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Dirichlet's problem on cracked polygonal domains

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ABSTRACT

This paper deals with solving Poisson's equation with Dirichlet's boundary conditions on cracked domains obtained by means of translations, symmetries and rotations of basic equilateral triangle. The method of large finite elements used gives satisfactory results. Numerical values obtained are highly accurate for both stress function u and its first derivatives except at the cracks end where significant variations are observed.

Keywords: Cracks, large elements, singularities.

INTRODUCTION

Poisson's equation is used in many areas in physics. For example, the study of the deformation of a horizontal elastic membrane under distributed load, that of the low torsion of the bar or the flow in a pipe [1-2]. It is also used in many other contexts dissemination of pollutants, heat transfer [3-4] electromagnetism [5] universal gravitation, speed potential, vorticity [6] etc. Solving the Poisson's equation with Dirichlet's boundary conditions on a domain with cracks is particularly difficult. The main difficulty stems from singularities located at their ends. Indeed, at these points σ_i, series that correspond to the solution of the homogeneous equation associated with Poisson's equation are: $\sum_{k=1}^{\infty} a_{ik} r_i^{\frac{k}{2}} \sin k\frac{\theta_i}{2}$ and the first terms is proportional to $r_i^{\frac{1}{2}}$, with derivatives going toward the infinite by the end of the cracks [7]. The usual methods of finite elements or finite differences give unsatisfactory results if used in their standard form. These methods as demonstrated by various authors [8 -13] can be significantly improved if they take the analytical form of the solution near singularities into account. We consider the case of two cracked polygons obtained from a basic equilateral triangle where the method of large singular finite elements is used to solving Poisson's equation. The rationale of the method and its convergence properties are demonstrated in [14].

MATERIALS AND METHODS

Let's consider a basic uncracked equilateral triangle, whose side measuring 2, under mixed boundary conditions. By simple symmetry of the basic equilateral triangle, we get a cracked diamond in figure 1 and by translations, rotations and symmetries, a hexagon (figures 2 and 3) with cracks from the center or from its vertices.

In dimensionless form, the mathematical model of the low torsion of a thin bar is reduced to solving a particular case of the Poisson's equation (1) associated with Dirichlet-like (2) homogenous boundary conditions for each cracked domain.

$$\Delta u(x,y) = -1 \quad (x,y) \in \Omega \tag{1}$$

$$u(x,y) = 0 \quad (x,y) \in \partial\Omega \tag{2}$$

We keep here to the case where Ω is a cracked domain whose boundary $\partial\Omega$ is made of a series of straight line segments, implying that the problem is singular at the polygon vertices. Indeed, at each vertex of the polygon, the laplacian of the function u is zero while in the immediate neighboring of the vertex, it must be -1. The function u is a potential of constraints from which non-zero components (3) and (4) of the stress tensor at any point of the bar can be deduced by derivation. $\partial\Omega$ is the domain boundary.

$$\tau_{xz} = 2G\alpha \frac{\partial u}{\partial y} \tag{3}$$

$$\tau_{yz} = -2G\alpha \frac{\partial u}{\partial x} \tag{4}$$

In expressions (1) to (4), x and y are cartesian coordinates of a point in Ω field of study and z the axis which together with x and y make a direct orthogonal reference mark. G is the sliding module and α the unit torsion angle.

Solving Poisson's equation on these cracked domains, with Dirichlet's homogenous boundary conditions is the same as the Poisson's equation on basic equilateral triangle for which boundary conditions are mixed (figures 1, 2 and 3).

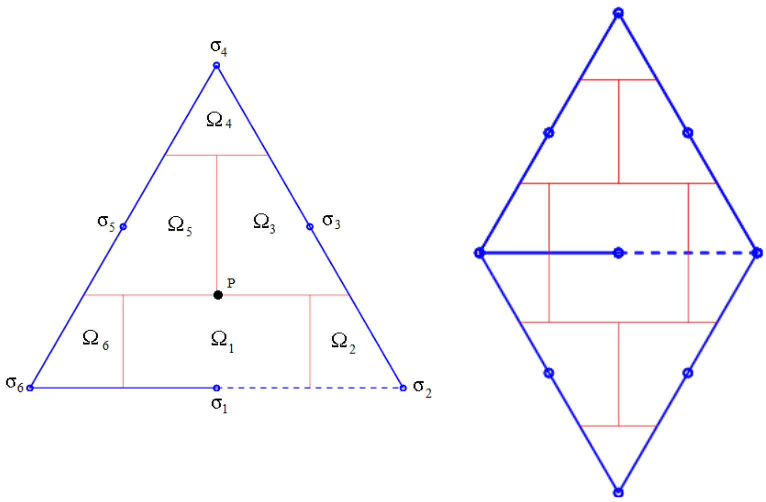

Figure 1 – Cracked diamond and its basic triangle whose side measures 2.

Implementing the method of large singular finite elements includes three steps [15]

Step 1: Splitting domains into sub-domains.
The first step of the method splits the basic equilateral triangle into six sub-domains Ω_i : three sub-domains with 60° opening angle, while the opening of the other three is 180 ° (figures 1, 2).

Step 2: Solving auxiliary problems.
The second step consists in solving auxiliary problems. The number of auxiliary problems is equal to that of the sub-domains Ω_i . Each sub-domain Ω_i $i = 1,\ldots,6$ is associated with an origin σ_i to the singularity, an angle α_i which is the opening angle in σ_i and in a local system of polar coordinates (r_i, θ_i) .

We now need to solve the six auxiliary problems associated with the basic triangle

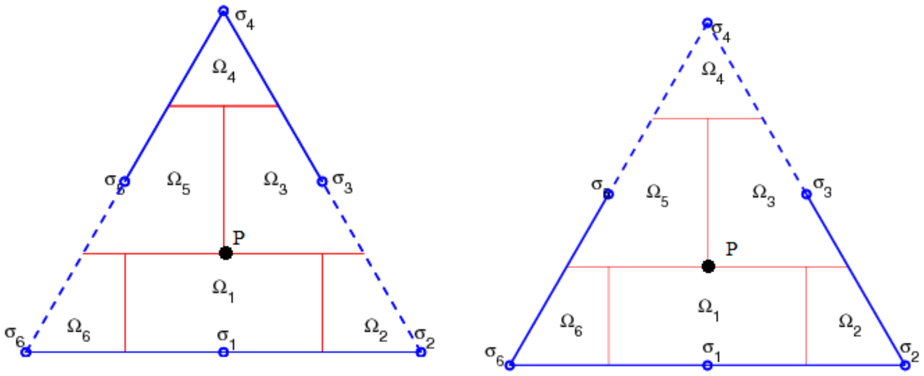

Figure 2 – Uncracked equilateral triangular domains with Neumann Dirichlet's boundary conditions for generating cracked hexagons. Solid lines correspond to Dirichlet's boundary conditions while dots correspond to Neumann's conditions. The length of the sides of the triangles is 2.

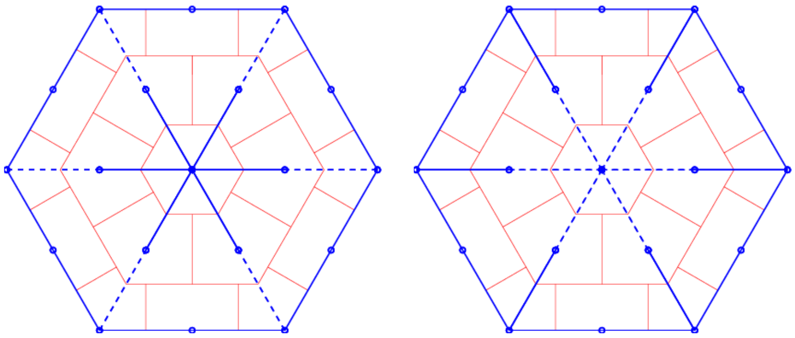

Figure 3 – Cracked hexagons produced by uncracked equilateral triangles. The length of the hexagons sides is 2 and cracks are of length unit.

a) Case of diamond (Figure 1).

First Auxiliary Problem

$$\Delta u_1(r_1,\theta_1) = -1 \quad (r_1,\theta_1) \in \Omega_1 \tag{5-a}$$

$$\frac{\partial u_1}{\partial n}(r_1,0) = 0 \tag{5-b}$$

$$u_1(r_1,\pi) = 0 \tag{5-c}$$

Second Auxiliary Problem

$$\Delta u_2(r_2,\theta_2) = -1 \quad (r_2,\theta_2) \in \Omega_2 \tag{6-a}$$

$$u_2(r_2,0) = 0 \tag{6-b}$$

$$\frac{\partial u_2}{\partial n}(r_2,\pi/3) = 0 \tag{6-c}$$

Third Auxiliary Problem

$$\Delta u_3(r_3,\theta_3) = -1 \quad (r_3,\theta_3) \in \Omega_3 \tag{7-a}$$

$$u_3(r_3,0) = 0 \tag{7-b}$$

$$u_3(r_1,\pi) = 0 \tag{7-c}$$

Fourth Auxiliary Problem

$$\Delta u_4(r_4,\theta_4) = -1 \quad (r_4,\theta_4) \in \Omega_4 \qquad (8\text{-a})$$

$$u_4(r_4,0) = 0 \qquad (8\text{-b})$$

$$u_4(r_4,\pi/3) = 0 \qquad (8\text{-c})$$

Fifth Auxiliary Problem

$$\Delta u_5(r_5,\theta_5) = 0 \quad (r_5,\theta_5) \in \Omega_5 \qquad (9\text{-a})$$

$$u_5(r_5,0) = 0 \qquad (9\text{-b})$$

$$u_5(r_5,\pi) = 0 \qquad (9\text{-c})$$

Sixth Auxiliary Problem

$$\Delta u_6(r_6,\theta_6) = -1 \quad (r_6,\theta_6) \in \Omega_6 \qquad (10\text{-a})$$

$$u_6(r_6,0) = 0 \qquad (10\text{-b})$$

$$u_6(r_6,\pi/3) = 0 \qquad (10\text{-c})$$

b) Case of hexagon (Figure 2).

For the triangle on the Left

- **First Auxiliary Problem**

$$\Delta u_1(r_1,\theta_1) = -1 \quad (r_1,\theta_1) \in \Omega_1 \qquad (11\text{-a})$$

$$u_1(r_1,0 = 0 \qquad (11\text{-b})$$

$$u_1(r_1,\pi) = 0 \qquad (11\text{-c})$$

- **Second Auxiliary Problem**

$$\Delta u_2(r_2,\theta_2) = -1 \quad (r_2,\theta_2) \in \Omega_2 \qquad (12\text{-a})$$

$$\frac{\partial u_2}{\partial n}(r_2,0) = 0 \qquad (12\text{-b})$$

$$u_2(r_2,\pi/3) = 0 \qquad (12\text{-c})$$

- **Third Auxiliary Problem**

$$\Delta u_3(r_3,\theta_3) = -1 \quad (r_3,\theta_3) \in \Omega_3 \qquad (13\text{-a})$$

$$u_3(r_3,0) = 0 \qquad (13\text{-b})$$

$$\frac{\partial u_3}{\partial n}(r_1,\pi) = 0 \qquad (13\text{-c})$$

- **Fourth Auxiliary Problem**

$$\Delta u_4(r_4,\theta_4) = -1 \quad (r_4,\theta_4) \in \Omega_4 \qquad (14\text{-a})$$

$$u_4(r_4,0) = 0 \qquad (14\text{-b})$$

$$u_4(r_4,\pi/3) = 0 \qquad (14\text{-c})$$

- **Fifth Auxiliary Problem**

$$\Delta u_5(r_5,\theta_5) = -1 \quad (r_5,\theta_5) \in \Omega_5 \qquad (15\text{-a})$$

$$\frac{\partial u_5}{\partial n}(r_5,0) = 0 \qquad (15\text{-b})$$

$$u_5(r_5,\pi) = 0 \qquad (15\text{-c})$$

- **Sixth Auxiliary Problem**

$$\Delta u_6(r_6,\theta_6) = -1 \quad (r_6,\theta_6) \in \Omega_6 \qquad (16\text{-a})$$

$$u_6(r_6,0) = 0 \qquad (16\text{-b})$$

$$\frac{\partial u_6}{\partial n}(r_6,\pi) = 0 \qquad (16\text{-c})$$

For the Triangle on the Right

- **First Auxiliary Problem**

$$\Delta u_1(r_1,\theta_1) = -1 \quad (r_1,\theta_1) \in \Omega_1 \qquad (17\text{-a})$$

$$u_1(r_1,0 = 0 \qquad (17\text{-b})$$

$$u_1(r_1,\pi) = 0 \qquad (17\text{-c})$$

- **Second Auxiliary Problem**

$$\Delta u_2(r_2,\theta_2) = -1 \quad (r_2,\theta_2) \in \Omega_2 \qquad (18\text{-a})$$

$$u_2(r_2,0) = 0 \qquad (18\text{-b})$$

$$u_2(r_2,\pi/3) = 0 \qquad (18\text{-c})$$

- **Third Auxiliary Problem**

$$\Delta u_3(r_3,\theta_3) = -1 \quad (r_3,\theta_3) \in \Omega_3 \qquad (19\text{-a})$$

$$\frac{\partial u_3}{\partial n}(r_3,0) = 0 \qquad (19\text{-b})$$

$$u_3(r_1,\pi) = 0 \qquad (19\text{-c})$$

- **Fourth Auxiliary Problem**

$$\Delta u_4(r_4,\theta_4) = -1 \quad (r_4,\theta_4) \in \Omega_4 \qquad (20\text{-a})$$

$$\frac{\partial u_4}{\partial n}(r_4,0) = 0 \qquad (20\text{-b})$$

$$\frac{\partial u_4}{\partial n}(r_4,\pi/3) = 0 \qquad (20\text{-c})$$

- **Fifth Auxiliary Problem**

$$\Delta u_5(r_5,\theta_5) = -1 \quad (r_5,\theta_5) \in \Omega_5 \qquad (21\text{-a})$$

$$u_5(r_5,0) = 0 \qquad (21\text{-b})$$

$$\frac{\partial u_5}{\partial n}(r_5,\pi) = 0 \qquad (21\text{-c})$$

- **Sixth Auxiliary Problem**

$$\Delta u_6(r_6,\theta_6) = -1 \quad (r_6,\theta_6) \in \Omega_6 \qquad (22\text{-a})$$

$$u_6(r_6,0) = 0 \qquad (22\text{-b})$$

$$u_6(r_6,\pi/3) = 0 \qquad (22\text{-c})$$

We check that auxiliary problems solutions for singularities σ_i with $i = 1, 2, ..., 6$ can be written taking boundary conditions into account for each sub-domain Ω_i from the splitting of basic equilateral triangles:

- With conditions on Dirichlet's homogenous limits

$$u_i(r_i,\theta_i) = \frac{r_i^2}{4}\left[\frac{\cos(2\theta_i - \alpha_i)}{\cos\alpha_i} - 1\right] + \sum_{n=1}^{\infty} a_{in} r_i^{\gamma_{in}} \sin(\gamma_{in}\theta_i) \qquad (23)$$

with $i = 3,4,5,6$ for diamond; $i = 1,4,$ for the triangle on the left ; $i = 1,2,6$ for the triangle on the right.

- With conditions on Dirichlet Neumann's mixed homogenous limits

$$u_j(r_j,\theta_j) = \frac{r_j^2}{4}\left[\frac{\cos 2(\theta_j - \alpha_j)}{\cos 2\alpha_j} - 1\right] + \sum_{n=1}^{\infty} a_{jn} r_j^{\gamma_{in}} \sin(\gamma_{jn}\theta_j) \qquad (24)$$

with $j = 2$ for diamond, $j = 3, 6$ for the triangle on the left and $j = 5$ for the triangle on the right.

- With conditions on Dirichlet Neumann's mixed homogenous limits

$$u_k(r_k,\theta_k) = \frac{r_{ki}^2}{4}\left[\frac{\cos 2\theta_k}{\cos 2\alpha_k} - 1\right] + \sum_{n=1}^{\infty} a_{kn} r_i^{\gamma_{in}} \sin(\gamma_{kn}\theta_k) \qquad (25)$$

with $k = 1$ for diamond, $k = 2,5$ for the triangle on the left and $k = 3$ for the triangle on the right ;

- With conditions on Neumann's homogenous limits

$$u_l(r_l,\theta_l) = -\frac{r_{li}^2}{4} + \sum_{n=1}^{\infty} a_{\ln} r_i^{\gamma_{in}} \sin(\gamma_{\ln}\theta_l) \qquad (26)$$

$l = 4$ for t he triangle on the right.

Provided that an appropriate value of the opening angle α_i is given to each singularity σ_i and that $\gamma_{in} = n\pi/\alpha_i\,; i = 1, 2, 3, ..., 6$ and $n = 1, 2, ..., \infty$. Opening angles are respectively π for rectangular sub-domains and π/3 for triangular domains.

Step 3: Connecting auxiliary solutions

The third step of the method consists in connecting these auxiliary solutions by imposing the continuity of the function and its normal derivative along the various sub-borders Γ_{kl} separating two adjacent sub-domains Ω_k and Ω_l. In practice, since it is not possible to solve an infinite system, we must limit the sums that appear in equations (23) to (26) to a finite number of terms. The number of terms used in the sums is chosen according to Decloux and Tolley's principle [14] which aims to represent the approximate solutions using functions whose degree is as uniform as possible. This is achieved by keeping more terms for the sub-domains with larger openings. The total number of parameters a_{kl} whose value can be freely chosen will be $(3\times 3 + 3\times 1)N = 12N$, (N being the number of coefficients a_{kl} used for triangular sub-domains whose opening angle is $\pi/3$). All approximate solutions will be of 3N degree for all sub-domains, keeping the number of terms proportional to the angle opening $\gamma_{in} = n\pi/\alpha_i$ where $\gamma_{in} = 3n\pi/\pi = 3n$ for triangular sub-domains and $\gamma_{in} = n\pi/\pi = n$ for rectangular sub-domains.

We connect the solutions of auxiliary problems in terms of continuous least squares; we must find coefficients a_{in} that allows minimizing the function.

$$I(a_{mn}) = \sum_{i<j} \int_{\Gamma_{ij}} \left[[u_i(a_{ik}) - u_j(a_{jl})]^2 + \left(\frac{\partial u_i(a_{ik})}{\partial n_i} + \frac{\partial u_j(a_{jl})}{\partial n_j} \right)^2 \right] ds \qquad (27)$$

The least squares method consists to minimize the previous integral with respect to unknown coefficients used in approximate solutions;

i.e. to write that $\dfrac{\partial I(a_{pq})}{\partial a_{kl}} = 0$ (28)

By minimizing the function (27) as compared to coefficients a_{in}, this leads to a linear algebraic system in constants a_{in}.

The accuracy of approximate solutions depends directly on the quality of the connection of auxiliary solutions. It is therefore natural to characterize its precision by measuring the imperfections of continuity conditions. This will be used to measure the overall error defined by (29).

$$\eta = \sum_{k<l} \frac{1}{S_{kl}} \int \left[(u_k - u_l)^2 + \left(\frac{\partial u_k}{\partial n_k} + \frac{\partial u_l}{\partial n_l} \right)^2 \right] ds_{kl} \qquad (29)$$

Where ds_{kl} is the arc length element of Γ_{kl} and S_{kl} its length.; n_k and n_l are the normals to the sub-border separating both adjacent sub-areas. If the overall error is null, the approximate solution got aligns with the exact solution.

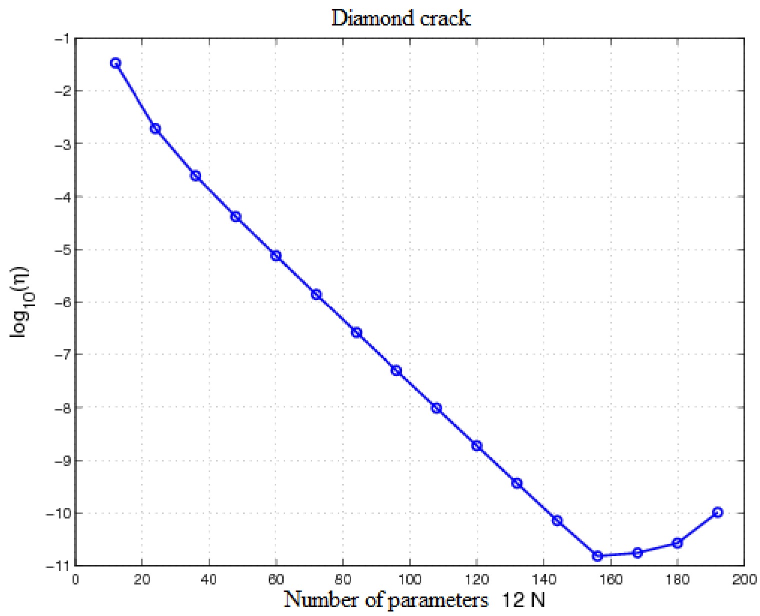

Figure 4- Evolution of the 10-base logarithm of the overall error according the total number of 12N coefficients a_{kl} kept.

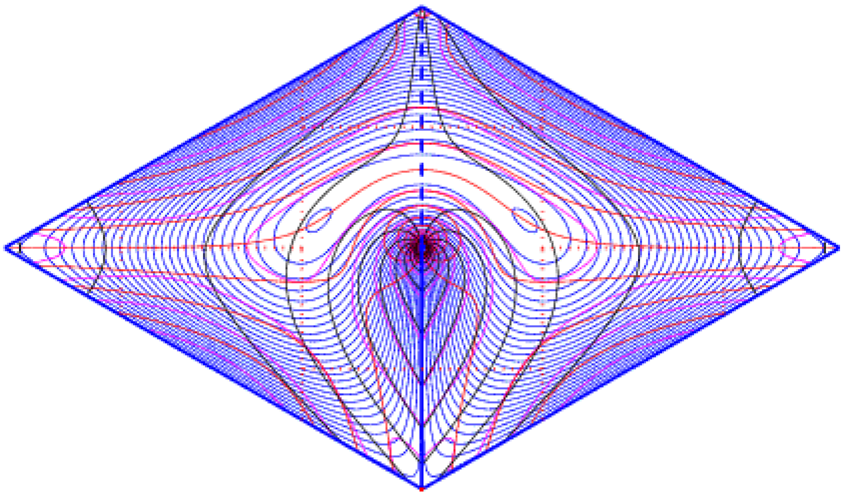

Figure 5: Cracked diamond: function u isolines in blue, its derivatives $\frac{\partial u}{\partial x}$ in red $\frac{\partial u}{\partial y}$ in black and $|\nabla u|$ in magenta.

SOLUTIONS AND DISCUSSION

For Diamond

Figure 4 shows that the convergence of the method is exponential. Numerical values obtained are highly accurate, both for function u and its derivatives. The values of u and of its derivatives calculated at P with coordinates (1, 0.5) are recorded in table 1. The value of u is calculated with 13 exact numbers while its derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are calculated with 11 exact numbers when the overall error is around $1.53\ 10^{-11}$. The overall error is calculated using 12N = 156, as shown in the graph in figure 4 where the 10-base logarithm of the overall error is presented as a function of 12N. We also show the function u isolines, its derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ and the length of the gradient on the figure 5. This shows a strong variation in its sizes near the crack tip.

12N	u	$\frac{\partial u}{\partial x}$	$\frac{\partial u}{\partial y}$
12	0.1259927250855	0.01806581186	0.009755004867
24	0.1302582668178	0.04032831256	−0.00431018859
36	0.1301993440539	0.03879061388	−0.00576708676
48	0.1301985893467	0.03855330956	−0.00586075354
60	0.1301988576386	0.03853742756	−0.00584953511
72	0.1301989118059	0.03853860789	−0.00584598366
84	0.1301989012605	0.03853875069	−0.00584571299
96	0.1301989018307	0.03853877272	−0.00584573768
108	0.1301989017921	0.03853877534	−0.00584574451
120	0.1301989018178	0.03853877542	−0.00584574302
132	0.1301989018214	0.03853877526	−0.00584574305
144	0.1301989018202	0.03853877527	−0.00584574308
156	0.1301989018201	0.03853877528	−0.00584574308
168	0.1301989018201	0.03853877528	−0.00584574308

Table 1 - Cracked Diamond - The value of the deflection and its derivatives at P with coordinates (1, 0.5) (see figure 1) based on the total number of coefficients kept in the series of auxiliary solutions.

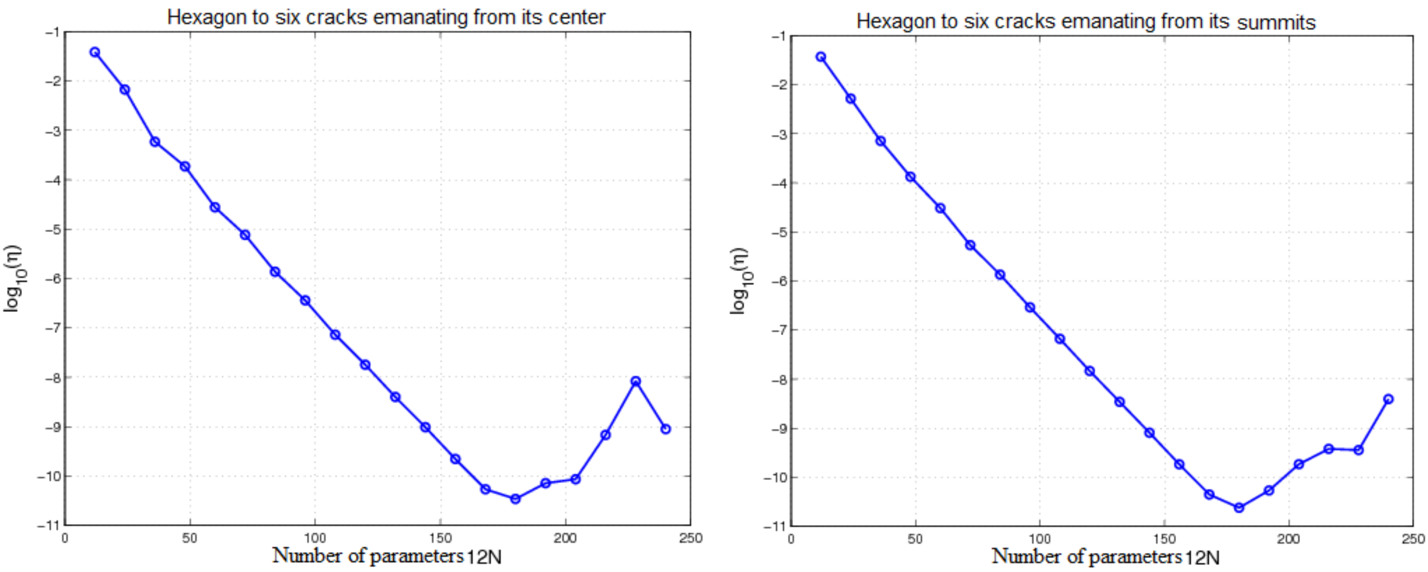

Figure 6- Cracked hexagons caused by uncracked equilateral triangles. Evolution of the overall error according to 12N.

Case of Hexagon

May cracks on hexagon start from the center or its summits, the convergence of the method of large singular finite elements is exponential as shows the graph in figure 6 where is represented the 10-base logarithm of the overall connection error according to N approximation level; 12N being the total number of coefficients a_{kl} kept in the series that characterize the solutions of the auxiliary problems. The lowest overall errors are got with $N = 15$ and account to $3.43\ 10^{-11}$ for the hexagon whose cracks start from the center and $2.39\ 10^{-11}$ for the other. This allows assuming that, in both cases, the deflection u (or the stress function) is known with at least 12 specific numbers, while its first partial derivatives are calculated with at least 10 exact numbers. Values of u and its first partial derivatives calculated at a point with coordinates (1, 0.5) with different values of N confirm this hypothesis. They are presented in tables 2 and 3.

The figure 7 presents the function u isovalues while figure 8 shows the perspective views and the length of the gradient vector of cracked hexagons.

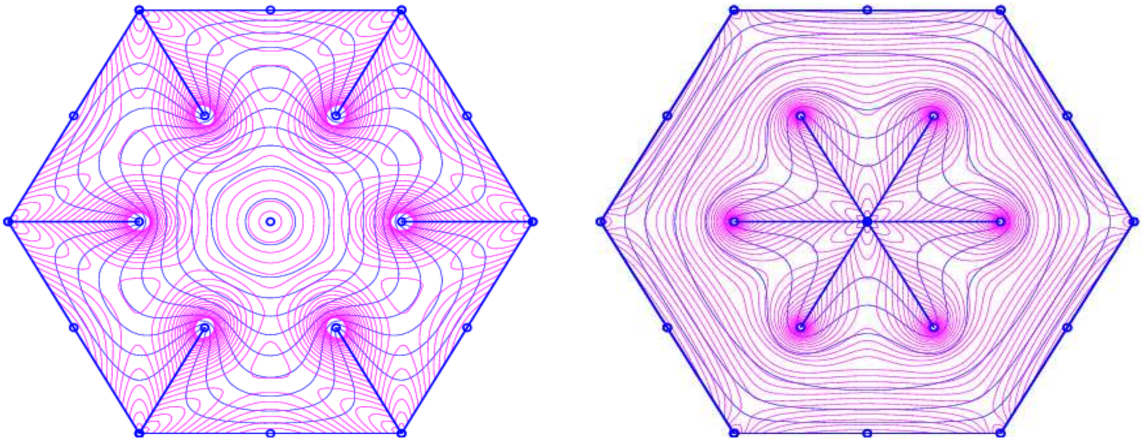

Figure 7: Cracked hexagon: isovalues of u in blue and $|\nabla u|$ in magenta.

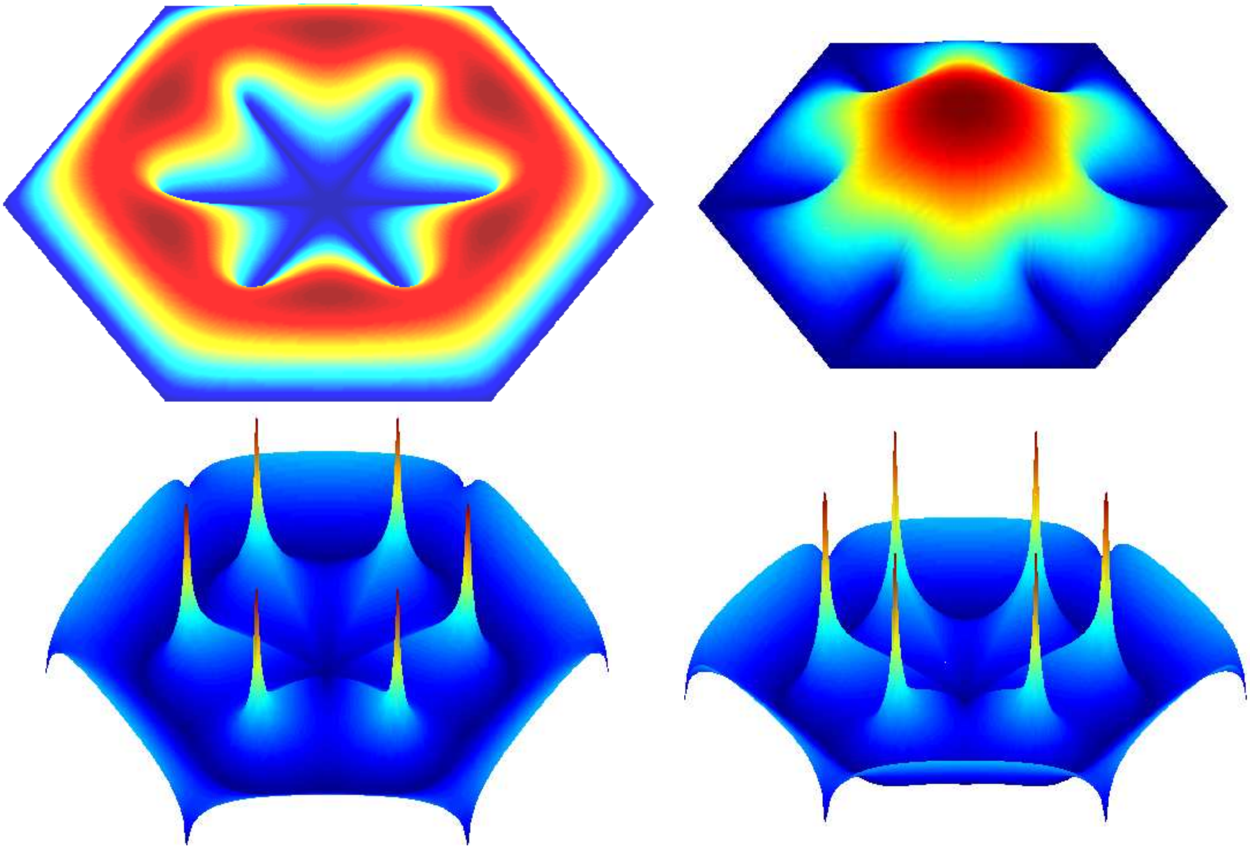

Figure 8: Cracked hexagons: perspective views of the function u and the module of its gradient

The values of the derivative of the function u according to x are not strictly zero at the intersection of the three sub-borders Γ_{13}, Γ_{15} and Γ_{35}. Their module varies from $2.4317\ 10^{-17}$ to $2.2490\ 10^{-12}$ for the hexagons with six cracks starting from summits; then from $7.3743\ 10^{-17}$ to 8.608310^{-13} for the other hexagon.

N	u	$\frac{\partial u}{\partial x}$	$\frac{\partial u}{\partial y}$
1	0.17006342770	-0.000000000	0.2087051751
2	0.17866526667	-0.000000000	0.2412513612
3	0.17897004264	0.0000000000	0.2406952475
4	0.17889636334	0.0000000000	0.2397542821
5	0.17889829991	-0.000000000	0.2397543559
6	0.17890088257	0.0000000000	0.2397911583
7	0.17890102103	-0.000000000	0.2397926476
8	0.17890095361	0.0000000000	0.2397911961
9	0.17890095020	0.0000000000	0.2397910708
10	0.17890095311	-0.000000000	0.2397911267
11	0.17890095361	0.0000000000	0.2397911348
12	0.17890095358	-0.000000000	0.2397911325
13	0.17890095358	0.0000000000	0.2397911321
14	0.17890095358	-0.000000000	0.2397911322
15	0.17890095358	0.0000000000	0.2397911322

Table 2: Hexagon with six cracks starting from its vertices. Values of the deflection and its derivatives at the intersection P with its sub-borders Γ_{13}, Γ_{15} and Γ_{35}.

N	u	$\frac{\partial u}{\partial x}$	$\frac{\partial u}{\partial y}$
1	0.136676426734	0.0000000000	0.0335210658
2	0.143258196906	0.0000000000	0.0492997390
3	0.144656388859	0.0000000000	0.0597282606
4	0.144641970953	0.0000000000	0.0597207715
5	0.144616888839	0.0000000000	0.0594374305
6	0.144617200413	0.0000000000	0.0594425040
7	0.144627734147	0.0000000000	0.0594517500
8	0.144617743641	-0.000000000	0.0594518717
9	0.144617727506	0.0000000000	0.0594514971
10	0.144617727114	-0.000000000	0.0594514987
11	0.144617727447	-0.000000000	0.0594515113
12	0.144617727445	0.0000000000	0.0594515118
13	0.144617727426	0.0000000000	0.0594515112
14	0.144617727426	-0.000000000	0.0595415112
15	0.144617727426	0.0000000000	0.0594515112

Table 3: Hexagon with six cracks starting from center. Values of the deflection and its derivatives at the intersection P with its borders Γ_{13}, Γ_{15} and Γ_{35}.

CONCLUSION

The study of cracked polygons obtained through translations, symmetries and rotations from a basic equilateral triangle using the method of large singular finite elements gives satisfactory results throughout the study area except at the end of the cracks where there are large variations of u and its first derivatives. This method takes the existence of the singularity into account by finding asymptotic solutions around them, which therefore allows getting, without additional formulation, derived values. The mode of convergence of the method is exponential. The lowest overall error is around 1.53 10^{-11} and obtained with N = 13 for the diamond. The lowest overall errors are obtained with N = 15 and stand for 3.43 10^{-11} for the hexagon with cracks starting from the center and 2.39 10^{-11} for the other. Numerical values obtained are extremely accurate for both the function u and its derivatives. For diamond, the constraint potential u is calculated with13 exact numbers while its derivatives are obtained with 11 exact numbers. In the case of hexagon, the constraint potential is calculated with at least 12 exact numbers, while its partial derivatives are calculated with at least 10 exact numbers.

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