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Effect of the plug flow on the flow characteristics of bile through diseased cystic duct: Casson Model Analysis

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ABSTRACT

Bile flow in the human biliary system plays a vital role in the pathogenesis of biliary diseases. In the present study, the effect of the plug flow in the cystic duct on the flow characteristics of bile is observed. Bile is taken as a Casson fluid. The expression for the flow characteristics like the resistance and the wall shear stress is studied. It is observed that as the size of stone and the core radius increases, the resistance to flow and the shear stress also increases.

INTRODUCTION

In order to understand the cause of biliary diseases, it is important to understand the mechanical behavior of the human biliary system. The presence of gallstone is one of the most common human biliary disease which is also known as Cholelithiasis. Clinical treatments for the diseases often involve the surgical removal of the gallbladder, known as cholecystectomy, which is the most often performed abdominal operation in the West. The role of human biliary system is to create, store, transport and release bile into the duodenum to aid digestion of fats. It consists of an organ and duct system. Bile, the liquid that flow in the biliary system is a yellow-brown fluid produced by the liver and is composed of three main components –cholesterol, bile salts and bilirubin. When the gallbladder is not functioning properly, components of bile are supersaturated leading to the formation of solid crystals called gallstones.Most common biliary diseases are:

- Cholelithiasis - the presence of gallstones.
- Cholecystitis – the inflammation of gallbladder.

Jungst et al.[1] suggested that the supersaturation and rapid nucleation of cholesterol in bile were of key importance in the pathogenesis of cholesterol gallstones. Ooi et al.[2] proposed the numerical study of steady flow in human cystic duct with idealized models were constructed, first with staggered baffles in a channel to represent the valves of Heister and lumen. The mathematical model accounting for the effects of geometry of the baffle configurations is presented by Al-Atabi et al.[3].It predicted the pressure drop in circular pipe fitted with segmental baffles. Bird et al.[4] proposed the variation in geometry of the cystic duct, obtained from acrylic resin casts of the neck and first part of the cystic duct in gallbladders removed for gallstones disease and obtained from patients undergoing partial hepatectomy for metastatic disease. Li et al.[5] presented the mechanical model for the human biliary system during the emptying phase, based on a clinical test in which gallbladder volume changes were measured in response to a standard stimulus and a recorded pain profile. Li et al.[6] presented the two one-dimensional models to estimate the pressure drop in the normal human biliary system for Reynolds number up to 20. Excessive pressure drop during bile emptying and refilling may result in incomplete bile emptying, leading to bile stasis and subsequent gallstones formation. The effects of biliary system geometry, elastic property of cystic duct and bile viscosity on pressure drop was studied. It was found that the maximum pressure drop occurs during bile emptying immediately after a meal, and was greatly influenced by the viscosity of bile and the geometric configuration of the cystic duct. To understand the physiological and pathological functions of the biliary system ,the work is presented by Luo et al.[7]. It was believed that the mechanical factors play an important role in the mechanism of the gallstone formation and biliary diseases.The study of the human biliary system to include two new factors :the non–Newtonian properties of bile, and elastic deformation of the cystic duct is presented by Li et al.[8]. A one – dimensional (1D) model was analyzed and compared with three–dimensional fluid structure interaction simulations. It was found that non – Newtonian bile raises resistance to the flow of bile, which can be increased enormously by the elastic deformation(collapse) of the cystic duct.

It has been shown that cholecystokinin (CCK) stimulation not only causes the gallbladder to contract, but also allows the cystic and common bile ducts to contract. The squeezing of the gallbladder causes bile to flow from gallbladder to the cystic duct and finally into the common bile duct. However, the sensitivity to CCK decreases from gallbladder to common bile duct. In this work the effect of the flow of bile in the cystic duct on the resistance to flow and the skin friction is studied. The present investigation has been focused to the problem of bile flow in the diseased cystic duct where the rheology of bile is described by Casson model. Computational result for the variation of flow resistance and wall shear stress are presented graphically.

1. Formulation and Solution of the Problem

The equation governing the flow of bile in the rigid cystic duct is taken in the form –

$$-\frac{dP}{dz} = \frac{1}{r}\frac{d(r\tau)}{dr} \tag{1}$$

The constitutive equation for the Casson fluid Model may be put as

$$-\frac{du_z}{dr} = \frac{1}{k}\left(\tau^{\frac{1}{2}} - \tau_0^{\frac{1}{2}}\right)^2 \quad ; \quad \tau \geq \tau_0$$

$$\frac{du_z}{dr} = 0 \quad ; \quad \tau \leq \tau_0 \tag{2}$$

In equations (1) and (2) z is the co-ordinate along the axis of the cystic duct in the flow direction, r is the co-ordinate in the radial direction and perpendicular to fluid flow, τ represents the shear stress of bile considered as Casson fluid , P is the pressure at any point ,(-dP/dz) is the pressure gradient , u_z stands for the axial velocity of bile , τ_0 is the yield stress and k stands for viscosity.

2A.Boundary conditions

The equations (1) and (2) are solved using the boundary conditions as follows-

a) τ is finite at r=0 (3)

b) u_z =0 at r=R(z) (4)

c) P= P_0 at z=0 and P = P_L at z= L (5)

Where the condition (3) and (4) are respectively known as regularity and no-slip conditions.

2B.Geometry used

The geometry of the cystic duct (Figure1) can be written as Young [10]-

$$R(z) = R_0\left[1-\frac{\delta}{2R_0}\left\{1+\cos\frac{2\pi}{L_2}\left(Z-L_1-\frac{L_2}{2}\right)\right\}\right] \quad ; \quad L_1 \le Z \le L_2+L_1 \qquad (6)$$

$$= 1 \quad ; \quad \text{elsewhere}$$

Where R (z) is the radius of the cystic duct in the obstructed region, R_0 is the radius of cystic duct in the normal region, δ is the maximum height of the stone assumed to be much smaller in comparison to the radius of cystic duct (δ <<R_0), L_1 is the length of cystic duct till the onset of stone, L_2 is the length of stone and L is the length of cystic duct.

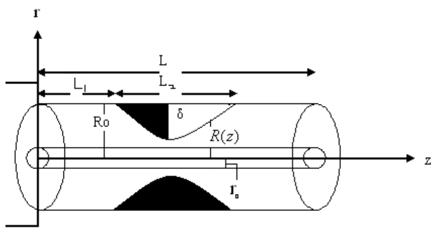


Figure 1: The geometry of the cystic duct with stone.

2C.Analysis of Model and Method of Solution

The model (1) and (2) is solved using the boundary conditions (3) and (4).On integrating (1) and using (3) the stress component can be found as:

$$\tau = -\frac{r}{2}\frac{dP}{dz} \qquad (7)$$

we can write, $\tau = \frac{pr}{2}$ where $p = \frac{-dP}{dz}$ (8)

Since, in the core region, $0 \le r \le r_0$, the yield stress exists and is given by Mazumdar [9]

$$\tau_0 = \frac{pr_0}{2} \qquad (9)$$

Where r_0 is the radius of the core region. Due to the existence of yield stress, the plug flow occurs. Also, in the core region =0

$$\Rightarrow \quad u_z = \text{constant}$$

Let $u_z = u_0$ be the velocity of bile in core region which shows that the velocity profile is straight.

Now to find the velocity of flow outside the core region, the equation (2) is re-arranged as

$$\sqrt{k\dot{\gamma}} = \sqrt{\tau} - \sqrt{\tau_0} \qquad (10)$$

Where $\dot{\gamma}$ = shear strain rate

Also $\dot{\gamma} = \frac{-du_z}{dr}$ (11)

Squaring equations (10) and using (11)

$$\dot{\gamma} = \frac{-du_z}{dr} = \frac{1}{k}(\sqrt{\tau} - \sqrt{\tau_0})^2$$

Using equations (7), (8) and (9) we get

$$\frac{du_z}{dr} = \frac{p}{2k}(2\sqrt{rr_0} - r - r_0) \quad (12)$$

On integrating (12), the equation can be written as

$$u_z = \frac{p}{2k}\left[\frac{2r^{3/2} * 2\sqrt{r_0}}{3} - \frac{r^2}{2} - r_0 r\right] + c \quad (13)$$

On using boundary condition (4), the above equation can be written as

$$u_z = \frac{p}{4k}\left[\left(R^2 - r^2\right) - \frac{8}{3}\sqrt{r_0}\left(\sqrt{R^3} - \sqrt{r^3}\right) + 2r_0\left(R - r\right)\right] \quad (14)$$

Where r represents $r_0 \le r \le R(z)$ and $R = R(z)$

At the core region, velocity=u_0 and radius=r_0, therefore equation (14) can be written as

$$u_0 = \frac{p}{4k}\left[R^2 + 2Rr_0 - \frac{1}{3}r_0^2 - \frac{8}{3}\sqrt{r_0 R^3}\right] \quad (15)$$

The equation (15) represents velocity distribution for all values of radius between 0 and r_0.

Rewriting equation (14) as

$$u_z = u_0 - \frac{p}{4k}\left[r^2 + 2r_0 r - \frac{8}{3}\sqrt{r_0 r^3} - \frac{1}{3}r_0^2\right]$$

The flux Q is given by

Q = flux in core region + flux outside of core region

$$= \pi R^2 u_0 - \frac{p\pi}{k}\left[\frac{1}{8}\left(R^4 - r_0^4\right) + \frac{r_0}{3}\left(R^3 - r_0^3\right) - \frac{8}{21}\sqrt{r_0}\left(R^{7/2} - r_0^{7/2}\right) - \frac{1}{12}r_0^2\left(R^2 - r_0^2\right)\right]$$

$$Q = \pi R^2 u_0 - \frac{p\pi R^4}{k}\left[\begin{array}{l}\frac{1}{8}\left(1 - \left(\frac{r_0}{R}\right)^4\right) + \frac{r_0}{3R}\left(1 - \left(\frac{r_0}{R}\right)^3\right) - \frac{8}{21}\sqrt{\frac{r_0}{R}}\left(1 - \left(\frac{r_0}{R}\right)^{7/2}\right) \\ -\frac{1}{12}\frac{r_0^2}{R^2}\left(1 - \left(\frac{r_0^2}{R^2}\right)\right)\end{array}\right]$$

Let $\frac{r_0}{R}$ =c $_0$

$$Q = \frac{p\pi R^4}{8k}\left[1+\frac{4}{3}c_0-\frac{16}{7}c_0^{1/2}-\frac{1}{21}c_0^4\right] \quad (16)$$

Therefore, $Q = \frac{p\pi R^4}{8k} f(c_0)$ (17)

Where, $f(c_0) = f(c_0(z)) = \left[1+\frac{4}{3}c_0-\frac{16}{7}c_0^{1/2}-\frac{1}{21}c_0^4\right]$ (18)

Case study: When there is no yield stress, that is τ_0 =0, then the casson equation reduces to

$\tau = k\dot{\gamma}$, which is the Newtonian case.

Therefore putting τ_0 =0 i.e c_0=0 into the equation (17) we get

$Q=Q_0=\frac{p\pi}{8k}R^4$, which coincides with the Poiseullie equation which says

$Q=\frac{(p_1-p_2)}{L}\frac{\pi}{8k}R^4$

Where Q_0 is the flux at r_0=0.

Thus Poiseullie model may be taken as a particular case of Casson's model.

Integrating equation (17) and using the condition (5) we get

$$P_0-P_L = \frac{8kQ}{\pi}\left[\frac{1}{R_0^4}\frac{L_1}{f_0}+(L-L_1-L_2)\frac{1}{f_0R_0^4}\right]+\int_{L_1}^{L_1+L_2}\frac{dz}{R^4(z)f(c_0)} \quad (19)$$

where $f_0 = 1+4/3\, r_0/R_0 - 16/7(r_0/R_0)^{1/2} - 1/21(r_0/R_0)^4$

The resistance to flow λ is defined by

$$\lambda = \frac{P_0 - P_L}{Q} = \frac{8kL}{\pi R_0^4 f_0}[1-L_2/L + f_0/L\int_{L_1}^{L_1+L_2}\frac{dz}{f(c_0)\frac{R^4(z)}{R_0^4}}] \quad (20)$$

In the absence of any constriction,(i.e.) $\delta = 0$;the resistance to flow λ_N is given by

$$\lambda_N = \frac{8kL}{\pi R_0^4 f_0} \quad (21)$$

In dimensionless form the flow resistance may be expressed as

$$\bar{\lambda} = \frac{\lambda}{\lambda_N} = [1-\frac{L_2}{L}+\frac{f_0}{L}\int_{L_1}^{L_1+L_2}\frac{dz}{f(c_0(z))\frac{R^4(z)}{R_0^4}}] \quad (22)$$

This expression gives the resistance to the flow of bile in the cystic duct due to the presence of stone in the duct.

Wall Shear stress (Skin friction)
Using (7) we get

$$\tau_R = \frac{-R}{2}\frac{dP}{dz} \quad \text{at} \quad r=R\,(z)=R$$

Substituting the value of $\frac{dP}{dz}$ from (17)

$$\tau_R = \frac{4kQ}{\pi R^3 f(c_0)}$$

When there is no stone in the cystic duct i.e. $R=R_0$, then the wall shear stress is given by

$$\tau_N = \frac{4kQ}{\pi R_0^3 f_0}$$

Now the non-dimensional expression for the shear stress may be put as

$$\tau_1 = \frac{\tau_R}{\tau_N}$$

$$= \frac{1+4/3\, r_0/R_0 - 16/7(r_0/R_0)^{1/2} - 1/21(r_0/R_0)^4}{(R/R_0)^3 + 4/3\, r_0/R_0.(R/R_0)^2 - 16/7\sqrt{(r_0/R)}.(R/R_0)^{5/2} - 1/21(r_0/R)^4(R/R_0)^{-1}} \quad (23)$$

At $z=L_1+\frac{L_2}{2}$, the shear stress is given by

$$\tau_2 = [\frac{f_0}{f(c_0)}.(1/(R/R_0)^3)]$$

where $R/R_0=1-\frac{\delta}{2R_0}(1+\cos(L_1+\frac{L_2}{2}-L_1-\frac{L_2}{2}))$

$$= (1-\frac{\delta}{R_0})$$

$$\tau_2 = \frac{1+(4/3)r_0/R_0-(16/7)\sqrt{r_0/R_0}-1/21(r_0/R_0)^4}{(1-\frac{\delta}{R_0})^3+\frac{4}{3}(r_0/R_0)(1-\frac{\delta}{R_0})^2-\frac{16}{7}\sqrt{r_0/R_0}(1-\frac{\delta}{R_0})^{5/2}-\frac{1}{21}(r_0/R_0)^4(1-\frac{\delta}{R_0})^{-1}} \quad (24)$$

The above expression gives the non-dimensional value of shear stress at the maximum height of the stone.

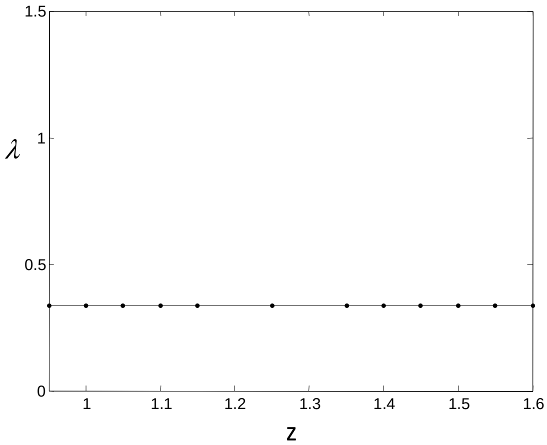
RESULTS AND DISCUSSION

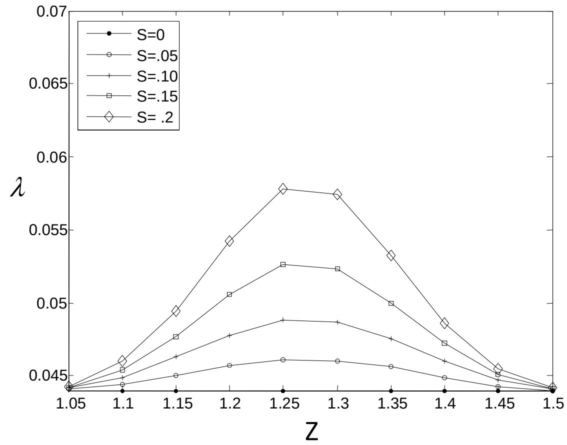
The analysis, equations (22), (23) and (24) derived in the preceding section have been computed. The computational results for the resistance and the shear stress have been plotted. Figure 2 illustrates the variation in resistance to flow with the axial distance in the non-uniform region. It is noticed that as the height of the stone increases the flow resistance also increases. When there is no stone in the cystic duct, the magnitude of resistance remains constant and the maximum resistance is observed at the throat of the duct (i.e. where the stone height is maximum). It is also noticed that as the core radius increases, the resistance to flow of bile also increases.

Figure 3 shows the variation of resistance to flow with the axial distance in the non uniform region of the cystic duct for different values of stone length. As the stone length increases the flow resistance also increases with maximum resistance at the throat of stone.

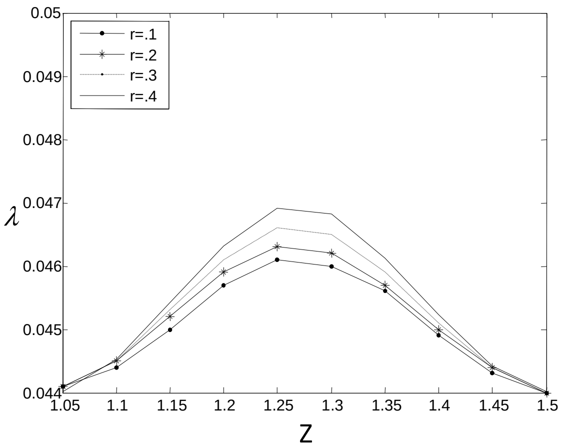
Figure 4 demonstrates the variation of shear stress with the axial distance in the non uniform region of the cystic duct for different values of the stone height and plug core radius. In both the graphs it is observed that the shear stress increases with the increase in stone height and the core radius of the duct. Moreover, the maximum stress is observed at the throat of stone.

Figure 5 illustrates the variation of shear stress with the stone height for different values of the core radius. As the stone height increases the stress also increases. Moreover stress (skin friction) increases with the increasing core radius.





(a) (b)



(c)

Figure 2: The variation of resistance to flow with the axial distance in the non uniform region of the cystic duct

a) For stone height=s=0
b) For variable stone height(s)
c) For variable core radius (r)

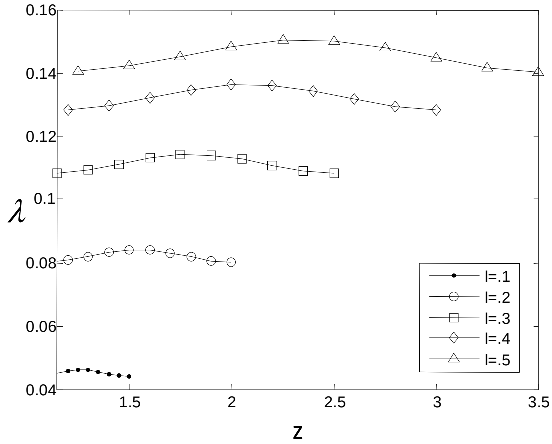
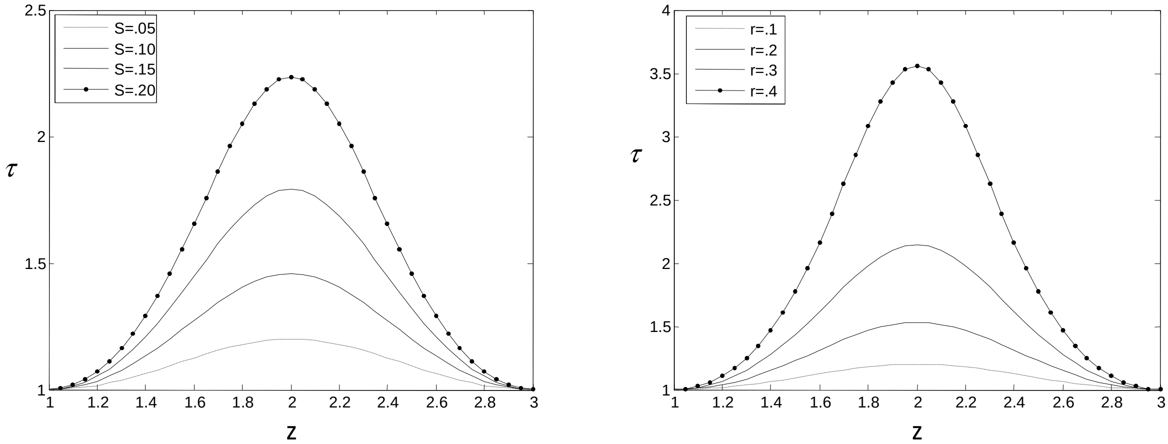


Figure 3: The variation of resistance to flow with the axial distance in the non uniform region of the cystic duct for stone length (l).



(a) **(b)**

Figure 4: The variation of shear stress with the axial distance in the non uniform region of the cystic duct.

a) For stone height(s)

b) For plug core radius(r)

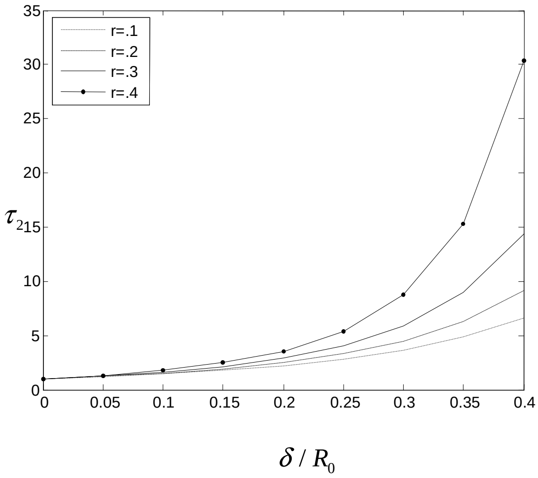


Figure 5: The variation of shear stress with the stone height for different values of the core radius

CONCLUSION

The one-dimensional rigid wall model of cystic duct has been proposed for estimating the flow characteristics of bile through the duct having axially symmetric stones. The presence of yield stress causes the plug flow in the core region and the incompressible bile here, behaves as a Casson fluid. The contribution of valves of heister in the cystic duct is negligible. The flowing bile is laminar, steady and axially symmetric. Using this model the effect of geometry and the plug core radius on resistance to flow and wall shear stress is studied. The resistance and wall shear stress increases as the size (height and length) of stone increases. The flow impedance is maximum at the maximum height of stone and minimum at the onset of uniform portion of the duct. With the increase in the core radius, the resistance to flow and shear stress increases

With the study of factors affecting the flow behavior of bile in the biliary system ,the preventive measures could be prescribed to patients at risk to gallstones.

REFERENCES

[1] Jungst D, Lang T, Huber P, Lang V, and Paumgartner G, *Journal of lipid* ***research*****1993**, Vol. **34**.
[2] Ooi R, Luo XY, Chin S, Johnson A and Bird NC,*Journal of biomechanics*,**2004, 37**: 1913-1922.
[3] Al-Atabi M, Chin SB, Al-Zuhair S, and Luo XY, *International Journal of Fluid Mechanics Research*,**2006**, Vol. **33**, No. 4.
[4] Bird N C, Ooi R, Chin SB, and Johnson A,*Clin Anat.*,2006, **19(6)**; 528-534
[5] Li W G, Luo XY, Bird N, and Johnson A,*J of Computational and Mathematical methods in Medicine*,**2008**, Volume 9, Issue 1, Pages 27-45.
[6] Li W, Chin SB, Johnson A, and Bird NC,*J of Biomechanics*,**2007**, Vol.**129**.
[7] Luo XY, Li W, Bird N, Chin SB, and Johnson A,*World J. Gastroentero*,**2007**, March 7**; 13(9):**1384- 1392.
[8] Li W, Luo XY, Chin SB, Hill NA, Johnson A, and Bird NC,Annals of Biomedical Engineering,**2008**, Vol.**36,** No.11.
[9] Mazumdar JN, Bio fluid mechanics;World Scientific Publishing Co.Pte.Ltd Singapore,**2004**.
[10] Young DF, *Journal of Engineering for Industry*, **1968**, 90, pp. 248 -254.