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Computer Extended Series Solution for the Flows in a Nonparallel Channels

Vishwanath B. Awati [1], N. M. Bujurke [2] and N.N. Katagi[3]

[1] Department of Mathematics, Maharani's Science College for Women, Bangalore, INDIA
[2] Department of Mathematics, Karnatak University, Dharwad, INDIA
[3]Department of Mathematics, Manipal Institute of Technology, Manipal, INDIA

ABSTRACT

The analysis of the nature of flow structure in a slowly varying channel is presented. The solution procedure, proposed and implemented, is valid for any smooth geometry. The expansion of the stream function in terms of $\lambda = R\varepsilon$, R the Reynolds number and ε the slope (small) parameter is considered. The coefficients generated in the expansion are universal (valid for any smooth geometry). This is accomplished using novel semi numerical schemes based on combinatorial concepts, as well as Mathematica. The converging Pade' sums of the series, for sufficiently large λ, gives analytic continuation of the series solution of non-linear partial differential equation for a variety of slowly varying geometries and shed useful light on flow structure. Comparison of predicted values of shear stress based on numerical and experimental findings are given and in the present analysis these are valid for much larger values of λ.

Keywords: Analytic continuation; Channel flow; Laminar boundary layer flow; Computer-extended-series; Pade' approximants.

INTRODUCTION

Viscous fluid flows in nonparallel channels constitute one of the important sections in Fluid mechanics due to its relevance to a variety of engineering applications. The detailed analysis of such problems giving an understanding of velocity and shear distribution can be performed up to any degree of accuracy especially at small Reynolds number. The study at large and moderately large Reynolds number requires careful consideration of the mathematical equations describing such flows. A large class of flows can be considered by taking different geometries such as non-parallel channel walls, constricted / dilation channels, exponential channels, etc. Flows in non-parallel channels require solution of non-linear partial differential equations wherein the geometry of the channel is coupled with relevant equations. There are large number of theoretical studies notably by Fraenkel [1-2], Blottner [3], Eagles and Smith [4], Pedley [5], Daniels & Eagles [6] where serious attempts are made in understanding flow structure in non-parallel channels using approximate and numerical schemes. In channel flows at moderately large Reynolds number, normally the perturbation expansion of stream function with $\varepsilon = 1/R$ enables one to solve partial differential equations in each approximation where in the first approximation corresponds to the solution of Prandtl's boundary layer equations with suitable boundary conditions. Van Dyke [7-8] gives an analysis of various problems of slow variations in continuum mechanics. Patterson [9-10] made systematic experimental investigation of flows in exponentially diverging channel flows. In the present paper one shed the light on these studies using computer extended series analysis which in favorable cases yields useful information about flows such as separation, reattachment, etc. normally much beyond the regions / ranges of pure numerical studies. Recently, Badari Narayana et al. [11] investigated the the oscillatory flow of Jeffrey fluid in an elastic tube. Sreenadh et al [12] developed a mathematical model to study the steady flow of Casson fluid through an inclined tube of non-uniform cross section with multiple stenosis. Krishna Kumari et al [13] studied the peristaltic pumping of a Casson fluid in an inclined channel under the effect of magnetic field

We develop two new schemes for the solution of governing non-linear partial differential equations. The nature of terms appearing in lower order solutions of a perturbation expansion of stream function enables one to propose a universal expansion scheme for the generation of subsequent higher order terms in the expansion. This can also be accomplished both using MATHEMATICA or novel semi numerical schemes based on combinatorial concepts and the algebra associated with partitions of *n* (Gupta [14]). Once a large number of universal coefficients are generated, a variety of special techniques can be employed to sum the series for moderately large values of the Reynolds number by confirming to specific geometries. In these types of studies Bujurke et al [15-19] were successful in accomplishing such an analysis based on computer-extended series solutions resulting into their analytic continuation much beyond the region of convergence. Bujurke et al [18] have also studied an analysis of the flow structure in a channel of variable cross section. A large class of problems in chemical engineering and fluid mechanics, the channel of variable cross-section has symmetry with respect to the axis. In the present study symmetric channel flow (of variable cross-section) is considered.

This paper is outlined as follows. A comprehensive analysis pertaining to the usefulness of the computer extended series analysis in the study of non-parallel channel flows to predict possible separation and reattachment at moderately higher Reynolds numbers is given. In section 2 we present the relevant nonlinear partial differential equations and boundary conditions of the problem for general smooth geometries *H(X)*. In section 3 we find lower-order terms manually, which show a definite pattern for obtaining higher order terms using a computer. Two methods for the generation of computer-extended series solution for arbitrary but smooth geometries are explained in section 4. Later in section 5 we study specific geometries and generate a sufficiently large (universal coefficients valid for any smooth geometries (15 effective terms)) number of terms in the series expansion. In section 6 we present discussion and conclusions based on the derivation given in earlier sections and indicate directions for further investigations.

2. FORMULATION OF THE PROBLEM

The steady laminar flow at moderately high Reynolds number through a channel of slowly-varying shape is considered. Far upstream the channel walls are parallel. The fluid is viscous and incompressible and far upstream the velocity is described by Poiseuille flow. Let *x, y* be dimensionless rectangular Cartesian coordinates in the streamwise and transverse directions, respectively, and let the walls of the channel be given by $y = \pm H(X)$ as shown in Fig.1 (Eagles and Smith (1980)), where $X = x\varepsilon$ and $\varepsilon << 1$ is slope parameter and *2H* is the channel width. Also, ψ , *(u, v)* and *p* are the stream function, velocity vector and the fluid pressure, respectively. These are made dimensionless with respect to *M, M/a* and $\left(\rho M^2/a^2\right)$ respectively, where *2M, a* and ρ are the volumetric flow rate per unit width of the oncoming Poiseuille flow, the undistributed channel width, and the fluid density, respectively. The oncoming flow is (Eagles and Smith [4])

$$\psi \to 3y-4y^3, \quad u \to 3-12y^2, v=0, \quad \frac{\partial p}{\partial x} \to -\frac{24}{R}, \ as \ x \to -\infty \tag{1}$$

Let *R* be the Reynolds number defined as

$$R = M/\nu \tag{2}$$

where ν is the kinematic viscosity.The equations governing ψ , *u, v* and *p* are the Navier-Stokes equations, and the boundary conditions are the no-slip conditions at the walls and downstream conditions of boundedness as $x \to \infty$. Equations are non-dimensionalized so as to ensure $\psi = \pm 1$ on the upper and lower walls respectively. Let $\lambda = R\varepsilon$ then for $\lambda << 1, (\varepsilon \to 0,\ R \to \infty)$ steady flow may be taken for which Poiseuille flow forms the first approximation and it is *O*(1). Suppose λ to be chosen a constant and is independent of ε in the limit $\varepsilon \to 0$ and $R \to \infty$. The stream function $\psi = \psi(X, y)$ is the first approximation and satisfies nonlinear boundary layer equation, so that, the steady flow deviates nonlinearly from Poiseuille flow and which includes the separation and reattachment of the flow. Let ψ be a function of *X* and *y*. The asymptotic expansion of the solution is (Eagles and Smith [4])

$$\psi = \psi\left(X, y\right) + O\left(\varepsilon^2\right) \tag{3}$$

and the associated velocities and pressure are

$$u = U\left(X, y\right) + O\left(\varepsilon^2\right) \tag{4}$$

$$v = \varepsilon V\left(X, y\right) + O\left(\varepsilon^3\right) \tag{5}$$

$$p = P\left(X\right) + O\left(\varepsilon^2\right) \tag{6}$$

The lowest order streamwise momentum equation reduces to boundary layer equation

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = -\frac{dp}{dx} + \frac{\psi_{yyy}}{\lambda} \tag{7}$$

The relevant boundary conditions are

$$\psi\left(-\infty, y\right) = 3y - 4y^3, \quad \psi_y\left(-\infty, y\right) = 3 - 12y^2, \quad \frac{dp}{dx}\left(-\infty\right) = -24\lambda^{-1} \tag{8}$$

$$\left.\begin{array}{ll} \psi = -1, \quad \psi_y = 0 \quad at \quad y = -H(X) \\ \psi = 1, \quad \psi_y = 0 \quad at \quad y = H(X) \end{array}\right\} \tag{9}$$

Also, symmetric conditions on the channel centreline $y = 0$ are $\psi = 0 = \psi_{yy}$. Upon introducing,

$$\eta = {y}/{H\left(X\right)} \quad and \quad \psi = F\left(X, \eta\right) \tag{10}$$

the governing equations and boundary conditions are transformed to (Eagles and Smith [4])

$$F_{\eta\eta\eta} + \lambda\left[H\left(F_{\eta\eta}F_x - F_\eta F_{\eta x}\right) + \frac{dH}{dX}F_\eta^{\ 2}\right] - \lambda H^3 \frac{dp}{dx} = 0 \tag{11}$$

$$F\left(X, \pm 1\right) = \pm 1, \qquad F_\eta\left(X, \pm 1\right) = 0 \tag{12}$$

$$\left.\begin{array}{l} F\left(-\infty, \eta\right) = \dfrac{3}{2}\eta - \dfrac{1}{2}\eta^3 \\ \dfrac{dp}{dx}\left(-\infty\right) = -24\lambda^{-1} \end{array}\right\} \tag{13}$$

Differentiating equation (11) with respect to η once, we obtain

$$F_{\eta\eta\eta\eta} + 2\lambda\frac{dH}{dX}F_\eta F_{\eta\eta} + \lambda H\left(X\right)\left(F_{\eta\eta\eta}F_x - F_\eta F_{\eta\eta x}\right) = 0\,. \tag{14}$$

3. SOLUTION OF THE PROBLEM

For small λ, a solution of (14) is considered in the form

$$F = F_0(\eta) + \sum_{n=1}^{\infty} \lambda^n F_n(X, \eta) \tag{15}$$

Substituting (15) in the equation (14) and equating like powers of η on both sides, we get equations of various orders whose solutions are

$$O(\lambda^0)\ :\qquad F_0=\frac{3}{2}\eta-\frac{1}{2}\eta^3$$

satisfying boundary conditions

$$\left.\begin{array}{l} F_0(X,\pm1)=\pm1,\qquad F_{0\eta}(X,\pm1)=0 \\ \text{For } n=1,\ 2,\ 3 \text{ the solutions satisfying the boundary conditions} \\ F_{n\eta}(X,\pm1)=0\quad,\qquad F_{n\eta}(X,\pm1)=0\quad \text{for } n\geq 1 \text{ are} \end{array}\right\} \tag{16}$$

$$O(\lambda)\ :\quad F_1=L_0(\eta)H'(X)$$

$$L_0(\eta)=\frac{1}{280}(15\eta-33\mu^3+21\eta^5-3\eta^7)$$

$$O(\lambda^2)\ :\ F_2=L_1(\eta)(H'(X))^2+L_1(\eta)H(X)H''(X)$$

$$L_1(\eta)=\frac{115}{17248}\eta-\frac{4111}{215600}\eta^3+\frac{57}{2800}\eta^5-\frac{51}{4900}\eta^7+\frac{3}{1120}\eta^9-\frac{1}{4400}\eta^{11}$$

$$L_2(\eta)=-\frac{1213}{4312000}\eta+\frac{3279}{4312000}\eta^3-\frac{3}{400}\eta^5+\frac{69}{19600}\eta^7-\frac{1}{1120}\eta^9+\frac{1}{12320}\eta^{11}$$

$$O(\lambda^3)\ :\ F_3=L_3(\eta)(H'(X))^3+L_4(\eta)H(X)H'(X)H''(X)+L_5(\eta)(H(X))^2H'''(X)$$

$$\begin{aligned}L_3(\eta)=&\frac{33897}{39239200}\eta-\frac{439093}{156956800}\eta^3+\frac{16493}{4312000}\eta^5-\frac{184599}{60368000}\eta^7+\frac{31}{19600}\eta^9\\&-\frac{603}{1232000}\eta^{11}+\frac{127}{1601600}\eta^{13}-\frac{1}{208000}\eta^{15}\end{aligned}$$

$$\begin{aligned}L_4(\eta)=&-\frac{67279}{71344000}\eta+\frac{49559}{17836000}\eta^3-\frac{28529}{8624000}\eta^5+\frac{68609}{30184000}\eta^7-\frac{167}{156800}\eta^9\\&+\frac{3}{9625}\eta^{11}-\frac{23}{457600}\eta^{13}+\frac{1}{320320}\eta^{15}\end{aligned}$$

$$\begin{aligned}L_5(\eta)=&\frac{58859}{39232000}\eta-\frac{341483}{784784000}\eta^3+\frac{1}{2000}\eta^5-\frac{19449}{6036800}\eta^7+\frac{11}{78400}\eta^9\\&-\frac{67}{1724800}\eta^{11}+\frac{1}{160160}\eta^{13}-\frac{9}{22422400}\eta^{15}\end{aligned}\tag{17}$$

Higher order terms can be obtained which involve lengthy / tedious algebra.

4. COMPUTER-EXTENDED SERIES

The higher order terms F_n involve more functions of X and an increasing number of terms of powers of η. Enumerating the functions of X is possible since for each n, different functions involve all the possible combination of n functions H and its derivatives multiplied together (e.g., F_2 involves HH'', H'^2; F_3 involves $H^2H''', H\ H'H''$ *and* H'^3; F_4 involves

$H^3H^{iv}, H^2H'H''', H^2H''^2, H\ H'^2H''$ and H'^4). Therefore, the functions of X in F_n span all the members of the set $G_{nj}\left(X\right)=H^{a_0}H'^{a_1}H''^{a_2}------H^{(n)a_n}$ (18)

where all a_m are non negative integers satisfying the diophantine equations

$$\sum_{m=0}^{n}a_m=\sum_{m=1}^{n}ma_m=n \tag{19}$$

Thus, the number of such combinations satisfying (19), is *p(n)* and the integer *j* in (18) runs from 1 to *p(n)*, where *p(n)* is the number of partitions of *n*. For any *n* partitions can be generated systematically in a variety of ways but the algorithm given by Gupta [14] is the most systematic. Thus we propose an elegant series expansion scheme with polynomial coefficients, which we find useful and efficient in the calculation of higher approximating terms of the series. We consider F_n as a finite double sum of known functions of X and η,

as $F_n(X,\eta)=\sum_{j=1}^{P(n)}G_{nj}(X)\sum_k t_{njk}(\eta)$

which can also be written as

$$F_n(X,\eta)=\sum_{k=1}^{2n}(1-\eta^2)^2\eta^{2k-1}g_{(n,k)}(X) \tag{20}$$

where $g_{n,k}(X)$ can in principle be expressed as sums of $G_{nj}(X)$. Equation (20) automatically satisfies the boundary conditions at $\eta=\pm1$. Substituting (20) into (14) and equating coefficients of various of power of η on both sides, a recurrence relation can be obtained for generating $g_{n,k}(X)$. The algebra associated with the calculations of universal functions $g_{n,k}(X)$, k= 2, 3, . . . 2n. j=1, 2, 3, . . .*p(n)* can be ordered if this is associated with the algebra of matrices formed with each row corresponding to partitions of a given number. This is an elegant algorithm in the collection and positioning of constants appearing in terms in the products ($F_\eta F_{\eta\eta}, F_{\eta\eta\eta}F_x, F_\eta F_{\eta\eta x}$ etc) which enables one to proceed from n = m to n = m+1, in the perturbation analysis, for m= 1, 2, 3, ... The coefficients estimated using this algorithm compare accurately with ones obtained using MATHEMATICA.

5. SPECIFIC GEOMETRIES AND UNIVERSAL COEFFICIENTS

We have calculated 15 effective terms. To this order the numbers of universal coefficients calculated are $4\sum_{m=1}^{15}mp_m=33956$ which are valid for any smooth geometry *H(X)*. For n=1, $g_{1,1}(X)=a\left(1,1,1\right)H'(X)$, $g_{1,2}(X)=a\left(1,1,2\right)H'(X)$ where $\mathrm{a}(1,1,1)=\frac{15}{280}$ $\mathrm{a}(1,1,2)=-\frac{3}{280}$.

For n=2, $g_{2,1}(X)=a\left(2,1,1\right)H'^2(X)+a\left(2,1,2\right)H(X)H''(X)$

$g_{2,3}(X)=a\left(2,3,1\right)H'^2(X)\ +a\left(2,3,2\right)H(X)H''(X),\quad g_{2,4}(X)=a\left(2,4,1\right)H'^2(X)\ +\ a\left(2,4,2\right)H(X)H''(X)$

where $a(2,1,1)=\frac{115}{17248}$, $a(2,1,2)=-\frac{1213}{431200}$, $a(2,2,1)=-\frac{309}{53900}$, $a(2,2,2)=\frac{853}{431200}$, $a(2,3,1)=-\frac{137}{61600}$, $a(2,3,2)=-\frac{9}{12320}$, $a(2,4,1)=-\frac{1}{4400}$, $a(2,4,2)=\frac{1}{12320}$.

Similarly for $n=3$, we have terms of the type

$$g_{3,1}(X) = a(3,1,1)H'^3(X) + a(3,1,2)H(X)H'(X)H''(X) + a(3,1,3)H^2(X)H'''(X)$$
$$g_{3,2}(X) = a(3,2,1)H'^3(X) + a(3,2,2)H(X)H'(X)H''(X) + a(3,2,3)H^2(X)H'''(X)$$
- -
$$g_{3,6}(X) = a(3,6,1)H'^3(X) + a(3,6,2)H(X)H'(X)H''(X) + a(3,6,3)H^2(X)H'''(X).$$

Universal coefficients (*a(n, k ,j), j =1, p(n)), k*=1,. . .2*n*) can be generated by ordering the algebra of finding coefficients in the products of terms in equation (14) after substituting (20) into this equation. It is an excellent algorithm wherein this algebra is associated with the algebra of matrices (which are formed with the partition of *m*) and proceed from $n = m$ to $n = m+1$ in the expansion. The skin friction at the channel walls is represented by the series

$$H^2\tau_0(X) = \left(\frac{\partial u}{\partial y}\right)(at\ y = \pm H(X)) = F_{\eta\eta}(at\ \eta = \pm 1) = = \left(F_{0\eta\eta} + \sum_{n=1}^{\infty}\lambda^n F_{n\ \eta\eta}\right)(at\ \eta = \pm 1) \qquad (21)$$

and the velocity profiles are given by

$$HU = F_\eta = F_{0\eta} + \sum_{n=1}^{\infty}\lambda^n F_{n\eta} \qquad (22)$$

For application of results, the specific channels considered one those of (Eagles and Smith [4] and Patterson [9-10])

$$H_1(X) = 1 + \frac{1}{2}\tanh(X) \qquad (23)$$

and

$$H_2(X) = 1 + \beta\, e^X,\ \beta = 0.4. \qquad (24)$$

DISCUSSION AND CONCLUSION

Computer extended series solution and other techniques are applied for the analysis of moderately high Reynolds number incompressible flows in plane channels whose width varies slowly in the streamwise direction. The main objective of this semi- analytical method is to see the possibility of enhancing the domain of validity of the series. In the low Reynolds number perturbation expansion, a large number of coefficients are generated. The complex expressions involving elementary functions appear in successive terms of the series and as such it is possible to calculate these up to 15 terms. To this order there are 33956 non-zero coefficients. These coefficients in turn give universal polynomials $L_n(\eta)$ which determine $F_n(X,\ \eta)$(*n*=0, 1, 2,. . ., 15). The series solution enables the prediction of skin friction $F_{\eta\eta}(at\ \eta = -1)$ for different λ and these are shown in Fig. 3 for a specific geometry. The velocity profiles represented by F_η for different values of *X* are shown in Figs. 4 to 6. The analytic continuation of the region and validity of series can be achieved by taking various Pade' approximants (Appendix 1). The coefficients of the series (21), representing skin friction $F_{\eta\eta}$ (at η = -1) are decreasing in magnitude but have no regular sign pattern of sign pattern. The Domb-Sykes [20] plot (Fig. 2) after extrapolation, confirms the radius of convergence of the series to be λ =2.28601, 5.85754 and 6.09069 for *X* = 0, 3 and 5 respectively. The direct sum of the series for skin friction is valid only up to the radius of convergence. We use Pade' approximants for summing the series which give a converging sum for sufficiently large λ (up to λ =30) whereas Eagles and Smith [4] could analyze the problem only up to λ =15.5. The flow described in this manner experiences separation from the channel wall at *X* = 0.05 for λ = 30. There is a critical point P_c, in the flow region where flows in its neighborhood reverse and P_c is found to shift downstream for larger values of λ. The skin friction at η = -1 decreases through zero at a point of separation and later converging (reattachment) flow is observed.

For the geometry given by $H_1(X)$, the coefficients of the series (23) representing velocity profiles are decreasing in magnitude but again have no regular sign pattern. We use Pade' approximants for summing the series and thereby the region of validity of the series is enhanced and these are shown in Figs. 4 to 6.

For the exponential diverging channel geometry $H_2(X)$ the calculated values of wall shear stress as a function of X (for different values of λ) are shown in Fig 7. A direct sum converges only for very small values of λ (and all negative values of X) but Pade' sums are found to converge for much larger λ. There is a symmetric shear distribution up to around $\lambda=3.2$ for small X beyond which it is asymmetric and separated flow is observed. Patterson [9-10] has also observed similar flow structure while conducting experiments of flows in an exponentially diverging channel.

Appendix 1:
Pade' approximants

The basic idea of Pade' summation is a process of replacing power series $\sum C_n \varepsilon^n$ by a sequence of rational fractions of the form

$$P_M^N(\varepsilon) = \frac{\sum_{n=0}^{N} A_n \varepsilon^n}{\sum_{n=0}^{M} B_n \varepsilon^n} \tag{i}$$

Without loss of generality take $B_0 = 1$. We determine the remaining $(M+N+1)$ coefficients $A_0, A_1, A_2, ---, A_N, B_1, B_2, B_3, ---, B_M$ so that first $(M+N+1)$ terms in Taylor series expansion of $P_M^N(\varepsilon)$ match with first $(M+N+1)$ terms of the power series $\sum C_n \varepsilon^n$. The resulting rational function $P_M^N(\varepsilon)$ is called Pade' approximant. For constructing Pade' approximants the full power series representation of a function is not necessary, only just the first $(M+N+1)$ terms are sufficient.The Pade' approximants perform an analytic continuation of the series outside its radius of convergence. With branch points it extracts a single-valued function by inserting branch cuts which it simulates by lines of alternating poles and zeros. Pade' approximant has been particularly successful in analysing series. Baker [21] has proved that diagonal Pade' approximants [N, N] $\left(P_N^N(\varepsilon)\right)$ are invariant under group of Euler transformations.

The Pade' approximants [N, M] with $M \geq P$ converge uniformly to $f(\varepsilon)$ as $N \to \infty$ everywhere inside the circle of meromarphy (the circle meromarphy is the largest circle such that all non regular points within it are at most poles or multiple poles of $f(\varepsilon)$) except at the points within a small circle on the poles. Throughout this region the rate of convergence is extremely rapid. Numerical experiments on various known functions show that convergence of the vicinity of the poles is still very fast even when $M < P$. There are many methods for the construction of Pade' approximants. One of the efficient methods for constructing Pade' approximants (diagonal $P_N^N(\varepsilon)$ and off diagonal $P_{N+1}^N(\varepsilon)$ in the Pade' table) is recasting of the series into continued fraction form and the truncating at various values of n to get required Pade' / rational approximant.A continued fraction is an infinite sequence of fractions whose $(N+1)$ member of $f_N(\varepsilon)$ has the form (Bender and Orszag [22])

$$f_N(\varepsilon) = \cfrac{D_0}{1+\cfrac{D_1\varepsilon}{1+\cfrac{D_2\varepsilon}{\cdots\cdots\cfrac{D_{N-1}\varepsilon}{1+D_N\varepsilon}}}} \tag{ii}$$

The coefficients D_N are determined by expanding the terminated continued fractions $f_N(\varepsilon)$ in a Taylor series and comparing the coefficients with those of the power series to be summed. An efficient procedure for calculating the coefficients D_N of the continued fraction may be derived from the algebraic identities of Bender & Orszag (1987), (8.4.2a) to (8.4.2c)). In contrast to representations by power series, continued fraction representations may converge in regions that contain isolated singularities of the function to be represented, and in addition the convergence is accelerated. Based on these D_N, we get terminated continued fractions of various orders (Bender & Orszag [22], (8.4.7),(8.4.8a),(8.4.8b)). The truncation of the continued fraction in (ii) yields the successive members of the Pade' sequence $P_0^1, P_1^0, P_1^1, P_2^1, P_2^2 \ldots\ldots$. This Pade sequence has some remarkable convergence properties when all the continued fraction coefficients are nonnegative. In general

$$\lim_{N\to\infty} P_{N+1}^N(\varepsilon) \le f(\varepsilon) \le \lim_{N\to\infty} P_N^N(\varepsilon)$$

or $$\lim_{N\to\infty} P_{N+1}^N(\varepsilon) = \lim_{N\to\infty} P_N^N(\varepsilon) \tag{iii}$$

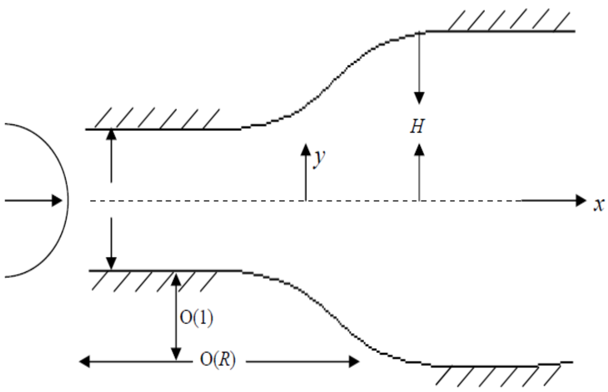

Fig. 1. Channel geometry $H_1(x)$ and non-dimensional co-ordinates.

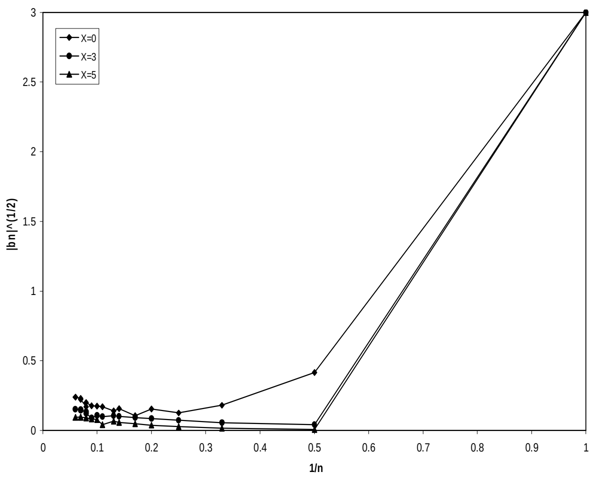

Fig. 2 Domb-Sykes plot for $F_{\eta\eta}$ at η =-1 for $H_1(X)$

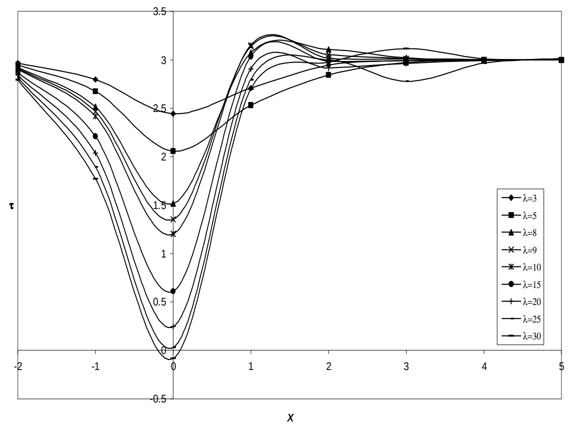

Fig. 3 Dimensionless wall shear magnitude ($F_{\eta\eta}$ at η = -1) as a function of X for $H_1(X)$ (converging Pade' sums) for various values of λ

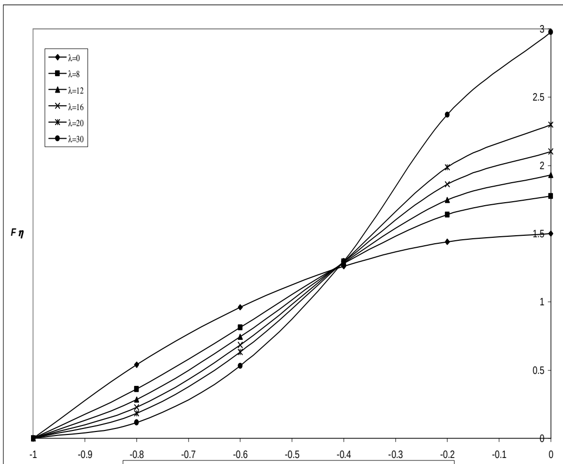

Fig. 4 Velocity Profiles for HU (=F_η)(converging Pade' sums) versus η for $H_1(X)$ for different values of λ at X=0

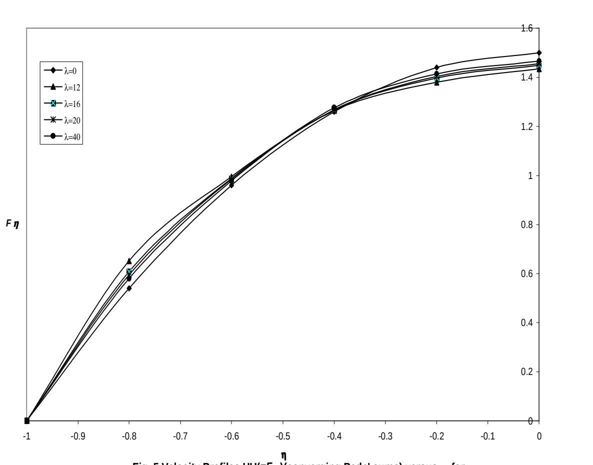

Fig. 5 Velocity Profiles HU(=F_η)(converging Pade' sums) versus η for $H_1(X)$ for different values of λ at X=1.488

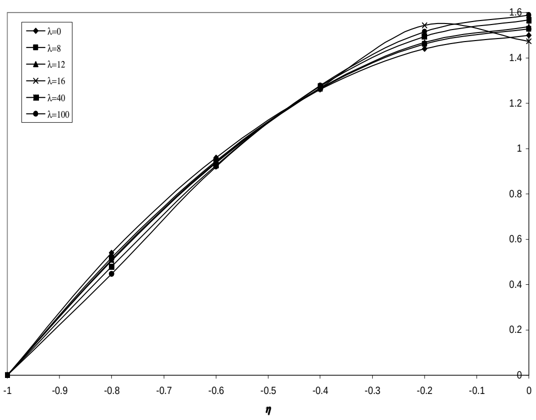

Fig. 6 Velocity Profiles for HU (=F_η) (converging Pade' sums) versus η for $H_1(X)$ for different values of λ at X=-1.512

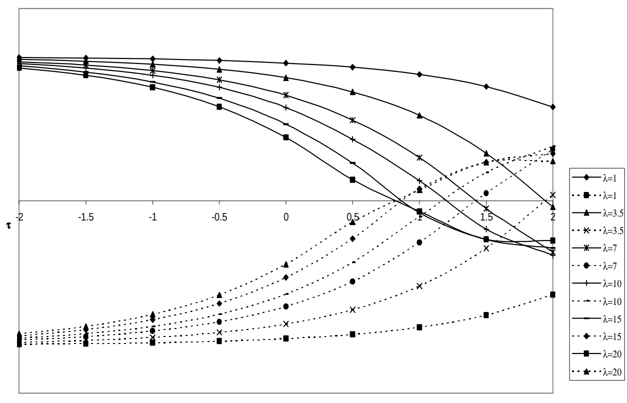

Fig.7 Dimensionless wall shear magnitude as a function of X for $H_2(X)$ (converging Pade' ums) for the various values of λ dotted lines correspond to η =1 and solid lines correspond to η =-1

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