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Dynamics in a discrete fractional order Lorenz system

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Dedicated to Dr. Major. M. Reni Sagayaraj, (Head, Department of Mathematics, Scared Heart College, Tirupattur, S.India)

ABSTRACT

In this paper, we are interested in the discrete version fractional order Lorenz system. A discretization process is applied to obtain a discrete version. Fixed points are computed and the stability properties are analyzed. Bifurcation and chaos for different values of the fractional order parameter are presented.

Key words: Fractional order, discretization, Lorenz System, phase portraits.

INTRODUCTION

The concept of fractional order differentiation and integration is nearly as old as calculus itself the fractional calculus is the generalization of integer calculus. In recent years fractional order differential equations and systems have attracted many researchers because of their applications in many areas of science and engineering; see, for example,[1, 3, 4]. Weprovide the basic definitions (Caputo) and properties of fractional order differentiation and integration.Let us recall the following definitions.

Definition. 1 The fractional – order integral of a function $f : [t_0, \infty) \to \mathbb{R}$ of order $\alpha \in \mathbb{R}^+ = [0, +\infty)$ is defined by

$$I_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) ds,$$

where$\Gamma(.)$ is the gamma function.

Definition. 2 For a function f given on the interval $[t_0, \infty)$, the α order Riemann-Liouville fractional derivative of f is defined by

$$\mathrm{D}_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_{t_0}^{t} (t-s)^{n-\alpha-1} f(s) ds, \; n-1 < \alpha < n, n \in \mathbb{N},$$

where $\mathbb{N} = \{1,2,3, \dots\}$.

Definition.3 For a function f given on the interval $[t_0, \infty)$, the α order Caputo fractional derivative of f is defined by

$${}^{c}\mathrm{D}_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \; n-1 < \alpha < n,$$

$${}^{c}\mathrm{D}_{t_0}^{\alpha} f(t) = f^{(n)}(t), \alpha = n, n \in \mathbb{N}.$$

2. DISCRETIZATION PROCESS

In [5, 6, 7, 8], a discretization process is introduced to discretize the fractional – order differential equations/systems. We noticed that when the fractional – order parameter $\alpha \rightarrow 1$, Euler's discretization method is obtained. The discretization method is applied to the logistic fractional-order differential equation, Riccati's fractional order differential equationand Chua's system [8]. Here, we are interested in applying the discretization method to Lorenz system of differential equations which is capable of generating chaotic behavior. In the early 1960s, Lorenz discovered the chaotic behavior of a simplified 3 – dimensional system. This system is well known and has been studied widely [2]. Let $\alpha \in (0,1)$ and consider the differential equation of fractional order

$$D^{\alpha}x(t) = f\big(x(t)\big), t > 0.$$
$$x(0) = x_0 \, , t \leq 0. \qquad (1)$$

The corresponding equation with a piecewise constant argument is

$$D^{\alpha}x(t) = f\left(x\left(r\left[\frac{t}{r}\right]\right)\right), t > 0.$$
$$x(0) = x_0, t \leq 0. \qquad (2)$$

Let $t \in [0, r)$, then $\frac{t}{r} \in [0,1)$.We get$D^{\alpha}x(t) = f(x_0), t \in [0,1)$.

Thus $x_1(t)x_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} f(x_0)$.

Let $t \in [r, 2r)$, then $\frac{t}{r} \in [1,2)$. We get $D^{\alpha}\, x(t) = f\big(x_1(t)\big), t \in [r, 2r)$.

Thus $x_2(t) = x_1(r) + \frac{(t-r)^{\alpha}}{\Gamma(1+\alpha)} f\big(x_1(r)\big)$.

Let $t \in [2r, 3r), then\ \frac{t}{r} \in [2,3)$. So we get $D^{\alpha}\, x(t) = f\big(x_2(2r)\big), t \in [2r, 3r)$

Thus $x_2(t) = x_2(2r) + \frac{(t-2r)^{\alpha}}{\Gamma(1+\alpha)} f\big(x_2(2r)\big)$.

Repeating the process, we get when $t \in [nr, (n+1)r)$, then $\frac{t}{r} \in [n, n+1)$.
So we get
$D^{\alpha}\, x(t) = f\big(x_n(nr)\big), t \in [nr, (n+1)r)$
Thus

$$x_{(n+1)}(t) = x_n(nr) + \frac{(t-nr)^{\alpha}}{\Gamma(1-\alpha)} f\big(x_n(nr)\big).$$

3. FRACTIONAL ORDER LORENZ SYSTEM AND DISCRETIZATION

The Lorenz equation is a model of thermally induced fluid convection in the atmosphere and published [9,10] by E.N Lorenz (an atmospheric scientist) of M.I.T. in 1963. In Lorenz's mathematical model of convection, three state variables are used *(x, y, z)*. They are not spatial variables but are more abstract. The variable *x* is proportional to the amplitude of the fluid velocity circulation in the fluid ring, positive representing clockwise and negative representing counterclockwise motion. The variable *y* is temperature difference between up and down fluids and *z* is the distortion from linearity of the vertical temperature profile. The model is continuous in time, but a modification of the continuous equation toa discrete quadratic recurrence equation known as the Lorenz map is also widely used.

$$x' = -\sigma x + \sigma y : y' = \gamma x - xz : \ z' = xy - \beta z$$

The Lorenz system includes three equations and three parameters with the following properties.

(i) Nonlinearity – the two nonlinearities are xy and xz.

(ii) Symmetry – Equations are invariant under $(x\,; y) \rightarrow (-x;\ -y)$. Hence if $\big(x(t); y(t); z(t)\big)$ is a solution, so is $\big(-x(t); -y(t); -z(t)\big)$;

(iii) Volume contraction – The Lorenz system is dissipative i.e. volumes in phase – space contract under the flow. The dimensionless parameters: *a* = Prandtl number is taken to be 10, *b* is related to the horizontal wave number of the convective motions and to be $\frac{8}{3}$. The remaining parameter γ is called the Rayleigh number. It is proportional to the difference in temperature from the warm base of a convection cell to the cooler top.

(iv)

Here we are concerned with the fractional order Lorenz system given by

$$\begin{aligned} D^{\alpha}x(t) &= -ax(t)+ay(t); \\ D^{\alpha}y(t) &= y(t)+\gamma x(t)-x(t)z(t); \\ D^{\alpha}z(t) &= (1-b)z(t)+x(t)y(t) \end{aligned} \tag{3}$$

where α is the fractional order. Now we are interested in discretizing fractional order Lorenz system given in the form

$$\begin{aligned} D^{\alpha}x(t) &= -ax\left(r\left[\frac{t}{r}\right]\right)+ay\left(r\left[\frac{t}{r}\right]\right) \\ D^{\alpha}y(t) &= y\left(r\left[\frac{t}{r}\right]\right)+\gamma x\left(r\left[\frac{t}{r}\right]\right)-x\left(r\left[\frac{t}{r}\right]\right)z\left(r\left[\frac{t}{r}\right]\right) \\ D^{\alpha}z(t) &= (1-b)z\left(r\left[\frac{t}{r}\right]\right)+x\left(r\left[\frac{t}{r}\right]\right)y\left(r\left[\frac{t}{r}\right]\right) \end{aligned} \tag{4}$$

with initial condition $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$. The proposed discretization method is explained in the following steps.

(1.) Let $t \in [0,r)$, then $\frac{t}{r} \in [0,1)$. So we get

$D^{\alpha}x(t) = -ax_0 + ay_0$; $D^{\alpha}y(t) = y_0 + \gamma x_0 - x_0 z_0$; $D^{\alpha}z(t) = (1-b)z_0 + x_0 y_0$

and the solution of (4) is given by

$$x_1(t) = x_0 + I^{\alpha} - ax_0 + ay_0 = x_0 + (-ax_0 + ay_o)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

$$y_1(t) = y_0 + I^{\alpha} y_0 + \gamma x_0 - x_0 z_0 = y_0 + (y_0 + \gamma x_0 - x_0 z_0)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

$$z_1(t) = z_0 + I^{\alpha}(1-b)z_0 + x_0 y_0 = z_0 + ((1-b)z_0 + x_0 y_o)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

(2.) Let $t \in [r,2r)$, then $\frac{t}{r} \in [1,2)$. We obtain

$D^{\alpha}x(t) = -ax_1 + ay_1$, $D^{\alpha}y(t) = y_1 + \gamma x_1 - x_1 z_1$, $D^{\alpha}z(t) = (1-b)z_1 + x_1 y_1$

and the solution of (4) is

$$x_2(t) = x_1 + I^{\alpha} - ax_1 + ay_1 = x_1 + (-ax_1 + ay_1)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

$$y_2(t) = y_1 + I^{\alpha} y_1 + \gamma x_1 - x_1 z_1 = y_1 + (y_1 + \gamma x_1 - x_1 z_1)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

$$z_2(t) = z_1 + I^{\alpha}(1-b)z_1 + x_1 y_1 = z_1 + ((1-b)z_1 + x_1 y_1)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

Repeating the process, we deduce the solution of (4) as

$$x_{n+1}(t) = x_n(nr) + \frac{(t-nr)^{\alpha}}{\Gamma(1+\alpha)}\left[-ax_n(nr) + ay_n(nr)\right]$$

$$y_{n+1}(t) = y_n(nr) + \frac{(t-nr)^{\alpha}}{\Gamma(1+\alpha)}[y_n(nr) + \gamma x_n(nr) - x_n(nr)z_n(nr)]$$

$$z_{n+1}(t) = z_n(nr) + \frac{(t-nr)^{\alpha}}{\Gamma(1+\alpha)}[(1-b)z_n(nr) + x_n(nr)y_n(nr)]$$

Let $t \to (n+1)r$, we obtain the following discretization

$$x_{n+1}((n+1)r) = x_n(nr) + \frac{(r)^{\alpha}}{\Gamma(1+\alpha)}\left[-ax_n(nr) + ay_n(nr)\right]$$

$$y_{n+1}((n+1)r) = y_n(nr) + \frac{(r)^{\alpha}}{\Gamma(1+\alpha)}[y_n(nr) + \gamma x_n(nr) - x_n(nr)z_n(nr)]$$

$$z_{n+1}((n+1)r) = z_n(nr) + \frac{(r)^{\alpha}}{\Gamma(1+\alpha)}[(1-b)z_n(nr) + x_n(nr)y_n(nr)]$$

which can be expressed as

$$\begin{aligned} x_{n+1} &= x_n + \frac{r^{\alpha}}{\Gamma(1+\alpha)}[-ax_n + ay_n] \\ y_{n+1} &= y_n + \frac{r^{\alpha}}{\Gamma(1+\alpha)}[y_n + \gamma x_n - x_n z_n] \\ z_{n+1} &= z_n + \frac{(r)^{\alpha}}{\Gamma(1+\alpha)}[(1-b)z_n + x_n y_n] \end{aligned} \tag{5}$$

4. FIXED POINTS, STABILITY AND NUMERICAL SIMULATIONS

Now we analyze the stability of the fixed points of the system (5) which has the following three fixed points.

- $F_0 = (0,0,0)$, Trivial points.
- $F_1 = (\sqrt{(b-1)(\gamma+1)}, \sqrt{(b-1)(\gamma+1)}, \gamma+1)$.
- $F_2 = (-\sqrt{(b-1)(\gamma+1)}, -\sqrt{(b-1)(\gamma+1)}, \gamma+1)$.

By considering a Jacobian matrix for interior fixed point and calculating their eigen values, we can investigate the stability of the interior fixed point based on the roots of the system characteristic equation [4]. Linearization of the system (5) about F_0 yield the characteristic equation:

$$\begin{aligned} P(\lambda) = \lambda^3 + \lambda^2[s(a+b-1)-3] + \lambda\big[3 + 2s(2-a-b) + s^2[(1-b)(1-a) - a(\gamma+1)]\big] \\ + [s(a+b-2) + s^2[b(1-a) + a(2+\gamma) - 1] + s^3 a(\gamma+1)(1-b) - 1] = 0 \end{aligned}$$

where $s = \frac{r^{\alpha}}{\Gamma(1+\alpha)}$. Let

$a_1 = s(a+b-1) - 3$
$a_2 = 3 + 2s(2-a-b) + s^2[(1-b)(1-a) - a(\gamma+1)]$
$a_3 = s(a+b-2) + s^2[b(1-a) + a(2+\gamma) - 1] + s^3 a(\gamma+1)(1-b) - 1$

From the Jury test, if $P(1) > 0$, $P(-1) < 0$, and $a_3 < 0, |b_3| > b_1, c_3 > |c_2|$ where

$b_3 = 1 - a_3^2, b_2 = a_1 - a_3 a_2, b_1 = a_2 - a_3 a_1, c_3 = b_3^2 - b_1^2$ and $c_2 = b_3 b_2 - b_1 b_2$, then the rootsof $P(\lambda)$ satisfy $|\lambda_j| < 1$ and thus F_0 is asymptotically stable. Suppose $P(1) < 0$ then $F_0$is unstable.Linearizing system (5) about F_1 or F_2 yields the characteristic equation:

$$P(\lambda) = \lambda^3 + \lambda^2[s(a+b-1)-3] + \lambda[3+2s(2-a-b)+s^2(b-1)(a+\gamma)] \\ + [s(a+b-2)+s^2(b-1)(1-a)+s^3 a(\gamma+1)(b-1)-1] = 0$$

where $s = \frac{r^\alpha}{\Gamma(1+\alpha)}$. Let

$a_1 = s(a+b-1)-3$

$a_2 = 3+2s(2-a-b)+s^2(b-1)(a+\gamma)$

$a_3 = s(a+b-2)+s^2(b-1)(1-a)+s^3 a(\gamma+1)(b-1)-1$

From the Jury test, if $P(1) > 0$, $P(-1) < 0$, and $a_3 < 0, |b_3| > b_1, c_3 > |c_2|$ where

$b_3 = 1 - a_3^2, b_2 = a_1 - a_3 a_2, b_1 = a_2 - a_3 a_1, c_3 = b_3^2 - b_1^2$ and $c_2 = b_3 b_2 - b_1 b_2$, then the rootsof $P(\lambda)$ satisfy $|\lambda_j| < 1$ and thus F_1 or F_2 is asymptotically stable. Suppose $P(1) < 0$ then F_1 or F_2 is unstable.

Numerical simulations are useful in understanding the dynamical behavior of the system. Numerical study of fractional order discrete dynamical systems provides an insight in to the dynamical characteristics. In this section, we present the time plots for $x(t), y(t), z(t)$, phase portraits and bifurcation diagrams for the system (5). Dynamic behavior of the system (5) about the interior fixed points under different sets of parameter values are presented below.

Example 1. Let us consider the parameters with values $\gamma = 0.02$; $a = 1.95$; $b = 1.98$, the initial conditions are $x = 0.5$; $y = 0.4$; $z = 0.2$ and the fractional derivative order $\alpha = 0.99$. Applying Jury test we get $P(1) = 0.0212 > 0, P(-1) = -1.9788 < 0$ and $a_3 = -0.9602 < 1$, thus $F_1$is asymptotically stable see Fig – 1.

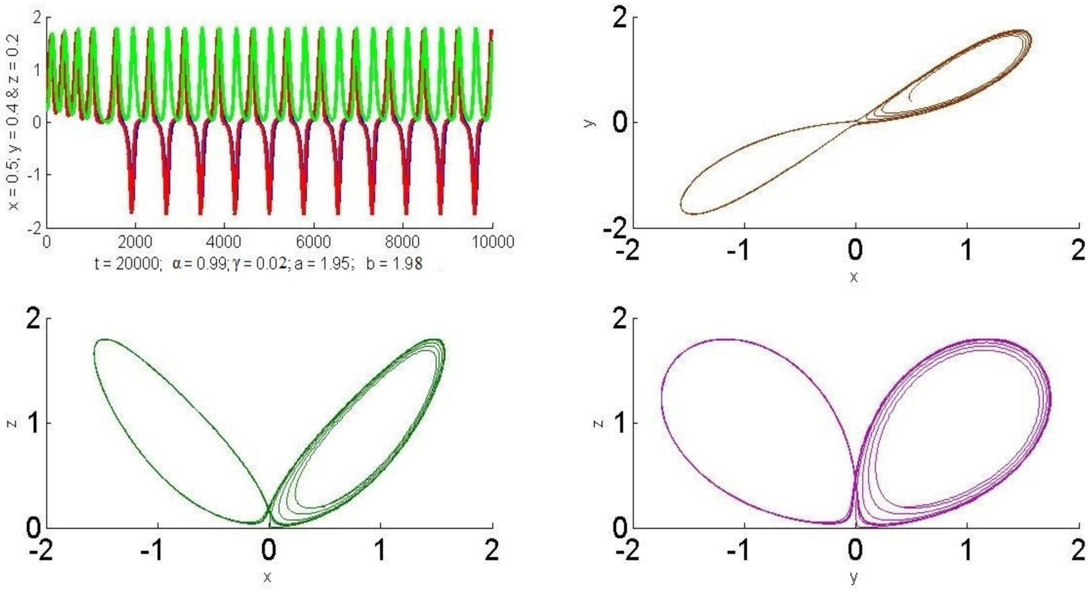

FIGURE 1. Time series and Phase diagram of fixed point F_1 with Stability

Bifurcation diagrams provide information about abrupt changes in the qualitative behavior in the dynamics of the system. The parameter values at which these changes occur are called bifurcation points. They provide information about the dependence of the dynamics on a certain parameter. If the qualitative change occurs in a neighborhood of an equilibrium point or periodic solution, it is called a local bifurcation. Any other qualitative change that occurs is considered as a global bifurcation.

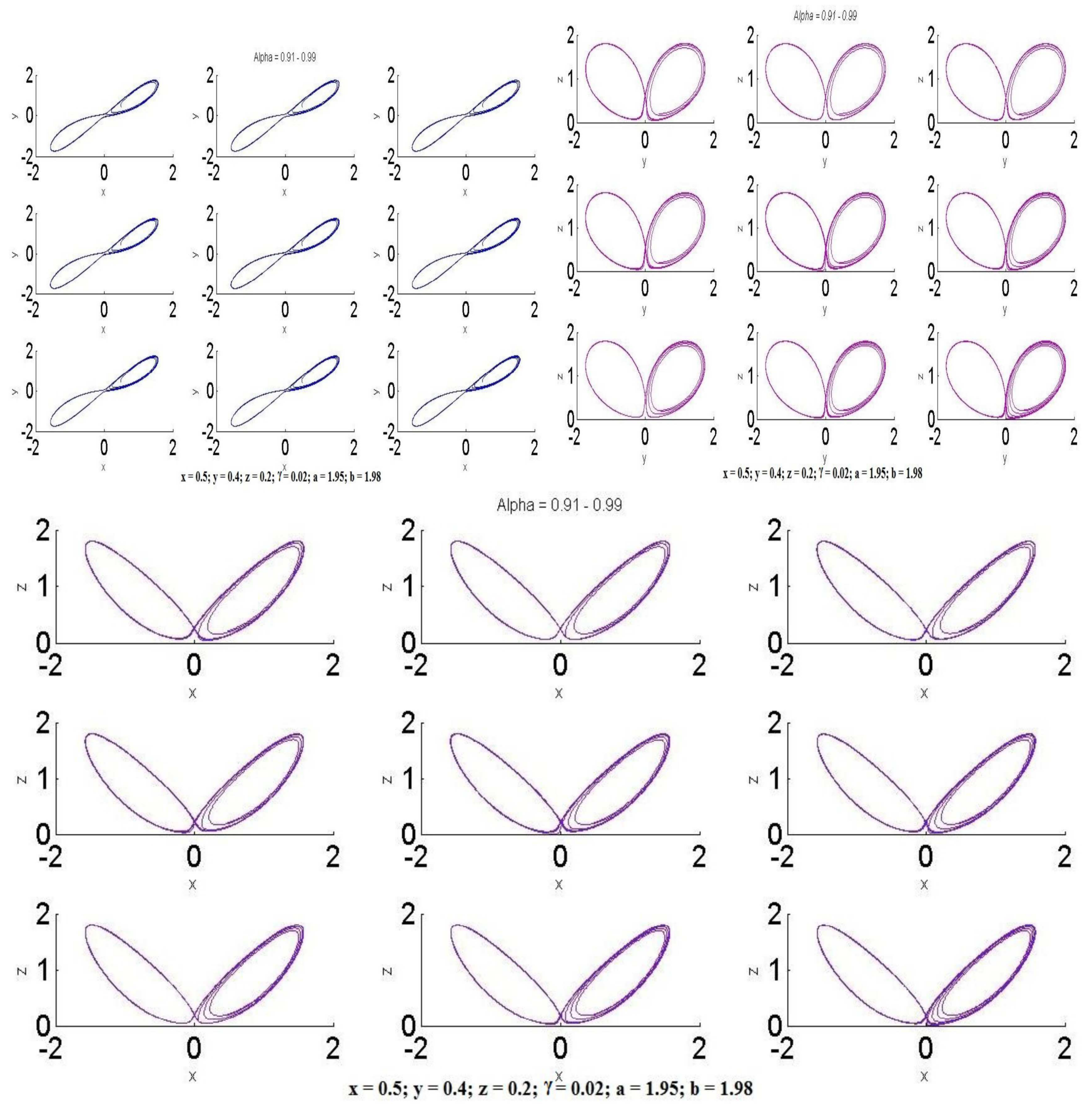

Figure 2. Different Phase diagrams of Fixed point F_1 with various values of α

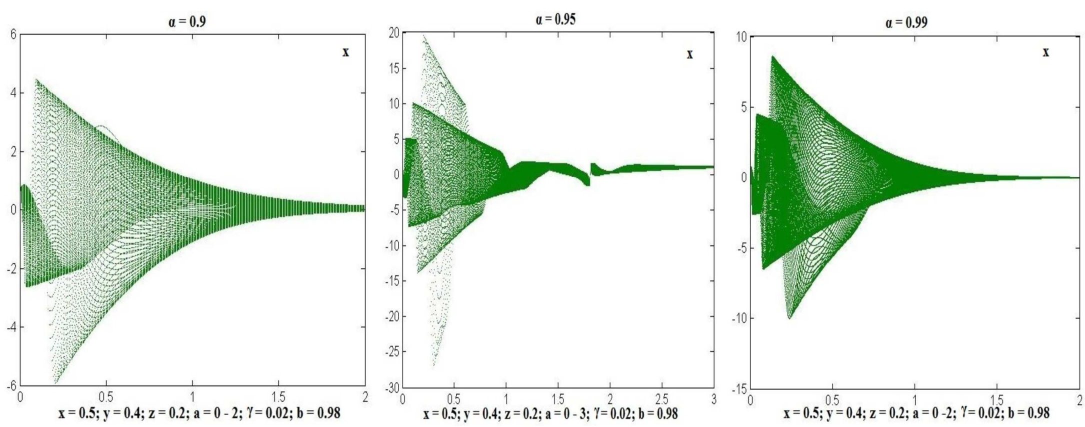

Figure 3. Bifurcation diagram for X with different values of α and the numerical values of a = 0 – 2, γ = 0.02, b = 0.98

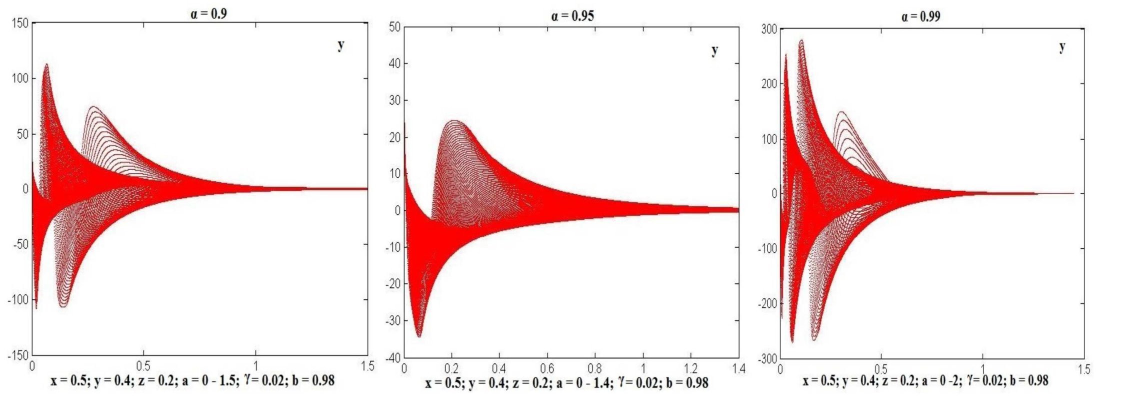

Figure 4. Bifurcation diagram for *Y* with different values of α and the numerical values of a = 0 – 2, γ = 0.02, b = 0.98

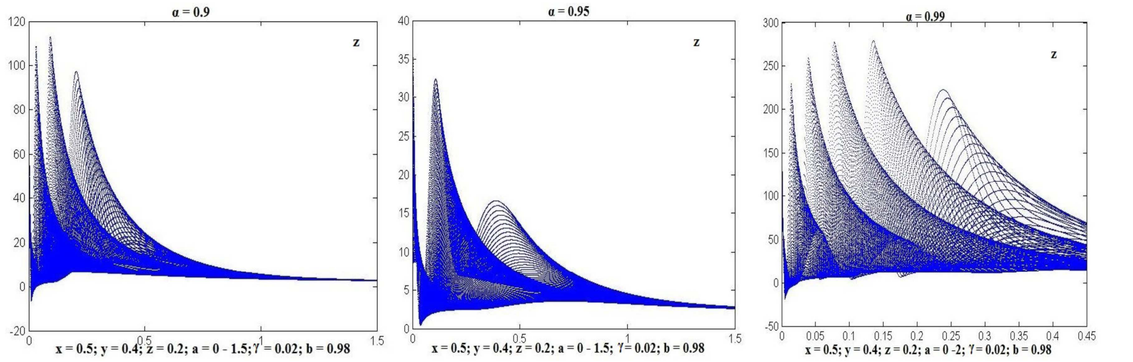

Figure 5. Bifurcation diagram for *Z* with different values of α and the numerical values of a = 0 – 2, γ = 0.02, b = 0.98

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