

## **Assumptions and errors in the Lorentz force equation in Electrostatics**

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### **ABSTRACT**

*All forces throughout the different branches of physics are ultimately derived from the Principle of Conservation of Energy, from the equation  $\mathbf{F} = d\mathbf{E}/d\mathbf{l}$ , or force=rate of change of energy over distance. Different disciplines derive equations from this to simplify their calculations. In electrostatics, however, the derivation of electrostatic forces from the energy conservation principles is rarely seen, and consequently the classical Lorentz force equation  $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$  can be erroneously applied in circumstances where its application violates this principle. This discussion paper analyses the Lorentz force equation by a rigorous analysis of the Principle of Conservation of Energy, and identifies situations where the Lorentz equation violates that Principle. It identifies the limits over which the Lorentz force equation is valid, and discusses how the exceptions need to be handled.*

**Keywords:** Electrostatics, Lorentz force equation.

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### **INTRODUCTION**

The Lorentz force equation [1] deals with the force on a charged particle when subjected to electric and magnetic fields. It is generally given as...

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

...where  $\mathbf{E}$  is the electric field vector,  $\mathbf{B}$  is the magnetic field vector,  $\mathbf{v}$  is the particle velocity,  $\times$  is the vector cross-product operator,  $q$  is the charge, and  $\mathbf{F}$  is the vector force generated on the particle. At the same time, for energy to be conserved, we must obey the equation...

$$\mathbf{F} = d\mathbf{U}/d\mathbf{L}$$

This describes the force  $\mathbf{F}$  experienced when potential energy  $U$  changes over distance  $d\mathbf{L}$ .

This paper analyses the Lorentz force equation by a rigorous analysis of the Principle of Conservation of Energy, and shows that there situations where the equation cannot apply, since it violates that principle. The electrostatic forces created within a system of charged particles are not caused directly by the effect on point charges of an electric field, but indirectly through

changes in the energy densities of electric fields when charges are brought together. This creates forces from the classic equation  $F=dU/dL$  where  $F$  is force,  $U$  is the potential energy and  $L$  is length.

## MATERIALS AND METHODS

Consider the first part of the Lorentz force equation (the electrostatic part) [1] - there are three conditions to consider.

1. Where the equation applies.
2. Where it does not apply, in that it claims there is no force where energy conservation expects a force.
3. Where it does not apply, in that it claims there is a force when energy conservation expects no force.

### 1. Where the Lorentz force equation applies

Consider a pair of distributed electric charges  $q_1$  and  $q_2$  whose electric fields extend to infinity, as happens with electrons and protons. Separate them by a distance  $L$  and chose the co-ordinates so that one charge is at  $[-L/2,0]$  and the other at  $[L/2,0]$ .

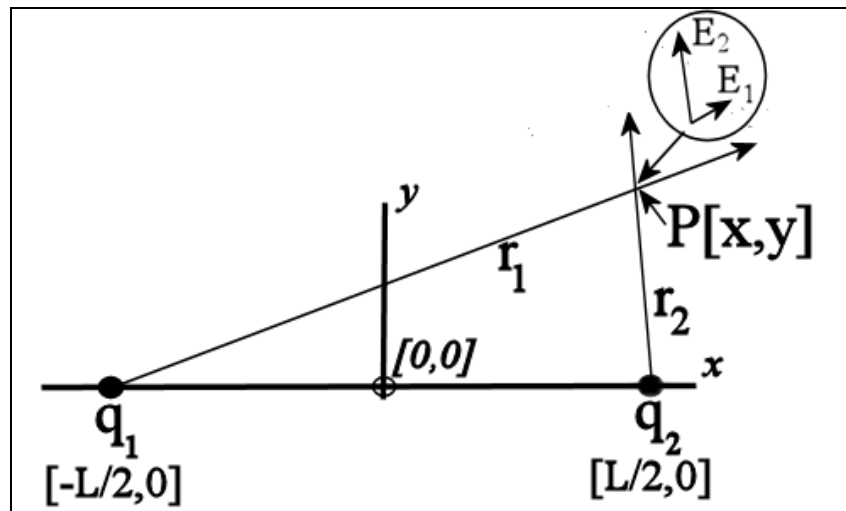


Figure 1

Since each charge's field extends to infinity their fields interact over all space. At any one specific point in space each charge will provide its own vector electric field  $E_1$  and  $E_2$  respectively. Each charge would *in isolation* have an associated energy density at this point in its field given by:-

$$dU/dv = \epsilon \cdot |E_1|^2 / 2$$

$$dU/dv = \epsilon \cdot |E_2|^2 / 2$$

(Here  $\epsilon$  is the permittivity,  $U$  is the potential energy, and  $dU/dv$  is the energy associated with that volume of space.) The individual fields will add vectorially at this point to give a resultant electric field  $E_R$  with associated energy density  $\epsilon \cdot |E_R|^2 / 2$ . Now if the resultant energy density is less than the sum of the individual "in isolation" energy densities, bringing the charges together in this arrangement has reduced the energy at that point, and this is associated with attractive forces at that point in space; if the resultant energy density is more, it takes work to move the charges into this arrangement and so this is associated with repulsive forces.

We can take the first's electric field strength at a distance  $r_1$  from the centre of it as  $q_1/4\pi\epsilon r_1^2$  and the second's field at a distance  $r_2$  from it as  $q_2/4\pi\epsilon r_2^2$ , with  $q_1$  and  $q_2$  being the respective strengths of the charges. From this we can look at how the energy density changes from the "in isolation" value at this point. By subtracting the "in isolation" energy density from all values we get the potential energy; this is zero at infinite separation and positive or negative when the charges are brought together. The potential energy density at a point in space for a separation  $\mathbf{L}$  is then...

$$dU/dv = \epsilon_0(|\mathbf{E}_R|^2 - |\mathbf{E}_1|^2 - |\mathbf{E}_2|^2)/2$$

To find the total potential energy at a separation  $\mathbf{L}$ , integrate over all space. Because of its symmetry we can simply multiply by a half-rotation round the  $q_1$ - $q_2$  axis, and then integrate over the other two axes...

$$\begin{aligned} U &= (\pi\epsilon_0/2) \iint (|\mathbf{E}_{1x} + \mathbf{E}_{2x}|^2 + |\mathbf{E}_{1y} + \mathbf{E}_{2y}|^2 - |\mathbf{E}_1|^2 - |\mathbf{E}_2|^2) y \, dy \, dx \\ &= (q_1 q_2 / 16\pi\epsilon_0) \iint ( (x^2 - L^2/4 + y^2) y / (x^4 - L^2 x^2/2 + L^4/16 + L^2 y^2/2 + 2x^2 y^2 + y^4)^{3/2} ) \, dx \, dy \\ &= q_1 q_2 / 4 \pi \epsilon_0 L \end{aligned}$$

Differentiating this gives the force at any separation  $\mathbf{L}$ :-

$$\mathbf{F} = dU/d\mathbf{L} = -q_1 q_2 / 4\pi\epsilon L^2$$

Since the electric field at a distance  $L$  from one charge is  $q_2/4\pi\epsilon L^2$  we can re-imagine this as one point charge lying in the distributed electric field of the other, at a radius  $\mathbf{r}$  from it, so replacing separation  $\mathbf{L}$  with radius  $\mathbf{r}$ , and switching from interacting charges to a "point-charge-in-an-external-field" concept where the force is positive if aligned with the external field  $\mathbf{E}_2$  (hence changing the sign of the force)...

$$\begin{aligned} \mathbf{F} &= -q_1 q_2 / 4\pi\epsilon L^2 = -q_1 (q_2 / 4\pi\epsilon L^2) \\ &\equiv q_1 (q_2 / 4\pi\epsilon \mathbf{r}^2) = q_1 \mathbf{E}_2 \end{aligned}$$

While this is a very useful and effort-saving technique, it must always be born in mind that it works *only* because the potential energy in this arrangement is  $-q_1 q_2 / 4\pi\epsilon L$ . This will not necessarily be the case if the integration limits are changed so that less than all of space is included, or if either or both of the electric fields do not obey the inverse-square law. Further, in any valid model of the universe all charges must have the same structure - the concept that one charge can be a point charge and the other a distributed field is no more than a mathematical convenience. This becomes quite surreal in a system of many charges where the concept that any one of them - and just one - can be a "test point charge" in the distributed fields of the others is clearly only a computational convenience, and not a true expression of the electrostatic model.

## 2. Where the Lorentz equation expects no force and the energy computation expects a force

What if the potential energy is not  $-q_1 q_2 / 4\pi\epsilon L$ ? In 1932 James Chadwick discovered the neutron [2]. This was an oddity in that it was found to have no long-range force, attracted other neutrons at very short range, and at even shorter range repelled them. This attractive-repulsive short-range force is extremely difficult to model using the Lorentz force equation, and attempts to do so have often led to complexity in the electric field structures used to model the inter-neutron force electrostatically. It has been shown that when approached from the Principle of Conservation of

Energy rather than the Lorentz force equation a simple inverse-square-law electrostatic field which is truncated sharply at a fixed radius exhibits the same attraction/repulsion character as a neutron [3]. The Lorentz equation fails to work here because the interaction between two neutrons is not over all space, as required in its derivation. It is beyond the scope of this paper to repeat such a detailed analysis here, but to summarise just *why* the Lorentz equation fails to demonstrate this behaviour consider Figure 2, which shows two such neutron models at three different separations, one drawn in blue and the other in black. The centre and the field periphery of each is drawn.

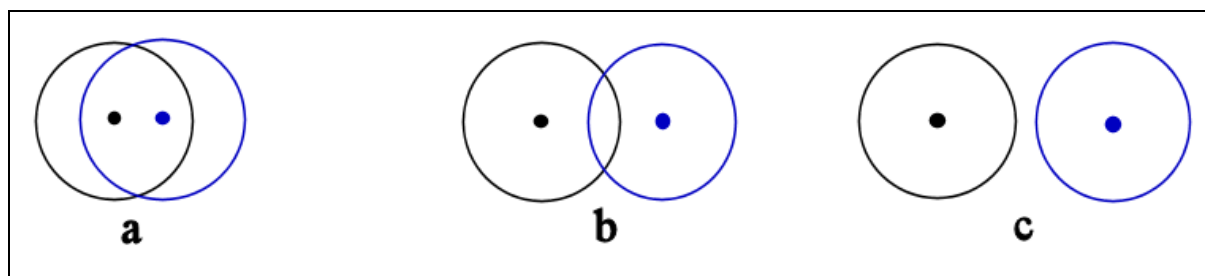


Figure 2

Figure 2a shows the electric fields overlapping with the centre of each inside the electric field of the other. In Figure 2b it shows them still overlapping, but separated so that the centre of each is outside the electric field of the other. In 1c it shows them completely separated with no overlap of the electric fields. The different scenarios are described below:-

- a. In Figure 2a the centre of each is inside the field of the other, and one expects from the Lorentz equation that the forces will be repulsive. This is in fact confirmed from energy computations, but the Lorentz force computation return a higher force than the energy computation because the field overlap does not extend to infinity, which is an implicit assumption in the Lorentz equation.
- b. In Figure 2b the centre of each is outside the field of the other, so the Lorentz computation, treating the centre of one as being a point charge lying in the distributed field of the other, returns zero force. However, there is still an overlap, and the energy computation returns a force because there of the interaction between the fields.
- c. In Figure 2c the centre of each is outside the field of the other so the Lorentz computation returns zero. Equally, there is no overlap of the fields so the energy computation also returns zero.

Lorentz force computations are modelled on the centre of a charged entity lying within the electric field of another, while energy computations look at the total field interactions to determine the energy changes in them. In geometries like this we can therefore expect disagreement between the methods.

### 3. Where the Lorentz force expects a force and the energy computation expects no force

Consider a homogeneous electric field. The normal way of creating this is with two plate electrodes facing each other, one positively charged (the anode) and the other negatively charged (the cathode), as shown in Figure 3a.

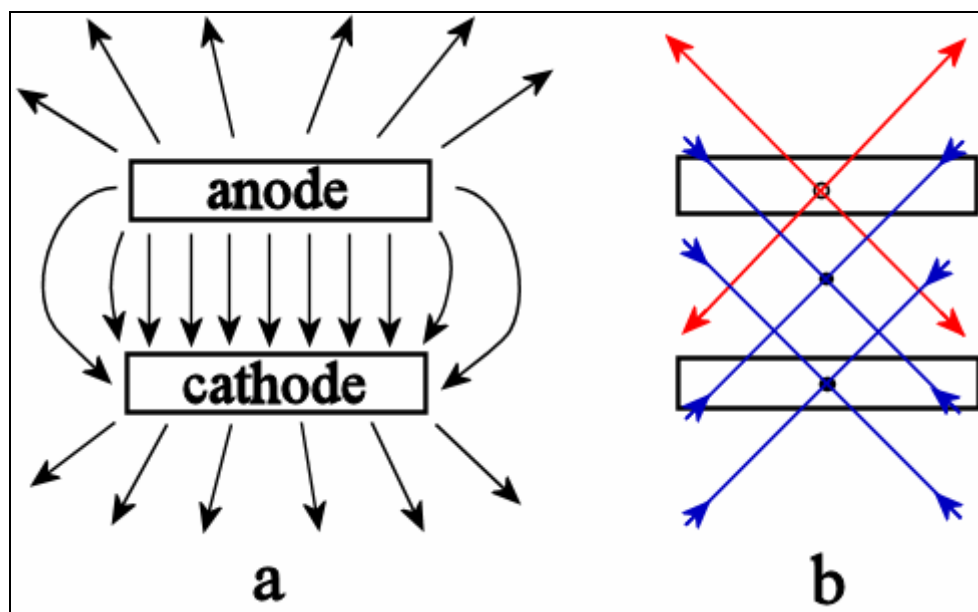


Figure 3

As can be seen, although the region between the anode and the cathode may be homogeneous, the anode must by necessity be surrounded by a positive electric field and the Cathode by a negative one. The field around the anode is generated by a surplus of protons (equivalent to a dearth of electrons) and the cathode by a surplus of electrons. However, the homogeneous field between the anode and the cathode is no more than the super-position of a massive number of polar charge distributions, and Figure 3b shows how the fields of a proton from the anode (in red to represent positive charge) and of an electron (in blue) from the cathode both interact with a test electron in the gap, their interaction covering all of space. The force felt by the test electron is the sum of all the repulsive forces between it and each individual electron on the cathode, plus the sum of all the attractive forces between it and each individual proton on the cathode. Since all these fields are inverse-square-law and extend to infinity, the Lorentz equation applies. (*Note that while the surface of each electrode is a Faraday cage, this is just a balancing mechanism to minimise the energy in the charge distribution on the electrode; it cannot block external fields penetrating the electrode or matter itself would fall apart.*)

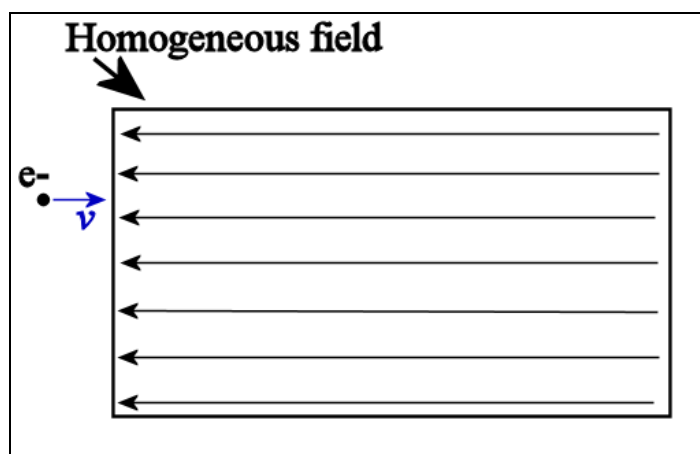


Figure 4

But what if a perfect homogeneous field could be created? In this mind experiment, construct a volume of space where there is a homogeneous electric field surrounded by a zero field as shown in Figure 4. Consider a beam of electrons entering the field at its positive edge.

According to the Lorentz equation each electron will accelerate continuously across this volume till it exits from the other side with added kinetic energy, simply because of the electric field. However, one has to ask from where the energy comes. No matter how many electrons are accelerated through the volume, the electric field and the energy in its volume remain unchanged. Clearly no energy comes from this source. Nor does it come from the electrons, for once they have exited the field they take all their electric field energy, plus their new kinetic energy. In the previous scenario where the homogeneous field is created by an anode of excess protons and a cathode of excess electrons, the energy to accelerate an electron placed in the field comes from the interaction between their fields and the reduction in potential energy as the electron changes its position with respect to protons and other electrons; here there is no such energy source. We can see this by assuming a near-infinite volume and moving a test electron around inside it, keeping it far enough away from the edges of the volume so that no significant part the test electron's electric field energy falls outside the volume to complicate the energy analysis. Wherever we place the electron inside this volume the interaction between it and the homogeneous field is virtually identical; there is no change in the energy wherever the electron is placed. Hence, without an energy differential, there is no motive force to accelerate the electron. Hence there is a marked difference between the Lorentz force equation and the energy conservation analysis.

It is also interesting to examine the negative and positive ends of the field using energy computations, as there is an energy differential there between the field and the surrounding space. The Lorentz equation is silent on these edges, merely reporting that the electron starts accelerating when its centre passes into the field, and continuously accelerates until its centre passes out of the field. However, energy analysis gives a very different picture. Consider Figure 5, which shows the electric field vector interaction between an electron and the positive edge of the homogeneous field, with the black vertical being the boundary of the homogeneous field, and the homogeneous field's electric field flux lines shown in red; the electron's flux lines are shown in blue. The field interactions at several points are shown by the circle cut-aways, with the electron's electric field vector in blue, the homogeneous field vector in red, and the resultant vector in black.

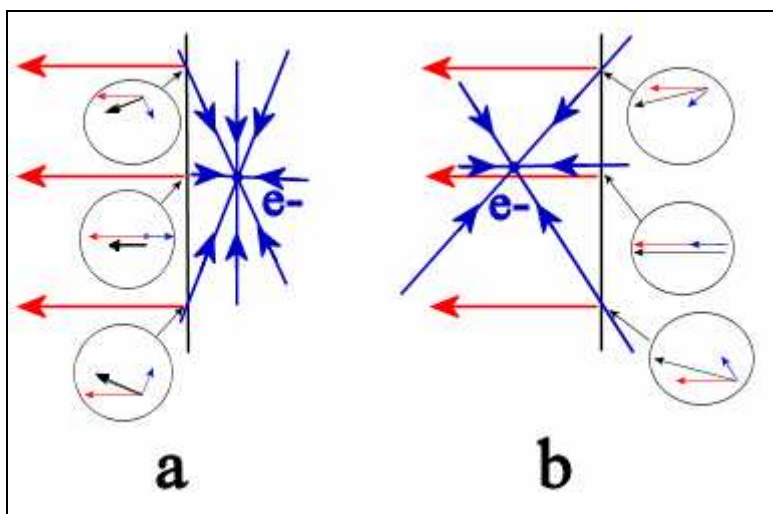


Figure 5

As the electron approaches the positive edge of the field from the right in Figure 5a it sees an attractive force because as the leading parts of the electron's field enter the homogeneous field they partially cancel its electric field to reduce the energy densities, providing attractive forces that attract the electron into the homogeneous field on the left. In Figure 5b, after the electron's centre has passed through the edge into the field and the trailing parts of the electron's field enter the homogeneous field they reinforce the electric field, increasing the energy densities and creating repulsive forces. This attractive/repulsive behaviour is similar to the neutron behaviour modelled in [3] and is a common feature of the energy analysis of bounded electric fields, and here it causes the electron to oscillate across the positive edge of the field. The situation is reversed on the negative edge of the field, where the electron is repelled by the edge no-matter which side of that edge it is on, whether inside or outside the perfect homogeneous field.

### The induction component of the Lorentz force equation

The induction component of the Lorentz force equation is  $\mathbf{F}=q(\mathbf{v}\times\mathbf{B})$ , conceptually based on the motion-induced electric field at a point being given by  $\mathbf{E}=(\mathbf{v}\times\mathbf{B})$ , equivalent to the force on the moving charge being  $q\mathbf{E}$ . Consider an electron moving through a fixed magnetic field, as shown in Figure 6; the signs in circles at the top represent vector tails, representing the magnetic field as pointing down into the page.

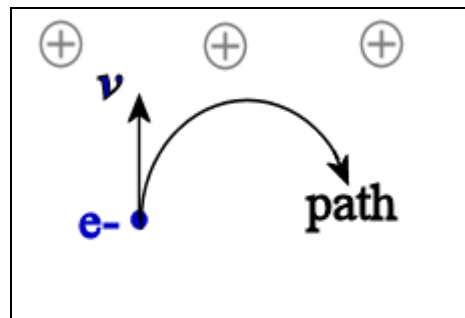


Figure 6

The electron is seen to follow a circular path and this is explained as follows. The electric field induced in the reference frame of the moving charge by its motion through the fixed magnetic field is at right angles to the direction of motion. This accelerates the charge transversely to its direction, and this change of direction changes the direction of the induced field so that it remains at right angles to the direction of motion. This causes the charge to follow a circular path as shown rather than be accelerated away to the right of Figure 5. The consequence of this is that no energy is expended in accelerating the charge as there is no change in kinetic energy. On the face of it this explanation seems reasonable.

However, for the sideways force to be present there must be a potential energy source, such that if the direction of the induction field did *not* change with the charge's change in direction this energy would be consumed in accelerating the electron away to the right of Figure 4. Where is this energy source? If the field is created by the opposing poles of a large magnet the induced electric field in the region between the poles is similar to the "perfect homogeneous electric field" described previously, with well-defined edges to the electric field at the edge of the magnet. There is a fringing field, but this merely softens the edge of the field. Moving the moving charge anywhere inside the magnet – so long as it is kept clear of the edges – produces no change in the energy densities around the charge, so in this explanation there is no potential energy to provide the force. The explanation that the charge experiences a force dependent purely on the transverse electric field, although apparently reasonable, is fundamentally flawed.

There is however one potential source of energy. It has been suggested that a charged particle entering a transverse electric field will be made to rotate by the entry forces [4]. Although the exact mechanism is unclear it may be that if the direction of the induction field did *not* change with the charge's change in direction this rotational energy would be consumed in providing the sideways acceleration.

### CONCLUSION

The Lorentz force equation is a very useful tool in simplifying the calculation of forces between charges, where those charges have an inverse-square-law polar distribution and extend to infinity. It is possible it can be used elsewhere provided the experimenter is aware of these limitations and can satisfy himself that the equation is still effective. However, in the most general case it may be necessary to return to first principles and integrate the energy density functions to find the total potential energy and differentiate that over distance to obtain forces. Such an approach has a firm foundation in the Principle of Conservation of Energy, and leads to results that match experiment better for configurations like the strong nuclear force. However, the complexity of the limits applied to the interacting space for bounded electric fields means that the technique will normally have to be integration by finite element analysis rather than the Calculus.

### REFERENCES

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