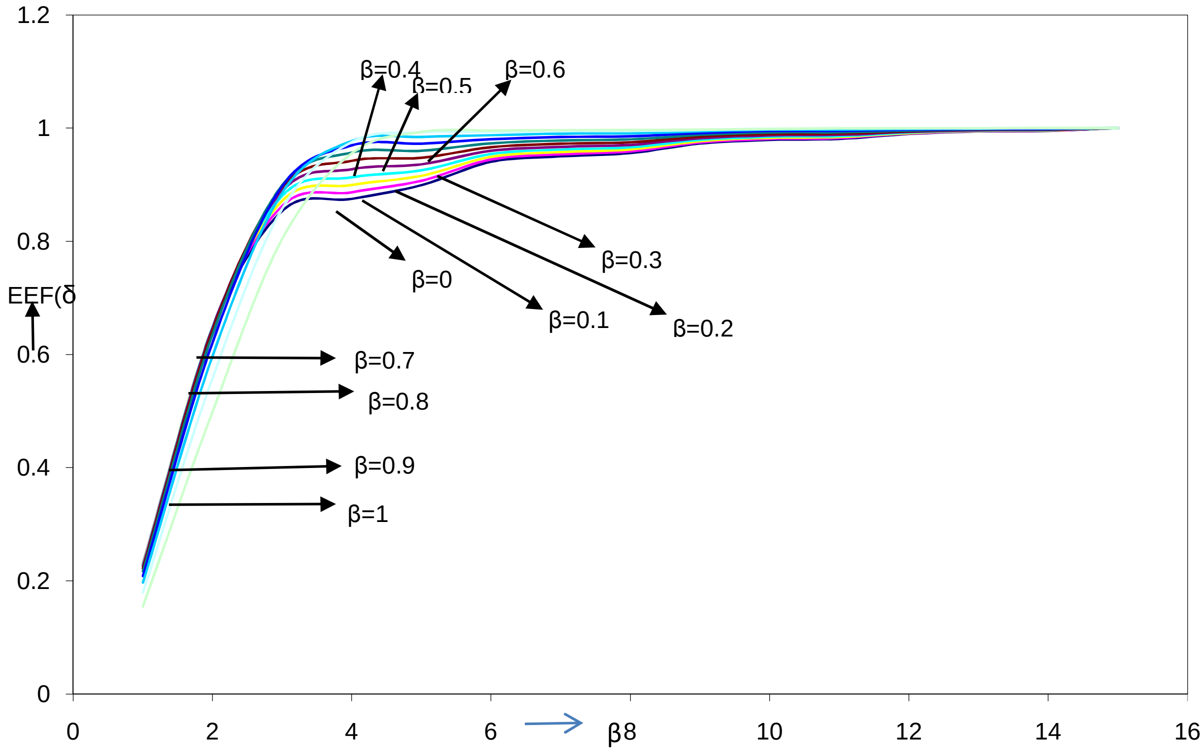


Table.1

ENCIRCLED ENERGY FACTOR FOR BARTLETT WINDOW FUNCTIONS

δ values	˜ β=0	β=0.1	β=0.2	β=0.3	β=0.4	β=0.5	β=0.6	β=0.7	β=0.8	β=0.9	β=1
1	0.2306	0.2301	0.2293	0.2278	0.2255	0.222	0.2166	0.2086	0.1967	0.1794	0.1548
2	0.6445	0.6466	0.6481	0.6486	0.6473	0.6435	0.6355	0.6212	0.597	0.5582	0.4988
3	0.8535	0.8623	0.8714	0.8803	0.8886	0.8953	0.8991	0.8972	0.8855	0.8576	0.8043
4	0.8749	0.8866	0.8992	0.9129	0.9273	0.9423	0.9568	0.9695	0.9773	0.9754	0.9557
5	0.8992	0.907	0.9157	0.9253	0.9359	0.9475	0.9599	0.9726	0.9842	0.9926	0.9934
6	0.9405	0.9446	0.9491	0.9542	0.9599	0.9662	0.9731	0.9803	0.9874	0.9932	0.9958
7	0.9501	0.9536	0.9576	0.962	0.9669	0.9723	0.9781	0.9842	0.9901	0.9948	0.9965
8	0.9559	0.9589	0.9623	0.9661	0.9703	0.975	0.98	0.9854	0.9907	0.995	0.997
9	0.9728	0.9747	0.9769	0.9793	0.9819	0.9848	0.988	0.9912	0.9943	0.9966	0.9973
10	0.979	0.9807	0.9826	0.9847	0.987	0.9895	0.9921	0.9947	0.9971	0.9987	0.9986
11	0.9809	0.9824	0.984	0.9859	0.9879	0.9902	0.9926	0.9951	0.9974	0.9992	0.9997
12	0.9896	0.9904	0.9912	0.9921	0.9932	0.9943	0.9956	0.9969	0.9983	0.9993	0.9999
13	0.9943	0.9947	0.9952	0.9957	0.9963	0.997	0.9977	0.9984	0.9991	0.9997	0.9999
14	0.995	0.9954	0.9958	0.9962	0.9968	0.9973	0.9979	0.9986	0.9992	0.9998	1
15	1	1	1	1	1	1	1	1	1	1	1

4. Displaced Energy Factor [$DEF(\delta)$]:

Wetherill (11) has defined the **"displaced energy"** as the difference of the encircled energy of the diffraction-limited system, in a specified circle, to the encircled energy due to an apodised system in the same specified circle. The mathematical form of this factor is, therefore,

(24) $$DEF(\delta) = EEF_A(\delta) - EEF_F(\delta)$$

Where, $EEF_A(\delta)$ and $EEF_F(\delta)$ represent the encircled energy for a diffraction -limited system and an apodised system respectively.

$$DEF(\boldsymbol{\delta}) = \frac{\int_0^{2\pi}\int_0^{\delta} |G_A(0,z)|^2 \, z dz d\phi}{\int_0^{2\pi}\int_0^{15} |G_A(0,z)|^2 \, z dz d\phi} - \frac{\int_0^{2\pi}\int_0^{\delta} |G_F(0,z)|^2 \, z dz d\phi}{\int_0^{2\pi}\int_0^{15} |G_F(0,z)|^2 \, z dz d\phi}$$
(25)

Since, the integration over ϕ introduces the same constant 2π in the numerator and the denominator, the above expression reduces to

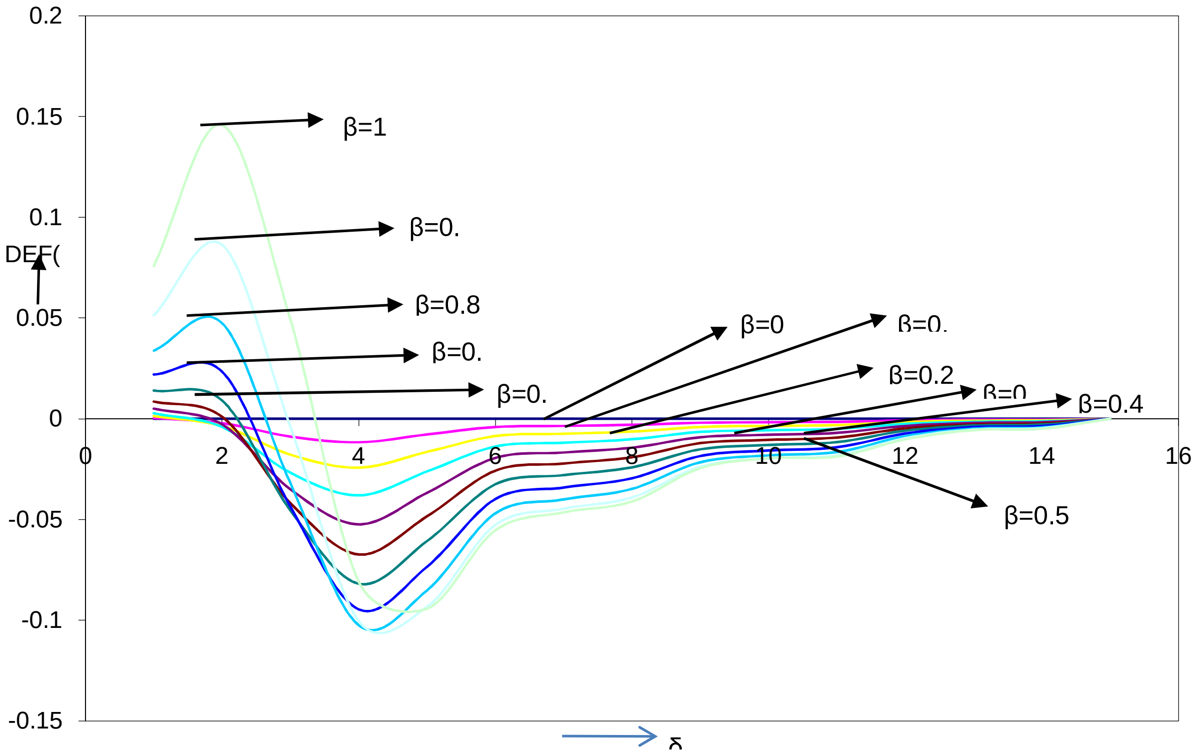
$$DEF(\delta) = \frac{\int_0^{\delta} |G_A(0,z)|^2 \, z dz}{\int_0^{15} |G_A(0,z)|^2 \, z dz} - \frac{\int_0^{\delta} |G_F(0,z)|^2 \, z dz}{\int_0^{15} |G_F(0,z)|^2 \, z dz}$$
(26)

Where the subscripts A and F stand for Airy (β=0) and filtered non-airy(β≠0) pupils respectively. The positive sign of this factor indicates that the energy displacement is outward while the negative sign indicates that the energy displacement is inward. This factor is useful to compare the energy distribution in the case of actual optical imaging systems to that of perfect systems. This is a more sensitive quality factor in the case of central obscuration in the aperture. It is, of course, less sensitive in the case of an image motion, where Strehl-ratio plays an important role.

Table.2

d values	**β=0**	**β=0.1**	**β=0.2**	**β=0.3**	**β=0.4**	**β=0.5**	**β=0.6**	**β=0.7**	**β=0.8**	**β=0.9**	**β=1**
1	0	0.0004	0.0013	0.0027	0.005	0.0086	0.0139	0.022	0.0339	0.0512	0.0758
2	0	-0.0021	-0.0036	-0.004	-0.0028	0.001	0.009	0.0233	0.0475	0.0863	0.1457
3	0	-0.0088	-0.0179	-0.0268	-0.0351	-0.0419	-0.0456	-0.0437	-0.032	-0.0041	0.0492
4	0	-0.0117	-0.0244	-0.038	-0.0525	-0.0674	-0.0819	-0.0946	-0.1024	-0.1005	-0.0809
5	0	-0.0078	-0.0165	-0.0261	-0.0368	-0.0484	-0.0608	-0.0734	-0.0851	-0.0934	-0.0942
6	0	-0.0041	-0.0086	-0.0137	-0.0194	-0.0257	-0.0326	-0.0398	-0.0469	-0.0527	-0.0553
7	0	-0.0035	-0.0075	-0.0119	-0.0168	-0.0222	-0.028	-0.0341	-0.04	-0.0447	-0.0464
8	0	-0.003	-0.0064	-0.0102	-0.0144	-0.0191	-0.0242	-0.0295	-0.0348	-0.0391	-0.0411
9	0	-0.0019	-0.0041	-0.0065	-0.0091	-0.0121	-0.0152	-0.0185	-0.0215	-0.0239	-0.0245
10	0	-0.0017	-0.0036	-0.0057	-0.008	-0.0105	-0.0131	-0.0157	-0.0181	-0.0197	-0.0196
11	0	-0.0015	-0.0031	-0.005	-0.007	-0.0092	-0.0117	-0.0141	-0.0165	-0.0183	-0.0188
12	0	-0.0007	-0.0016	-0.0025	-0.0036	-0.0047	-0.006	-0.0073	-0.0086	-0.0097	-0.0102
13	0	-0.0004	-0.0009	-0.0014	-0.002	-0.0027	-0.0034	-0.0042	-0.0049	-0.0054	-0.0056
14	0	-0.0004	-0.0008	-0.0012	-0.0018	-0.0023	-0.0029	-0.0036	-0.0042	-0.0048	-0.005
15	0	-1E-16	-1E-16	0	-1E-16	-1E-16	-1E-16	-1E-16	-1E-16	-1E-16	0

Figure.3



RESULTS AND DISCUSSION

We have used the expression (26) to evaluate the displaced energy. The results have been shown in the tabular form in table-2 and in the graphical form in the figure-3. So far as the variation of DEF with β is concerned, we find that only for lower values of δ, $DEF(\delta)$ is positive, whereas for higher values δ, $DEF(\delta)$ is negative irrespective of the value of β. We have noticed earlier that $\beta = 1$ gives the best possible result for the family of filters we have considered.

CONCLUSION

The pupil function of the chosen apodised system can be mathematically expressed as:

$$f(r) = (1 - \beta r) \quad \text{For} \quad 0 < r \le 1;$$
$$= 0 \qquad \text{for} \qquad r > 1;$$

Where β is the apodisation parameter which controls the transmission of the transmitted light through the optical system; $\beta = 0$ corresponds to the diffraction-limited perfect system with uniform transmission of unity within the pupil function which is known as the Airy system **r** Studies on encircled energy factor and its corollaries reveal that in most of the cases, the DEF (δ) is negative, i.e., the energy displacement is inward as towards the centre of the diffraction pattern. is the normalized distance of a point within the pupil circle, $x^2 + y^2 = r^2 \leq 1$ from its centre. **In observational astronomy**, the experimental determination of a PSF is often very straight-forward due to the ample supply of point sources like **stars or quasars**. The form and the shape of the PSF may vary widely depending on the instrument and the context in which it is used. The theoretical model presented in this thesis should be able to handle capably these varying situations.

In space telescopes: For radio telescopes and diffraction-limited space telescopes, the dominant terms in the EEF may be inferred from the configuration of this aperture in the **Fourier domain**. In practice, there may be multiple terms contributed by various components in a complex optical system. A complete description of the EEF will also include diffusion of light or photo-electrons in the detector, as well as tracking errors in the space-craft or the telescope. In such situations, the pupil-function has to be suitably modified to fit into our theoretical model.

Acknowledgements
Authors express their gratitude to University Grants Commission, New Delhi, India for sanctioning DRS-III/SAP-I to the Department of Mathematics, Osmania University, Hyderabad, A.P., India wherein the facilities are used for under taking this work

REFERENCES

[1] Rayleigh Lord, (**1879**), *Phil. Mag*. 8, 261,
[2] G. Stokes, C. W. (**1966**) an Introduction to Coherent Optics and Holography, (Academic Press, New York).
[3] Sraubel, R., Pieter Zeeman Verhandclingen, The Hague, Nij hoft, (**1935**).
[4] Linfoot, R., Wolf, E. (**1951**) *Rep. Prog. Phys*: 14, Proc. Phys. , Soc
[5] Lansraux, G., (**1947**) *Revved optique*. 26, 24
[6] Lansraux, G., Boivin, A. (**1961**) *Can. J. Phys*., 39, 158.