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Common fixed point theorems in fuzzy metric space

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ABSTRACT

In this paper, the concept of occasionally weakly compatible maps in fuzzy metric space has been introduced to prove common fixed point theorems which generalize the result of Sharma [13]. We also cited an example in support of our result.

Keywords: Common fixed points, fuzzy metric space, compatible maps, occasionally weakly compatible mappings and weak compatible mappings.
AMS Subject Classification : *Primary 47H10, Secondary 54H25.*

INTRODUCTION

The notion of a probabilistic metric space corresponds to the situation when we do not know the distance between the points but know only probabilities of possible value of this distance. Since the 16^{th} century, probability theory has been studying a kind of uncertainty randomness, that is, the uncertainty of the occurrence of an event; but in this case, the event itself is completely certain and the only uncertain thing is whether the event will occur or not and the causality is not clearly known. Following the study on certainty and on randomness, the study of mathematics began to explore the restricted zone - fuzziness. Fuzziness is a kind of uncertainty i.e., for some events, it cannot be completely determined that in which cases these events should be subordinated to, (they have already occurred or not yet), they are in non-black or non-white state. We can say that the law of excluded middle in logic cannot be applied any more. Zadeh [18] introduced the concept of fuzzy set as a new way to represent vagueness in our everyday life. A fuzzy set A in X is a function with domain X and values in [0, 1]. Since then, many authors regarding the theory of fuzzy sets and its applications have developed a lot of literatures.

However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable. We can divide them into following two groups: The first group involves those results in which a fuzzy metric on a set X is treated as a map where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek [10] and modified by George and Veeramani [4]. Recently, Grabiec [5] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [14] introduced the concept of compatible mappings in Fuzzy metric space and proved the common fixed point theorem. Jungck et. al. [8] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Cho [2, 3] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in fuzzy metric space. In 2011, using the concept of compatible maps of type (A) and type (β), Singh et. al. [15, 16] proved fixed point theorems in a fuzzy metric space. Recently in 2012, Jain et. al. [6, 7] and Sharma et. al. [12] proved various fixed point theorems using the concepts of semi-compatible mappings, property (E.A.) and absorbing mappings.

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2. Preliminaries

Definition 2.1. [11] A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0, 1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. [11] The 3-tuple $(X, M, *)$ is said to be a *Fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions :

for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Note that $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [11] Let (X, d) be a metric space. Define $a * b = \min \{a, b\}$ and $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$. Then $(X, M, *)$ is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [11] A sequence $\{x_n\}$ in a Fuzzy metric space $(X, M, *)$ is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

A Fuzzy metric space $(X, M, *)$ is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [14] Self mappings A and S of a Fuzzy metric space $(X, M, *)$ are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X
as $n \to \infty$.

Definition 2.5. [15] Two self maps A and B of a fuzzy metric space $(X, M, *)$ are said to be weak compatible if they commute at their coincidence points, i.e. $Ax = Bx$ implies $ABx = BAx$.

Definition 2.6. Self maps A and S of a Fuzzy metric space $(X, M, *)$ are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Proposition 2.1. [16] In a fuzzy metric space $(X, M, *)$ limit of a sequence is unique.

Proposition 2.2. [14] Let S and T be compatible self maps of a Fuzzy metric space $(X, M, *)$ and let $\{x_n\}$ be a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X. Then $STx_n \to Tu$ provided T is continuous.

Proposition 2.3. [14] Let S and T be compatible self maps of a Fuzzy metric space $(X, M, *)$ and $Su = Tu$ for some u in X then
$STu = TSu = SSu = TTu$.

Lemma 2.1. [5] Let $(X, M, *)$ be a fuzzy metric space. Then for all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.

Lemma 2.2. [1] Let $(X, M, *)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$

$M(x, y, kt) \geq M(x, y, t) \; \forall \; t > 0$
then $x = y$.

Lemma 2.3. [16] Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. If there exists a number $k \in (0, 1)$ such that

$M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t) \; \forall \; t > 0$ and $n \in N$.

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.4.[9] The only t-norm $*$ satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t-norm, that is

$a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

In 2002, Sharma [13] proved the following theorem :

Theorem 2.1. Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$. Let A, B, S, T, P and Q be mappings from X into itself satisfying the following conditions:

(2.1) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$,
(2.2) $AB = BA$, $ST = TS$, $PB = BP$, $SQ = QS$, $QT = TQ$,
(2.3) Pairs (P,AB) and (Q,ST) are compatible of type (α) (or compatible of type (A)),
(2.4) A, B, S and T are continuous,
(2.5) There exists a number $k \in (0, 1)$ such that

$$M(Px, Qy, kt) \geq M(ABx, Px, t) * M(STy, Qy, t) * M(STy, Px, \beta t) * M(ABx, Qy, (2 - \beta)t) * M(ABx, STy, t)$$

for all $x, y \in X$; $\beta \in (0, 2)$ and $t > 0$.

Then A, B, S, T, P and Q have a unique common fixed point in X.

3. Main Result.
Now we prove the following results:

Theorem 3.1. Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$. Let A, B, S, T, P and Q be mappings from X into itself satisfying

(3.3.1) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$;
(3.3.2) $AB = BA$, $ST = TS$, $PB = BP$, $SQ = QS$, $QT = TQ$;
(3.3.3) Pairs (P, AB) and (Q, ST) are occasionally weakly compatible;
(3.3.4) There exists a number $k \in (0, 1)$ such that

$M(Px, Qy, kt) \geq M(ABx, Px, t) * M(STy, Qy, t) * M(STy, Px, \beta t) * M(ABx, Qy, (2 - \beta)t) * M(ABx, STy, t)$, for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$.

If the range of the subspaces P(X) or AB(X) or Q(X) or ST(X) is complete, then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. By [13], $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, so $\{y_n\}$ converges to a point $z \in X$. Since $\{Px_{2n}\}$, $\{Qx_{2n+1}\}$, $\{ABx_{2n+1}\}$ and $\{STx_{2n+2}\}$ are subsequences of $\{y_n\}$, they also converge to the same point z.

Since $P(X) \subset AB(X)$, there exists a point $u \in X$ such that $ABu = z$. Then, using (3.3.4)

$M(Pu, z, kt) \geq M(Pu, Qx_{2n+1}, kt)$

$\geq M(ABu, Pu, t) * M(STx_{2n+1}, Qx_{2n+1}, t) * M(STx_{2n+1}, Pu, \beta t)$

$* M(ABu, Qx_{2n+1}, (2 - \beta)t) * M(ABu, STx_{2n+1}, t).$

Proceeding limit as $n \to \infty$ and setting $\beta = 1$,

$M(Pu, z, kt) \geq M(Pu, z, t) * M(z, z, t) * M(z, Pu, \beta t) * M(z, z, t) * M(z, z, t)$

$= M(Pu, z, t) * 1 * M(Pu, z, t) * 1 * 1;$

$\geq M(Pu, z, t).$

By Lemma (2.2),

$Pu = z.$

Therefore, $ABu = Pu = z$.

Since $Q(X) \subset ST(X)$, there exists a point $v \in X$ such that $z = STv$. Then, again using (3.3.4)

$M(Pu, Qv, kt) \geq M(ABu, Pu, t) * M(STv, Qv, t) * M(STv, Pu, \beta t)$

$* M(ABu, Qv, (2 - \beta)t) * M(ABu, STv, t)$

Proceeding limit as $n \to \infty$, we have for $\beta = 1$, $Qv = z$.

Therefore, $ABu = Pu = STv = Qv = z$.

Since pair (P, AB) is occasionally weakly compatible, therefore, $Pu = ABu$ implies that $PABu = ABPu$ i.e., $Pz = ABz$.

Now we show that z is a fixed point of P. For $\beta = 1$, we have

$M(Pz, Qv, kt) \geq M(ABz, Pz, t) * M(STv, Qv, t) * M(STv, Pz, \beta t)$

$* M(ABz, Qv, (2 - \beta)t) * M(ABz, STv, t)$
$= 1 * 1 * M(z, Pz, t) * M(Pz, z, t) * M(Pz, z, t).$

Therefore, we have by Lemma 2.2,
$Pz = z.$
Hence
$Pz = z = ABz.$

Similarly, pair of map {Q, ST} is occasionally weakly compatible, we have
$Qz = STz = z.$
Now we show that $Bz = z$, by putting $x = Bz$ and $y = x_{2n+1}$ with $\beta = 1$ in for (3.3.4) we have
$M(PBz, Qx_{2n+1}, kt) \geq M(AB(Bz), P(Bz), t) * M(STx_{2n+1}, Qx_{2n+1}, t)$
$* M(STx_{2n+1}, PBz, t) * M(AB(Bz), Qx_{2n+1}, t)$
$* M(AB(Bz), STx_{2n+1}, t).$
Proceeding limits as $n \to \infty$ and using Lemma 2.2, we have $Bz = z$. Since $ABz = z$, therefore, $Pz = ABz = Bz = z = Qz = STz$.

Finally, we show that Tz = z, by putting x = z and y = Tz with $\beta = 1$ in (3.3.4).
$M(Pz, Q(Tz), kt) \geq M(ABz, Pz, t) * M(ST(Tz), Q(Tz), t)$
$* M(ST(Tz), Pz, t) * M(ABz, Q(Tz), t)$
$* M(ABz, ST(Tz), t)$.

Therefore, Tz = z.

Hence, ABz = Bz = STz = Tz = Pz = Qz = z.

Uniqueness follows easily.

If we put B = T = I, the identity map on X, in Theorem 3.3.1, we have the following:

Corollary 3.3.1. Let (X, M, *) be a complete fuzzy metric space with $t * t \geq t$ for all $t \in (0, 1)$ and let A, S, P and Q be the mapping from X into itself such that
(3.3.5) $P(X) \subset A(X)$, $Q(X) \subset S(X)$.
(3.3.6) The pairs (A, S) and (Q, S) are occasionally weakly compatible.
(3.3.7) There exists a number $k \in (0, 1)$ such that
$M(Px, Qy, kt) \geq M(Ax, Px, t) * M(Sy, Qy, t) * M(Sy, Px, \beta t) * M(Ax, Qy, (2 - \beta)t) * M(Ax, Sy, t)$;
for all $x, y \in X$, $\beta \in (0, 2)$ with $t > 0$.

If the range of the one subspaces is complete then A, S, P and Q have a unique common fixed point in X.

If we put A = B = S = T = I in Theorem 3.3.1, we have the following:

Corollary 3.3.2. Let (X, M, *) be a complete fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$ and let P and Q be occasionally weakly compatible mapping from X into itself. If there exists a constant $k \in (0, 1)$ such that
$M(Px, Qy, kt) \geq M(x, Px, t) * M(y, Qy, t) * M(y, Px, \beta t)$
$* M(x, Qy, (2 - \beta)t) * M(x, y, t)$;
for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$.

If the range of the one subspaces is complete then P and Q have a unique common fixed point in X.

If we put P = Q, A = S and B = T = I in Theorem 3.3.1, we have the following:

Corollary 3.3.3. Let (X, M, *) be a complete fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$ and let P, S be occasionally weakly compatible maps on X such that $P(X) \subset S(X)$ and satisfy the following condition:
$M(Px, Py, t) \geq M(Sx, Px, t) * M(Sy, Py, t) * M(Sy, Px, \beta t)* M(Sx, Py, (2 - \beta)t) * M(Sx, Sy, t)$,
for all $x, y \in X$, $\beta \in (0, 2)$ and $t > 0$. If the range of the one subspaces is complete then P and S have a unique common fixed point in X.

Example 3.3.1. Let X = [0, 1] with usual metric d and for each $t \in [0, 1]$.
Define

$$M(x, y, t) = \frac{t}{t + |x - y|}, \quad M(x, y, 0) = 0 \text{ for all } x, y \in X.$$

Clearly (X, M, *) is a complete fuzzy metric space where * is defined by a * b = ab.
Let A, B, S, T, P and Q be defined by Ax = x, Bx = x/2, Sx = x/5, Tx = x/3,
Px = x/6 and Qx = 0 for all $x, y \in X$.
Then $P(X) = [0, 1/6] \subset [0, 1/2] = AB(X)$ and $Q(X) = 0 \subset [0, 1/5] = STx$.
If we take k = 1/2, t = 1 and $\beta = 1$, we see that all conditions of Theorem 2.3.1 are satisfied.
Moreover, the pair {P, AB} and {Q, ST} are occasionally weakly compatible.

CONCLUSION

Theorem 3.1 is a generalization of the result of Sharma [13] in the sense that condition of compatibility of type (A) of the pairs of self maps has been restricted to occasionally weakly compatible self maps and continuity of the mappings have been completely removed.

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