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Common fixed point theorems for four self maps on a fuzzy metric space, satisfying common E. A. property

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ABSTRACT

In this paper, we prove common fixed point theorems for four self maps by using weak compatibility in fuzzy metric spaces. Our result extend, generalized several fixed point theorems on metric and fuzzy metric spaces.

Key words: fixed point; fixed point theorem; weakly compatible; E.A. property
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INTRODUCTION

The evolution of fuzzy mathematics commenced with the introduction of the notion of fuzzy sets by Zadeh [14], in 1965, as a new way to represent the vagueness in everyday life. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences, image processing, control theory, communication etc. With the concept of fuzzy sets, the fuzzy metric space was introduced by Kramosil and Michalek [6]. Grabiec [3] proved the contraction principle in the setting of the fuzzy metric space which was further generalization of results by Subrahmanyam [11] for a pair of commuting mappings. Also George and Veeramani [2] modified the notion of fuzzy metric spaces with the help of continuous t-norm by generalizing the concept of probabilistic metric space to fuzzy situation. In 1999, Vasuki [12] introduced the concept of R-weakly commutative of mappings in fuzzy metric spaces and Pant [8] introduced the notion of reciprocal continuity of mappings in metric spaces. Also Jungck and Rhoades [5] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. Aamri and Moutawakil [1] generalized the notion of non compatible mapping in metric space by E.A. property.

Varsha [13] proved a common fixed point theorem for four mappings satisfying the general contractive condition along with the notion of E.A. property and weakly compatible mapping in fuzzy metric space.

In this paper, we continue studying the effect of E.A. property on the existence and uniqueness of common fixed point of four self maps on a fuzzy metric space.

1. PRELIMINARIES:

Definition 2.1 (Schweizer.B and Sklar.A [9]) A binary operation $*: [0,1]\ X\ [0,1] \rightarrow [0,1]$ is a continuous t-norm if * satisfies the following conditions:
(i) * is commutative and associative
(ii) * is continuous
(iii) a * 1 = a for all a ∈ [0, 1]
(iv) $a * b \leq c * d, when\ ever\ a \leq c\ and\ b \leq d,$ for each a, b, c, d ∈ [0,1]

Examples of t – norm are *(a, b) = min {a, b} and *(a, b) = a.b

Definition 2.2 (George, A. and Veeramani, P. [2]) $(X, M,*)$ is said to be a fuzzy metric space if X is a non empty set, * is a continuous t-norm and M is a fuzzy set on X^2 x $(0, \infty)$ satisfying the following conditions, for all x, y, z $\in X,\ t, s > 0$

(2.2.1) $M(x, y, t) > 0$
(2.2.2) $M(x, y, t) = 1\ iff\ x = y$
(2.2.3) $M(x, y, t) = M(y, x, t)$
(2.2.4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(2.2.5) $M(x, y, .): (0, \infty) \to (0, 1]$ is continuous.

Definition 2.3 (Jungck G and B. E. Rhoades, [5]) Let X be a non empty set and f and g be self maps of X. A point x $\in X$ is called a coincidence point of f and g iff fx = gx. We shall call w = fx = gx a point of coincidence of f and g.

Definition 2.4 (Jungck G and B. E. Rhoades, [5]) A pair of maps S and T is called weakly compatible pair if they commute at their coincidence points. That is Sx = Tx implies TSx = STx.

Definition 2.5 (Mishra S N., Mishra N and Singh , S.L [7]) Let $(X, M,*)$ be fuzzy metric space. A sequence $\{x_n\}$ in X is said to converge to a point x ∈ X if and only if for each $\epsilon > 0, t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \epsilon$ for all n≥ n_0

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M,*)$ is said to be a Cauchy sequence if and only if for each $\epsilon > 0, t > 0$ there exist $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all n, m ≥ n_0

Definition 2.6 (Aamir M and D.El Moutawakil, [1]) Let f and g be two self-maps of a fuzzy metric space$(X, M,*)$. We say that f and g satisfy the property E.A. if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\leftrightarrow\infty} fx_n = \lim_{n\leftrightarrow\infty} gx_n = z\ for\ some\ z \in X$$

Definition 2.7 (Aamir M and D.El Moutawakil, [1]): Let A, B, S, T: X → X where $(X, M,*)$ is a fuzzy metric space. Then the pair (A,S) and (B,T) is said to satisfy common EA property if there exist two sequences $\{x_n\}\ and\ \{y_n\}$ such that

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = z\ for\ some\ z \in X$$

Varsha et. al. [13] proved the following theorem.
Theorem 2.7 [13]: $(X, M,*)$ be a fuzzy metric space and A, B, S and T be self maps on X satisfyimg the following conditions:

(1) A(X) ⊂ $T(X) and\ B(X) \subset S(X)$ and $S(X)$ is closed.

(2) $M(Ax, By, t) \geq \varphi(min \begin{pmatrix} M(Sx, Ty, t), \\ sup_{t_1+t_2=\frac{2}{k}t} \min\{M(Sx, Ax, t_1), M(Ty, By, t_2), \\ sup_{t_3+t_4} = \frac{2}{k} t \max\{M(Sx, By, t_3), M((Ty, Ax, t_4)\} \end{pmatrix})$

for all $x, y \in X,\ t > 0\ and\ for\ some\ 1 \leq k < 2$. Suppose the pairs (A, S) and (B, T) satisfy common E.A. property and (A, S) and (B, T) are weakly compatible. Also suppose that S(X) and T(X) are closed subsets of X. Then A, B, S, T have a unique common fixed point in X.

However, in the above statement, the nature of φ is not specified.

In the next section, we prove a theorem similar to Theorem 2.7 satisfying 2.2.6 and 2.2.7 when 0 < k < 1 and $\varphi: [0, 1] \to [0, 1]$ is continuous with $\varphi(t) > t\ for\ 0 < t < 1$.

2. Main result:

In the rest of the paper we assume that a fuzzy metric space $(X, M,*)$ satisfies the following:

(2.2.6) $M(x, y, t) \to 1\ as\ t \to \infty\ for\ all\ x, y \in X$
(2.2.7) If $\{x_n\}\ and\ \{y_n\}$ are sequence in X such that $x_n \to x\ and\ y_n \to y$ then

$M(x_n, y_n, t) \to M(x, y, t) as\ n \to \infty$

Theorem 3.1: Let $(X, M,*)$ be a fuzzy metric space and A, B, S and T be self maps on X satisfying the following conditions:

1 A(X) $\subset T(X) and\ B(X) \subset S(X)$ and $S(X)\ and\ T(X)$ are closed.

2 $M(Ax, By, t) \geq \varphi\left\{min\begin{pmatrix} M(Sx, Ty, t), M(Ty, By, t), M(Sx, Ax, t), \\ max\left\{M(Sx, By, t), M\left(Ty, Ax, \left(\frac{2}{k}-1\right)t\right)\right\}, \\ \max\{M\left(Sx, By, \left(\frac{2}{k}-1\right)t\right), M((Ty, Ax, t)\} \end{pmatrix}\right\}$ (3.1.1)

For all $x, y \in X,\ t > 0\ and\ for\ some\ 0 < k < 1.$ where $\varphi: [0,1] \to [0,1]$is a continuous function and $\varphi(t) > t\ if\ 0 < t < 1.$ Suppose the pairs (A, S) and (B, T) satisfy common E.A. property and (A, S) and (B, T) are weakly compatible. Then A, B, S, and T have a unique common fixed point in X.

Proof: Since (A, S) and (B, T) satisfy common E.A. property, there exist sequences $\{x_n\}\ and\ \{y_n\}$ such that

$$\lim_{n\to\infty} M(Ay_{n,}z, t) = \lim_{n\to\infty} M(Bx_{n,}z, t) = \lim_{n\to\infty} M(Sy_{n,}z, t) = \lim_{n\to\infty} M(Tx_{n,}z, t) = 1$$

$for\ some\ z\ \in X\ and\ for\ all\ t > 0$

so that $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = z.$

Since S(X) and T(X) are closed subsets of X, there exist u, v ∈ X such that Su = z and Tv = z
We show that Au= z. Put x = u and y = x_n in (3.1.1)

$$M(Au, Bx_n, t) \geq \varphi\left\{min\begin{pmatrix} M(Su, Tx_n, t), M(Tx_n, Bx_n, t), M(Su, Au, t), \\ max\left\{M(Su, Bx_n, t), M\left(Tx_n, Au, \left(\frac{2}{k}-1\right)t\right)\right\}, \\ \max\{M\left(Su, Bx_n, \left(\frac{2}{k}-1\right)t\right), M((Tx_n, Au, t)\} \end{pmatrix}\right\}$$

On letting $n \to \infty$ we get

$$\text{M (Au, z, t)} \geq \varphi\left\{min\begin{pmatrix} M(z, z, t), M(z, z, t), M(z, Au, t), \\ max\left\{M(z, z, t), M\left(z, Au, \left(\frac{2}{k}-1\right)t\right)\right\}, \\ \max\{M\left(z, z, \left(\frac{2}{k}-1\right)t\right), M(z, Au, t)\} \end{pmatrix}\right\}$$

$$= \varphi\left(M(z, Au, t)\right) > M(z, Au, t)\ if\ Au\ \neq z, a\ contradiction.$$

$There\ fore\ Au = z.$
In a similar way we can show that Bv = z.

Thus u is a coincidence point of A and S, v is a coincidence point of B and T.

Since (A, S) is weakly compatible ASu = SAu so that Az = Sz and (B, T) is weakly compatible BTv = TBv so that Bz = Tz

Now we show that z is a common fixed point of A and S

$$M(Az, z, t) = M(Az, Bv, t) \geq \varphi\left\{min\begin{pmatrix} M(Sz, Tv, t), M(Tv, Bv, t), M(Sz, Az, t), \\ max\left\{M(Sz, Bv, t), M\left(Tv, Az, \left(\frac{2}{k}-1\right)t\right)\right\}, \\ \max\{M\left(Sz, Bv, \left(\frac{2}{k}-1\right)t\right), M((Tv, Az, t)\} \end{pmatrix}\right\}$$

On letting $n \to \infty$ we get

$$= \varphi\left\{min\begin{pmatrix} M(Az, z, t), M(z, z, t), M(Az, Az, t), \\ max\left\{M(z, zv, t), M\left(z, Az, \left(\frac{2}{k}-1\right)t\right)\right\}, \\ \max\{M\left(z, zv, \left(\frac{2}{k}-1\right)t\right), M(z, Az, t)\} \end{pmatrix}\right\}$$

$=\varphi\left(M(Az, z, t)\right) > M(Az, z, t)\ if\ Az\ \neq z,$ a contradiction.

Therefore Az = z.

Hence Az = Sz = z. Similarly Bz = Tz = z. Hence Az = Sz = Tz= Bz = z.

Thus z is a common fixed point for A, B, S and T.

Uniquesness:

If x and y are fixed points of A, B, S and T, then

$$M(x,y,t) = M(Ax,By,t) \geq \varphi\left\{min\begin{pmatrix} M(Sx,Ty,t), M(Ty,By,t), M(Sx,Ax,t), \\ max\left\{M(Sx,By,t), M\left(Ty,Ax,\left(\frac{2}{k}-1\right)t\right)\right\}, \\ \max\{M\left(Sx,By,\left(\frac{2}{k}-1\right)t\right), M((Ty,Ax,t)\} \end{pmatrix}\right\}$$

$$= \varphi\left\{min\begin{pmatrix} M(x,y,t), M(y,y,t), M(x,x,t), \\ max\left\{M(x,y,t), M\left(y,x,\left(\frac{2}{k}-1\right)t\right)\right\}, \\ \max\{M\left(x,y,\left(\frac{2}{k}-1\right)t\right), M(y,x,t)\} \end{pmatrix}\right\}$$

$= \varphi\left(M(x,y,t)\right) > M(x,y,t)$, a contradiction if x $\neq y$.

Hence x = y.
Consequently, A, B, S, and T have unique common fixed point.

Theorem 3.2 Let $(X, M, *)$ be a fuzzy metric space and A, B, S and T be self maps on X satisfyimg the following conditions:
1 A(X) $\subset T(X) and\ B(X) \subset S(X)$ and $S(X)\ and\ T(X)$ are closed.
2 $M(Ax,By,t) \geq$

$$\varphi\left\{min\begin{pmatrix} M(Sx,Ty,t), \min\left\{M\left(Ty,By,\left(\frac{2}{k}-1\right)t\right), M(Sx,Ax,t)\right\}, \min\{M(Ty,By,t), M\left(Sx,Ax,\left(\frac{2}{k}-1\right)t\right)\} \\ max\left\{M(Sx,By,t), M\left(Ty,Ax,\left(\frac{2}{k}-1\right)t\right)\right\}, \\ \max\{M\left(Sx,By,\left(\frac{2}{k}-1\right)t\right), M((Ty,Ax,t)\} \end{pmatrix}\right\} \quad (3.2.1)$$

for all $x, y \in X,\ t > 0\ and\ for\ some\ 0 < k < 1$. where φ: [0,1] →[0,1] is a continuous function and $\varphi(t) > t\ \ if\ 0 < t < 1$. Suppose the pairs (A, S) and (B, T) satisfy common EA property and (A, S) and (B, T) are weakly compatible. Then A, B, S, T have a unique common fixed point in X.
Proof: The proof is similar to that of Theorem 3.1

Corollory 3.3: $(X, M, *)$ be a fuzzy metric space and A, B, S and T be self maps on X satisfying the following conditions:

(1) A(X) $\subset T(X) and\ B(X) \subset S(X)$ and $S(X)$ and T(X) are closed.

(2) $$M(Ax,By,t) \geq \varphi\left(min\begin{pmatrix} M(Sx,Ty,t), \\ sup_{t_1+t_2=\frac{2}{k}t} \min\{M(Sx,Ax,t_1), M(Ty,By,t_2), \\ sup_{t_3+t_4} = \frac{2}{k}t \max\{M(Sx,By,t_3), M((Ty,Ax,t_4)\} \end{pmatrix}\right) \quad 3.3.1$$

for all $x, y \in X,\ t > 0\ and\ for\ some\ 0 < k < 1$. where φ: [0,1] → [0,1] is a continuous and increasing function and $\varphi(t) > t\ \ if\ 0 < t < 1$. Suppose the pairs (A, S) and (B, T) satisfy common E.A. property and (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

Proof: Since φ is increasing we observe that (3.3.1) > (3.2.1). Consequently from Theorem 3.2 the result follows.

Note: In view of Corollory 3.3, we observe that Theorem 2.7 follows from Theorem 3.2, with the understanding that $0 < k < 1$ and $\varphi: [0,1] \rightarrow [0,1]$ is continuous and increasing. (It may be observed that in the statement of Theorem 2.7, the nature of φ is not mentioned)

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