Available online at www.pelagiaresearchlibrary.com

Pelagia Research Library

Advances in Applied Science Research, 2015, 6(7):224-229

ISSN: 0976-8610
CODEN (USA): AASRFC

Common fixed point theorems in Menger space with special reference to coincidence points

Arihant Jain[1], V. K. Gupta[2] and Ramesh Bhinde[3]

[1]*Department of Applied Mathematics, Shri Guru Sandipani Institute of Technology and Science, Ujjain (M.P.)India*
[2]*Department of Mathematics, Govt. Madhav Science College, Ujjain (M.P.)India*
[3]*Department of Mathematics, Govt. P. G. College, Alirajpur (M.P.) India*

ABSTRACT

The aim of this paper is to prove some common fixed point theorems for the class of compatible maps to larger class of occasionally weakly compatible maps without appeal to continuity in Menger spaces and we also give a set of alternative conditions in place of completeness of the space. We improve and extend the results of Dedeic & Sarapa [3] and Rashwan & Hedar [17].

Keywords: Menger space, Common fixed points, Compatible maps, Occasionally Weakly Compatible maps.

INTRODUCTION

A common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem was proved by Jungck [7]. Banach fixed point theorem has many applications but suffers from one drawback, the definition requires continuity of the function. There then the existing literature contains papers involving contractive definition that do not require the continuity of the function. This result was further generalized and extended in various ways by many authors. Sessa [21] defined weak commutativity and proved common fixed point theorem for weakly commuting mappings. Further, Jungck [8] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity. Since then various fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings in the compatible pair, have been obtained by many authors in different spaces. It has been known from the paper of Kannan [10] that there exists maps that have a discontinuity in the domain but which has a fixed point. Moreover the maps involved in every case were continuous at the fixed point.

In 1998, Jungck & Rhoades [9] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not be true. Recently, Singh & Mishra [22] and Chugh & Kumar [2] proved some interesting results in metric spaces for weakly compatible maps without assuming any mapping continuous.

The notion of probabilistic metric spaces, which is generalization of metric space, was introduced by Menger [11] and the study of these spaces was expanded rapidly with the pioneering work of Schweizer & Sklar [18, 19]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis. The existence of fixed points for compatible mappings on probabilistic metric spaces is shown by Mishra [12]. Recently, many authors including Pathak, Kang & Baek [14], Stojakovic [23, 24, 25], Hadzic [4, 5], Dedeic & Sarapa [3], Rashwan & Hedar [17], Mishra [12], Radu [15, 16], Sehgal & Bharucha-Reid [20] and Cho, Murthy & Stojakovic [1] have proved fixed point theorems in Menger spaces.

In this paper, we prove some common fixed point theorems for occasionally weakly compatible mappings in Menger spaces without using the condition of continuity. We also give a set of alternative conditions in place of completeness of the space. We improve results of Dedeic & Sarapa [3] and Rashwan & Hedar [17].

MATERIALS AND METHODS

2. Preliminaries.

Definition 2.1.[11] A mapping $\mathcal{F} : R \to R^+$ is called a *distribution* if it is non-decreasing left continuous with $\inf \{ \mathcal{F}(t) \mid t \in R \} = 0$ and $\sup \{ \mathcal{F}(t) \mid t \in R\} = 1$.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0 & , \quad t \le 0 \\ 1 & , \quad t > 0 \end{cases}.$$

Definition 2.2. [6] A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if it satisfies the following conditions :

(t-1) $t(a, 1) = a$, $t(0, 0) = 0$;

(t-2) $t(a, b) = t(b, a)$;

(t-3) $t(c, d) \ge t(a, b)$; for $c \ge a$, $d \ge b$,

(t-4) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. [6] A *probabilistic metric space (PM-space)* is an ordered pair $(X, \mathcal{F})$ consisting of a non empty set X and a function $\mathcal{F} : X \times X \to L$, where L is the collection of all distribution functions and the value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F_{u, v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v}(x) = F_{v,u}(x)$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$,

for all $u,v,w \in X$ and $x, y > 0$.

Definition 2.4. [6] A *Menger space* is a triplet $(X, \mathcal{F}, t)$ where $(X, \mathcal{F})$ is a PM-space and t is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y) \ge t \{F_{u, v}(x), F_{v, w}(y) \}$, for all $u, v, w \in X$, $x, y \ge 0$.

Definition 2.5. [18] A sequence $\{x_n\}$ in a Menger space $(X, \mathcal{F}, t)$ is said to be *convergent* and *converges to a point* x in X if and only if for each

$\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$F_{x_n, x}(\varepsilon) > 1 - \lambda$ for all $n \ge M(\varepsilon, \lambda)$.

Further the sequence $\{x_n\}$ is said to be *Cauchy sequence* if for

$\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$F_{x_n, x_m}(\varepsilon) > 1- \lambda$ for all $m, n \ge M(\varepsilon, \lambda)$.

A Menger PM-space $(X, \mathcal{F}, t)$ is said to be *complete* if every Cauchy sequence in X converges to a point in X.

A complete metric space can be treated as a complete Menger space in the following way :

Proposition 2.1. [12] If (X, d) is a metric space then the metric d induces mappings

$\mathcal{F} : X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$, where

$H(k) = 0$, for $k \leq 0$ and $H(k) = 1$, for $k > 0$.

Further if, $t : [0,1] \times [0,1] \to [0,1]$ is defined by $t(a,b) = \min \{a, b\}$.

Then

$(X, \mathcal{F}, t)$ is a Menger space. It is complete if (X, d) is complete.

The space $(X, \mathcal{F}, t)$ so obtained is called the *induced Menger space.*

Definition 2.6. [6] Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for $x \in X$ implies $ASx = SAx$.

Definition 2.7. [12] Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be *compatible* if $F_{ASx_n, SAx_n}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.8. [6] Self maps A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Remark 2.1. [6] The concept of occasionally weakly compatible maps is more general than that of compatible maps.

RESULTS AND DISCUSSION

Theorem 3.1. Let A, B, S and T be self mappings on a Menger space $(X, \mathcal{F}, t)$ where t is continuous and $t(x, x) \geq x$ for all $x \in [0,1]$, satisfying the conditions:

(3.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(3.2) There exists $k \in (0,1)$ such that

$F_{Au, Bv}(kx) \geq t(F_{Au,Su}(x), t(F_{Bv, Tv}(x), t(F_{Au, Tv}(\alpha x), F_{Bv, Su}(2x - \alpha x))))$

for all $u, v \in X$, $x > 0$ and $\alpha \in (0,2)$.

If

(3.3) one of A(X), B(X), S(X) and T(X) is a complete subspace of X,

then

(i) A and S have a coincidence point, and

(ii) B and T have a coincidence point. Further if

(3.4) the pairs {A, S} and {B, T} are occasionally weakly compatible, then A, B, S and T have a unique fixed point in X.

We need the following lemma proved by Mishra [12] for our first result.

Lemma 3.1. [12] Let A, B, S and T be self mappings of the Menger space $(X, \mathcal{F}, t)$, where t is continuous and $t(x, x) \geq x$ for all $x \in [0,1]$, satisfying the conditions (3.1) and (3.2). Then the sequence $\{y_n\}$ defined by condition (3.4) is a Cauchy sequence in X.

Proof of Theorem 3.1. Since $A(X) \subset T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

(3.5) $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for n = 1,2, ...

Let $\{y_n\}$ be the sequence in X defined above. By using Lemma 3.1, $\{y_n\}$ is a Cauchy sequence in X.

Now suppose that S(X) is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in S(X) and has a limit z in S(X). Let $p \in S^{-1}z$. Then Sp = z.

We shall use the fact that the subsequence $\{y_{2n}\}$ also converges to z. By (3.2), we have

$$F_{Ap,\, Bx_{2n+1}}(kx) \;\geq\; t(F_{Ap,Sp}(x), t(F_{Bx_{2n+1},\, Tx_{2n+1}}(x), t(F_{Ap,\, Tx_{2n+1}}(\alpha x), F_{Bx_{2n+},\, Sp}(2x - \alpha x)))).$$

Taking $n \to \infty$ and $\alpha \to 1$, we have

$$F_{Ap,z}(kx) \geq t(F_{Ap,\, z}(x), t(F_{z,\, z}(x), t(F_{Ap,\, z}(x), F_{z,\, z}(x))))$$

$$\geq F_{Ap,\, z}(x),$$

which means that Ap = z. Hence Ap = Sp = z, i. e., p is a coincidence point of A and S. This proves (i).

Since $A(X) \subset T(X)$, Ap = z implies that $z \in T(X)$. Let $q \in T^{-1}z$. Then Tq = z.

It can easily verified by using similar arguments of the previous part of the proof that Bq = z. This proves (ii).

If we assume that T(X) is complete, then argument analogous to the previous completeness argument establishes (i) and (ii).

The remaining two cases pertain essentially to the previous cases. Indeed, if B(X) is complete, then by condition (3.1), $z \in B(X) \subset S(X)$. Similarly if A(X) is complete the $z \in A(X) \subset T(X)$. Thus (i) and (ii) are completely established.

Now, we assume that condition (3.4) holds. Since the pair {A, S} is occasionally weakly compatible therefore A and S commute at the coincidence point. i. e., ASp = SAp or Az = Sz. Similarly BTq = TBq or Bz = Tz.

Now, we prove that Az = z. By (3.2), we have

$$F_{Az,\, Bx_{2n+1}}(kx) \;\geq\; t(F_{Az,Sz}(x), t(F_{Bx_{2n+1},\, Tx_{2n+1}}(x), t(F_{Az,\, Tx_{2n+1}}(\alpha x), F_{Bx_{2n+1},\, Sz}(2x - \alpha x)))).$$

Taking $n \to \infty$ and $\alpha \to 1$, we have

$$F_{Az,z}(kx) \geq t(F_{Az,\, z}(x), t(F_{z,\, z}(x), t(F_{Az,\, z}(x), F_{z,\, Az}(x)))) \geq F_{Az,\, z}(x).$$

Therefore, Az = z. Hence Az = z = Sz.

Similarly, we have Bz = z = Tz. This means that z is a common fixed point of mappings A, B, S and T.

For uniqueness of common fixed point let $w \neq z$ be another fixed point of mappings A, B, S and T.

Then by condition (3.2) and taking $\alpha \to 1$, we have

$$F_{z,\, w}(kx) \;\geq\; t(F_{z,\, z}(x), t(F_{w,\, w}(x), t(F_{z,\, w}(x), F_{w,\, z}(x)))) \geq F_{z,\, w}(x)$$

which means that z = w. This completes the proof.

Remark 3.1. We note that Theorem 3.1 is still true if we replace the condition (3.2) by the following condition:
(3.6) there exists $k \in (0, 1)$ such that

$$F_{Au,\, Bv}(kx) \geq \min\{F_{Au,Su}(x), F_{Bv,\, Tv}(x), F_{Au,\, Tv}(\alpha x), F_{Bv,\, Su}(2x - \alpha x)\}$$

for all $u, v \in X$, $x > 0$ and $\alpha \in (0,2)$.

Theorem 3.2. Let A, B, S and T be self mappings on a Menger space (X, $\mathcal{F}$, t), where t is continuous and $t(x, x) \geq x$ for all $x \in [0,1]$, satisfying the conditions (3.1), (3.3), (3.4) and (3.7) there exists $k \in (0,1)$ such that

$F_{Au,Bv}(kx) \geq \min\{F_{Au,Su}(x), F_{Bv,\,Tv}(x), F_{Su,\,Tv}(x)\}$,

for all $u, v \in X$, $x > 0$. Then all the conclusions of Theorem 3.1 are true.

Proof. If the condition (3.7) is satisfied, then for any $\alpha \in (0,2)$, we have on the lines of Dedeic and Sarapa [3]

$F_{Au,Bv}(kx) \geq \min\{F_{Au,Su}(x), F_{Bv,Tv}(x), F_{Su,Tv}(x)\}, \geq \min\{F_{Au,Su}(x), F_{Bv,Tv}(x), F_{Au,Tv}(x), F_{Bv,Su}(2x - \alpha x)\}$.

Then using the Remark 3.1, the Theorem 3.2 is still true.

The metric version of Theorem 3.1 is as follows:

Theorem 3.3. Let A, B, S and T be self mappings on a metric space (X, d) satisfying the following conditions:
(3.8) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
(3.9) $d(Ax, By) \leq \max\{d(Ax, Sx), d(By, Ty), d(Sx, Ty), ½[d(Ax, Ty) + d(Sy, By)]\}$

for all $x, y \in X$.

If

(3.10) One of A(X), B(X), S(X) or T(X) is a complete subspace of X, then
(i) A and S have a coincidence point, and
(ii) B and T have a coincidence point. Further if
(3.11) the pairs {A,S} and {B,T} are occasionally weakly compatible,
then A, B, S and T have a unique fixed point in X.

CONCLUSION

Theorem 3.1 improves result of Rashwan & Hedar [17] and theorem 3.2 improves and extends the main result of Dedeic & Sarapa [3].

REFERENCES

[1] Cho, Y. J., Murthy, P. P. and Stojakovic, M., *Com. Korean Math. J.* 7 **(1992)**, 325-339.
[2] Chugh, R. and Kumar, S., *Proc. Indian Acad. Sci.* 111 (2), **(2001),** 241-247.
[3] Dedeic, R. and Sarapa, N., *Math. Japon.* 34 (6), **(1989)**, 919-923.
[4] Hadzic, O., Publ. Inst. Math. (Beograd) 20 **(1979)**, 107-112.
[5] Hadzic, O., *Boll. Un. Mat. Ital. B(6)* 1 (1), **(1982)**, 381-391.
[6] Jain, A. and Chaudhary, B., *International Journal of Research and Reviews in Applied Sciences*, Vol. 14 (3), **(2013)**, 662-670.
[7] Jungck, G., *Amer. Math. Monthly* 83 **(1976)**, 261-263.
[8] Jungck, G., *Internat. J. Math. Math. Sci.* 9 **(1986)**, 771-779.
[9] Jungck, G. and Rhoades, B. E., *Indian J. Pure Appl. Math.* 29 (3), **(1998)**, 227–238.
[10] Kannan, R., *Bull. Calcutta Math. Soc.* 60 **(1968)**, 71–76.
[11] Menger, K., *Proc. Nat. Acad. Sci. U. S. A.* 28 **(1942)**, 535–537.
[12] Mishra, S. N., *Math. Japon.* 36 **(1991)**, no. 2, 283–289.
[13] Pant, B.D. and Chauhan, S., *Int. Journal of Math. Analysis*, 3 (28), **(2009)**, 1389-1398.
[14] Pathak, H. K., Kang, S. M. and Baek, J. H., *Commun. Korean Math. Soc.* 10 **(1995),** no. 1, 67–83.
[15] Radu, V., *An Univ. Timisoara Ser. Stiint. Mat.*22 **(1984)**, 83–88.
[16] Radu, V., *An Univ. Timisoara Ser. Stiint. Mat.* 23 **(1985),** 61–65.
[17] Rashwan, R. A. and Hedar, A., *Demonstratio Math.* 31 (3), **(1998)**, 537–546.
[18] Schweizer, B. and Sklar, A., *Pacific J. Math.* 10 **(1960),** 313–334.
[19] Schweizer, B. and Sklar, A., *Statistical metric spaces*, North-Holland Series in Probability and Applied Mathematics 5, North Holland, Amsterdam **1983**.
[20] Sehgal, V. M. and Bharucha-Reid, A. T., *Math. Systems Theory* 6 **(1972)**, 97–102.
[21] Sessa, S., *Publ. Inst. Math.* 32 **(1982)**, 46, 149–153.
[22] Singh, S. L. and Mishra, S. N., *Indian J. Pure Appl. Math.* 28 (5), **(1997)**, 611–615.

[23] Stojakovic, M., *Kobe J. Math*. 2 (1), **(1985)**, 1–9.
[24] Stojakovic, M., *Bull. Austral. Math. Soc.* 36 **(1987)**, 73–88.
[25] Stojakovic, M., *Glasnik Mat.* 23 (1), **(1988)**, 203–211.