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# Birnbaum Saunders Distribution with Application in Environmental Sciences

**Kishore K. Das and Bhanita Das***

*Department of Statistics, Gauhati University, Assam, India*

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**ABSTRACT**

*In this paper, we have considered transformed Birnbaum Saunder's distribution to obtain weighted and length biased distribution. To obtain a valid weighted probability distribution the transformation was made. The statistical distribution of Zinc concentration in drinking water quality data follows LBWBSD. Here, we develop LBD of weighted Birnbaum Saunders Distribution (WBSD). Moment properties, estimation of parameter, and hazard function of LBD of WBSD along with its behavior have been considered. Finally, we illustrate the applicability of LBD of WBSD in environmental sciences using drinking water quality data based on Zinc concentration.*

**Keywords**: WBSD, length biased distribution (LBD), moments, estimation of parameter, hazard function, drinking water quality.

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## INTRODUCTION

Although the effectiveness for analysis of univariate data where the variability of measurement may be affected by many factors such as sampling, analytical procedure and physio-chemical environment, simple statistical applications viz. mean, mode, SD, skewness, kurtosis etc. on environmental data provides only the pattern of distribution (symmetric and asymmetric) of data (Stephenson (**2003**)). Several distribution theories and order statistics related to drinking water quality data have been reported by various authors e.g. Vanrolleghum *et al.*, (**1996**) and Leiva *et al.*, (**2008**). But similar approaches using water quality data in Assam is lacking. As such an attempt has been made distribution theory with application in water quality data.

In environmental studies, observations fall in the non-experimental, non-replicated and nonrandom categories. Therefore, making random selection from the observed population is impossible. Thus, problems of model specification and data interpretation acquire great importance. A way of confronting this problem is by considering observations which are selected with probability proportional to their length. The resulting distribution is called the length-biased distribution (Patil (**2002**)). On the other hand, non-observation and damage of data set results reduction of value or adoption of a sampling plan which gives unequal probabilities to the various units and cannot be considered as a random sample from the theoretical distribution. For these types of data sets the weighted distribution can be applied (Rao (**1985**). Majority of water quality data are non uniform data therefore some work has been done in this paper using the concept of weighted distribution (WD) and length biased (LB) distributions.

Patil (**2002**) reported that if $X$ be a non negative random variable (r.v.) with pdf $f_X(x)$, then the weighted r.v. is denoted by $X_w$ with weight function $w(x)$ whose density function is called the weighted distribution. The pdf of weighted distribution is given by

$$f_{X_w}(x)=\frac{w(x)f_X(x)}{E[w(x)]} \quad ;x>0$$

Assuming that $E[w(x)]<\propto$ i.e. the first moment of w(x) exists. The length biased distribution is a particular case of weighted distribution which can be denoted by a r.v t which has pdf expressed as

$$f_T(t)=\frac{t\ f_X(t)}{\mu}\,;t>0\,;\mu>0$$

Where $\mu=E[x]$. The length biased distribution has been applied in various fields such as biometry, ecology, environmental sciences, and reliability and survival analysis. A review of this distribution and their applications are included in Gupta and Kirmani (**1990**), Das *et al.*, (**2011**) and Das and Roy (**2011**). Leiva *et al.*, (**2008**) reported that length biased Inverse Gaussian distribution and Birnbaum Saunder's distribution have been developed by various authors using environmental data and worked successfully. As such, in this paper, weighted Birnbaum Saunder's distribution (WBSD) as well as length biased distributions (LBD) has been derived and the behavior of these newly developed distributions also have been studied setting various properties.

The Birnbaum Saunders (BS) distribution is an important positively skewed probability model, which has attractive properties. Due to the interesting theoretical arguments the BS model is appropriate for describing cumulative degradation processes. The Birnbaum Saunders distribution was derived from a model showing that failure is due to the development and growth of a dominant crack Birnbaum and Saunders (**1969**). The two parameter Birnbaum Saunders distribution was originally proposed by Birnbaum and Saunders (**1969**). Although the BS distribution was originally proposed as a failure time distribution for fatigue failure under the assumption that the failure is due to the development and growth of a dominant crack a more general deviation was provided by Desmond (**1985**) based on a biological model. Some work on the BS distribution has been found in Balakrishnan *et. al.*, (**2007**) Chang and Tang (**1993**) and Chang and Tang (**1994**), Duphis and Mills (**1998**), Owen (**2006**) and Xie and Wei (**2007**) A concise review of different developments of BS distribution is found in the book by Johnson *et al.*, (**1995**). Mann *et al.*, (**1974**)] mentioned that hazard function of the BS distribution is not an increasing function. For details about theoretical aspects and old and new applications of the BS model, including for example the environmental area, see Johnson *et al.*, (**1995**). Balakrishnan *et. al.*, (**2007**) and Sanhueza *et al.*, (**2008**)

Birnbaum and Saunder's distribution (**1969a**) have proposed the distribution with pdf

$$p_x(x;\alpha,\beta)=\frac{1}{2\alpha\beta}\left(\frac{x}{\beta}\right)^{\frac{-1}{2}}\left[1+\left(\frac{x}{\beta}\right)^{-1}\right]\frac{1}{2\pi}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}\right)-2+\left(\frac{\beta}{x}\right)\right], x>0;\alpha,\beta>0 \quad (1)$$

to represent the distribution of lifetime of components under specified conditions of wear (a "fatigue life" distribution).

In the rest of the paper, in section **1.1** WBSD and LBWBSD have been derived. Various properties of LBWBSD have been studied in section **1.2**. Section **1.4**, and section **1.5** attempts estimation of parameters of LBWBSD, and fitting of LBWBSD. Finally, conclusion has been given in section **1.6**.

## 1.1 Weighted and Length Biased Version of Birnbaum Saunder's Distribution

In this section, Weighted and Length Biased Version of transformed Birnbaum Saunder's distribution has been derived.

Using transformations $u=\frac{\beta}{x}$ to equation (1) we have,

$$f(u)=\frac{\exp\left(\frac{1}{\alpha^2}\right)}{2\alpha\sqrt{2\pi}}u^{-\frac{3}{2}}(1+u)\exp\left[-\left\{\frac{1}{2\alpha^2}\left(1+u^2\right)\right\}\right],\quad u>0,\alpha>0 \quad (2)$$

Again using weight function $w(u)=\exp\left(-\frac{1}{2\alpha^2 u}\right)(1+u)^{-1}, u>0$ we have the weighted Birnbaum- Saunder's (WBS) and length biased Birnbaum Saunder's distribution respectively as,

$$f_{U_w}(u)=\frac{\left(\frac{1}{2\alpha^2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}u^{-\frac{1}{2}}\exp\left(-\frac{u}{2\alpha^2}\right),\quad u>0,\alpha>0 \quad (3)$$

$$f_{U_w}(u)=\frac{\left(\frac{1}{2\alpha^2}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}u^{\frac{1}{2}}\exp\left(-\frac{u}{2\alpha^2}\right),\quad u>0,\alpha>0 \quad (4)$$

## 1.2 Properties of LBD of WBSD

The rth order moment of LBD is given by

$$\mu_r' = \frac{\Gamma\left(r+\frac{3}{2}\right)}{\left(\frac{1}{2\alpha^2}\right)^r \Gamma\left(\frac{3}{2}\right)} \tag{5}$$

moment generating function (mgf) of LBD is given by

$$M_u(t) = \left(1-2\alpha^2 t\right)^{-\frac{3}{2}} \tag{6}$$

The cumulant generating function (cgf) is given by

$$K_u(t) = -\frac{3}{2}\log\left(1-2\alpha^2 t\right) \tag{7}$$

From the expression given in (6) and (7) we can find out moments, mean, variance, Skewness and kurtosis of the LBD of WBS distribution as follows

$$\kappa_1 = \mu_1' = 3\alpha^2, \kappa_2 = \mu_2 = 6\alpha^4, \kappa_3 = \mu_3 = 24\alpha^6, \kappa_4 = \mu_4 + 3\kappa_2^{\ 2} = 144\alpha^8, \beta_1 = 2.67, \beta_2 = 4.75$$

The cumulative distribution function for LBD of WBSD is given by

$$F_L(U_w) = \frac{\left(\frac{1}{2\alpha^2}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} \sum_{j=0}^{\infty} \frac{\left(-\frac{1}{2\alpha^2}\right)^j}{j!} \left[\frac{v^{(j+\frac{3}{2})}}{(j+\frac{3}{2})}\right]$$

The corresponding survival function, hazard function and reverse hazard function are given by

$$S_L(U_w) = 1 - \frac{\left(\frac{1}{2\alpha^2}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} \sum_{j=0}^{\infty} \frac{\left(-\frac{1}{2\alpha^2}\right)^j}{j!} \left[\frac{v^{(j+\frac{3}{2})}}{(j+\frac{3}{2})}\right]$$

$$\lambda_L(U_w) = \frac{\frac{\left(\frac{1}{2\alpha^2}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} u^{\frac{1}{2}} \exp\left(-\frac{u}{2\alpha^2}\right)}{1 - \frac{\left(\frac{1}{2\alpha^2}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} \sum_{j=0}^{\infty} \frac{\left(-\frac{1}{2\alpha^2}\right)^j}{j!} \left[\frac{u^{\left(j+\frac{3}{2}\right)}}{\left(j+\frac{3}{2}\right)}\right]}$$

$$\tau_L(U_w)=\frac{\dfrac{\left(\dfrac{1}{2\alpha^2}\right)^{\frac{3}{2}}}{\Gamma\left(\dfrac{3}{2}\right)}u^{\frac{1}{2}}\exp\left(-\dfrac{u}{2\alpha^2}\right)}{\dfrac{\left(\dfrac{1}{2\alpha^2}\right)^{\frac{3}{2}}}{\Gamma\left(\dfrac{3}{2}\right)}\sum_{j=0}^{\infty}\dfrac{\left(-\dfrac{1}{2\alpha^2}\right)^j}{j!}\left[\dfrac{u^{\left(j+\frac{3}{2}\right)}}{\left(j+\dfrac{3}{2}\right)}\right]}$$

## 1.3 Behavior of LBWBSD

The probability density function $f(u)$ is twice differentiable function for a r.v. u. Considering $\eta(u)=\dfrac{-f'(u_w)}{f(u_w)}$ we see that the first derivative of $\eta(u)$ exist, i.e. $\eta'(u)$ exist and

$$\eta'(u)=-\frac{1}{2u}+\frac{1}{2\alpha^2}$$

Again, $\eta'(u)<0,$ for $u>0,$ therefore, according to Glaser's (**1980**) lemma the hazard function of LBWBSD is monotonically increasing function (IFR) in $u$.

Moreover, considering derivative of the hazard function we have

$$\lambda_L{}'(U_w)=\frac{c\exp\left(-\dfrac{u}{2\alpha^2}\right)\left[\dfrac{1}{2u^{\frac{1}{2}}}-\dfrac{u^{\frac{1}{2}}}{2}\right]}{1-c\displaystyle\sum_{j=0}^{\infty}\dfrac{\left(-\dfrac{1}{2\alpha^2}\right)^j}{j!}\left[\dfrac{u^{\left(j+\frac{3}{2}\right)}}{\left(j+\dfrac{3}{2}\right)}\right]}+\frac{c^2\exp\left(-\dfrac{u}{2\alpha^2}\right)\displaystyle\sum_{j=0}^{\infty}\dfrac{-\left(\dfrac{1}{2\alpha^2}\right)^j}{j!}u^{(1+j)}}{\left[1-\displaystyle\sum_{j=0}^{\infty}\dfrac{-\left(\dfrac{1}{2\alpha^2}\right)^j}{j!}\dfrac{u^{\left(j+\frac{3}{2}\right)}}{\left(j+\dfrac{3}{2}\right)}\right]}$$

where, $c=\dfrac{\left(-\dfrac{1}{2\alpha^2}\right)^{\frac{3}{2}}}{\Gamma\left(\dfrac{3}{2}\right)}$, By observing $\lambda_L{}'(U_w)$ it is seen that

a. The LBWBSD has a decreasing failure rate (DFR) if $c>0$
b. The LBWBSD has a increasing failure rate (IFR) if $c<0$
c. The LBWBSD has a constant failure rate (CFR) if $c=0$

### 1.4 Estimation

Using the method of moments the estimation of parameter for LBD is given as $\hat{\alpha} = \sqrt{\frac{m_1'}{3}}$, where $m_1'$ is the sample moment.

### 1.5 Application

Sampling was done using a stratified simple random procedure from 1st March to 31$^{st}$ October, 2009 in three administrative sub-divisions of Nagaon district of Assam, India. A total of 78 water samples (66 Tube Well, 8 Deep Tube Well, 2 Ring Well and 2 Supply Water) were collected randomly from 9 villages from three sub-divisions viz. Nagaon, Kaliabor, and Hojai. Three villages from Hojai sub-division viz. Nilbagan, Udali and Dimoru, five stations from Nagaon sub-division (Kampur Town, South Haibargaon, Halowabhakatgaon, Baruabli and Barghat) and greater Kaliabor town area were selected randomly for water sampling which is in 5 km minimum distance apart. Out of the total bore well or ring well water sources 74% sources are only source of water and 26% sources are complementary source of water.

Water samples were collected after 10 minutes of initial pumping in pre cleaned polythene containers of one liter capacity, were rinsed out 3 - 4 times and transported to laboratory at 10$^0$C. The containers were filled up to the mouths and then tightly stopper to avoid contact with air or to prevent agitation during transport. The storage and preservation of samples were done with standard procedure (APHA, **1995**). Water temperature and pH are measured at the time of collection of the sample. In the laboratory water samples were vacuum filtered using 0.45 µm pre-washed and pre-weighed membranes and the filtrates containing dissolved metals were poured in to clean plastic containers then acidified with concentrated nitric acid to pH 2 to 3. Since the government of Assam has declared 14 districts of Assam as drought/drought like condition1n the year 2009 including Nagaon district also the rainfall received (% departure from the normal) in this district during the above period was -27%. Maximum and minimum temperature recorded 36$^0$C and 22$^0$C respectively during the study period. The water quality parameter estimation and calibration of equipments were done using standard methods and techniques APHA (**1995**).

**Zinc** is an essential element and is necessary for the effective functioning of various enzyme systems, deficiency of which leads to growth retardation, immaturity and anemia. Symptoms of zinc toxicity in human include vomiting, dehydration, electrolyte imbalance, abdominal pain, nausea, lethargies, dizziness and lack of muscular coordination.

**Table 1: The Observed Frequencies and Expected Frequencies of Zinc Concentration.**

| Class Interval | Observed Frequency($O_i$) | Expected Frequency($E_i$) |
|---|---|---|
| Below 0.50 | 24 | 26 |
| 0.50-0.90 | 18 | 21 |
| 0.90 -1.30 | 17 | 12 |
| 1.30-1.70 | 5 | 5 |
| 1.70-2.10 | 2 | 2 |

The chi-square test of goodness of fit is applied to drinking water quality data. The no. of class intervals over which the computations were found was 5 for Zinc concentration. Fitting of LBD of WBSD was performed by Simpson's Rule of numerical integration with the help of C++.

## CONCLUSION

Table 1. shows that esimated value of the parameter LBWBSD is $\hat{\alpha} = 0.28$ and $\chi^2 = 2.653$ for LBWBSD. In this article, the LBD of weighted Birnbaum-Saunder's (WBSD) was developed. This new distribution turns out to be quite flexible for modeling water quality. An application to the real data showed that this two new model is flexible alternative to other well known models. Thus, we have developed a new probability model which might be of use for practitioners in the environmental sciences. Here we postulated that LBD of WBSD is appropriate for modeling water quality.

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