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Advances in Applied Science Research, 2011, 2 (4):320-327

ISSN: 0976-8610
CODEN (USA): AASRFC

Applicability of Length Biased Weighted Generalized Rayleigh Distribution

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ABSTRACT

The length-biased form of the Weighted Generalized Rayleigh distribution (WGRD) known as length-biased Weighted Generalized Rayleigh distribution is considered. From the length-biased WGRD, well known distributions are generated by expanding suitable functions of the parameters. Properties of length-biased WGRD are discussed. The parameters of the length-biased WGRD are obtained by the method of moments and fitted accordingly.

Keywords: Length-biased, Generalized Rayleigh distribution, Weighted Generalized Rayleigh Distribution, moments.

INTRODUCTION

Weighted distribution concept can be traced from the study of Fisher [2]. Rao [14] introduced distribution of the type $f^w(x) = \frac{w(x)f(x)}{W}$ where $W = \int w(x)f(x)$ with an arbitrary non-negative function w(x) which may exceed unity and gave practical examples where w(x) = x or x^α are appropriate. He called distributions with arbitrary weight w(x) weighted distributions. Zelen [23] introduced weighted distributions to represent what he broadly perceived as length-biased sampling. A study of size biased sampling and related from invariant weighted distributions was made by Patil and Ord [11]. Weighted distributions in general and length-biased distributions in particular are very useful and convenient for the analysis of lifetime data. Weighted distributions occur frequently in research related to reliability, biomedicine, ecology and several other areas. Various works are done to characterize relationships between original distributions and their length-biased forms. A table for some basic distributions and their length-biased forms is given by [12]

Rayleigh distribution is an important distribution in statistics and operations research. It is applied in several areas such as health, agriculture, biology, and other sciences. If x_i, $1 \leq i \leq n$, are Gaussian variates and r^2 is the sum of their squares, then we called r a generalized Rayleigh random variable. Surles and Padgett [19] (see also Surles and Padgett; 2005 [20]) introduced two-parameter Burr Type X distribution and correctly named as the generalized Rayleigh distribution. The two-parameter generalized Rayleigh distribution is a particular member of the generalized Weibull distribution, originally proposed by Mudholkar and Srivastava [9] (see also Mudholkar, Srivastava and Freimer [10]). Several aspects of the one-parameter (scale parameter equals one) generalized Rayleigh distribution were studied by Sartawi and Abu-Salih [16], Jaheen [6], [7], Ahmad et al. [1], Raqab [15] and Surles and Padgett [21]. Urban Hjorth [4], proposed another generalization of Rayleigh distribution. Voda [22] proposed a new generalized Rayleigh distribution.

MATERIALS AND METHODS

Secondary data has been used for the study. The data has been collected from Regional Meteoroligical Centre, LGBI Airport, Guwahati [3]. The data relates to the monthly mean minimum temperature for a period of seven years i.e. from 2003-2010 of Dhubri, Assam, India. To perform the analysis of the data we have derived the length-biased form of the Weighted Generalized Rayleigh distribution.

To introduce the concept of a weighted distribution, suppose x is a nonnegative random variable (rv) with its natural probability density function (pdf) f(x). Let the weight function be w(x) which is a non-negative function. Then the weighted density function f(x) is obtained as

$$f(x) = \frac{w(x)f(x)}{\int_{-\infty}^{\infty} w(x)f(x)} \qquad x > 0 \qquad (1)$$

assuming that $E(X) = \int_{-\infty}^{\infty} w(x)f(x) < \infty$ i.e the first moment of w(x) exists.
By taking weight as w(x) = x we obtain length-biased distribution.

Length-Biased Weighted Generalized Rayleigh Distribution (WIGD)
The probability density function (pdf) of the Generalized Rayleigh Distribution is given by

$$f(x) = \frac{2}{(2\sigma^2)^{N/2}\Gamma(N/2)} x^{N-1} exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad x > 0, \sigma > 0 \quad (2)$$

This generalized form of the Rayleigh distribution is also referred in literature as the chi distribution with N degrees of freedom and scale parameter σ.

We consider the weight function as

$$w(x) = x^{2c-N} exp\left(-x^2\left(c\sigma^2 - \frac{1}{2\sigma^2}\right)\right) \qquad (3)$$

Hence using (2) and (3) in (1) we obtain the density function of the WGRD of the form

$$f(x) = \frac{2(c\sigma^2)^c x^{2c-1} exp\left(-x^2(c\sigma^2)\right)}{\Gamma(c)} \quad x > 0, c > 0, \sigma > 0 \qquad (4)$$

Taking weight as w(x) =x, from (4) we obtain length-biased WGRD of the form

$$f(x) = \frac{2(c\sigma^2)^{c+\frac{1}{2}} x^{2c} exp\left(-x^2(c\sigma^2)\right)}{\Gamma\left(c+\frac{1}{2}\right)} \quad x > 0, c > 0, \sigma > 0 \qquad (5)$$

Figure 1 shows the variations of the pdf for different values of c and σ

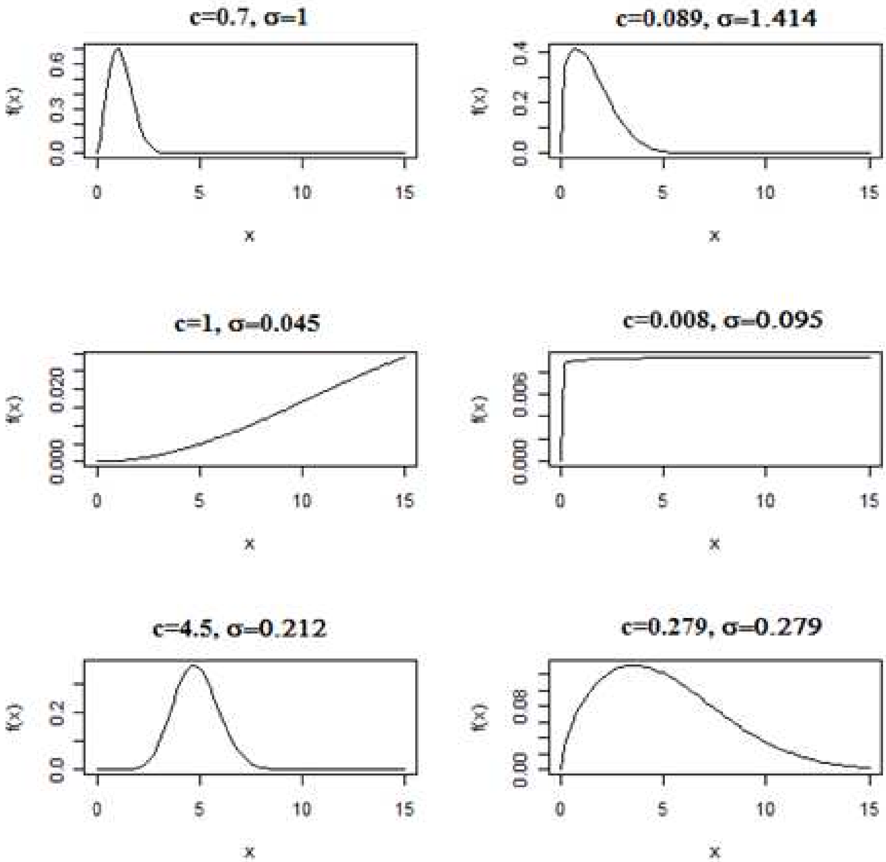

Figure 1. Probability density function of the Length-Biased Weighted Generalized Rayleigh distribution.

The case when c = 1/2,$\sigma^2 = \frac{1}{\sigma^2}$, (5) gives the Rayleigh density function of the form

$$f(x) = \frac{x\ exp\left(-\frac{x^2}{2\sigma^2}\right)}{\sigma^2} \quad x > 0, \sigma > 0$$

Properties of the Length-Biased Weighted Generalized Rayleigh Distribution

The cumulative distribution function (cdf) is

$$F(x) = \frac{(x^2 c\sigma^2)^{c+\frac{1}{2}}}{\Gamma\left(c+\frac{1}{2}\right)} \sum_{m=0}^{\infty} \frac{(-1)^m (c\sigma^2)^m x^{2m}}{m!\ \left(c+m+\frac{1}{2}\right)}$$

The survival function or reliability function is

$$R(x) = \frac{\Gamma\left(c+\frac{1}{2}\right) - (x^2 c\sigma^2)^{c+\frac{1}{2}}}{\Gamma\left(c+\frac{1}{2}\right)} \sum_{m=0}^{\infty} \frac{(-1)^m (c\sigma^2)^m x^{2m}}{m!\left(c+m+\frac{1}{2}\right)}$$

The hazard rate or instantaneous failure rate is

$$h(x) = \frac{2(c\sigma^2)^{c+\frac{1}{2}} x^{2c} exp\left(-x^2(c\sigma^2)\right)}{\Gamma\left(c+\frac{1}{2}\right) - (x^2 c\sigma^2)^{c+\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m (c\sigma^2)^m x^{2m}}{m!\left(c+m+\frac{1}{2}\right)}}$$

The rth moment about the origin is

$$\mu_r' = \frac{\Gamma\left(c+\frac{r+1}{2}\right)}{\Gamma\left(c+\frac{1}{2}\right)(c\sigma^2)^{\frac{r}{2}}}$$

For the case, r = 1, 2, 3, 4 we have

$$\mu_1' = \frac{\Gamma(c+1)}{\Gamma\left(c+\frac{1}{2}\right)(c\sigma^2)^{\frac{1}{2}}}$$

$$\mu_2' = \frac{\Gamma\left(c+\frac{3}{2}\right)}{\Gamma\left(c+\frac{1}{2}\right)(c\sigma^2)} = \frac{\left(c+\frac{1}{2}\right)}{(c\sigma^2)}$$

$$\mu_3' = \frac{\Gamma(c+2)}{\Gamma\left(c+\frac{1}{2}\right)(c\sigma^2)^{\frac{3}{2}}}$$

$$\mu_4' = \frac{\left(c+\frac{3}{2}\right)\left(c+\frac{1}{2}\right)}{(c\sigma^2)^2}$$

The first four central moments are

$$\mu_1 = 0$$

$$\mu_2 = \frac{1}{(c\sigma^2)}\left[\left(c+\frac{1}{2}\right) - \frac{\Gamma^2(c+1)}{\Gamma^2\left(c+\frac{1}{2}\right)}\right]$$

$$\mu_3 = \frac{\Gamma(c+1)}{\Gamma\left(c+\frac{1}{2}\right)(c\sigma^2)^{\frac{3}{2}}}\left[(c+1) - 3\left(c+\frac{1}{2}\right) + 2\frac{\Gamma^2(c+1)}{\Gamma^2\left(c+\frac{1}{2}\right)}\right]$$

$$\mu_4 = \frac{1}{(c\sigma^2)^2}\left[\left(c+\frac{3}{2}\right)\left(c+\frac{1}{2}\right) - 4\frac{\Gamma(c+1)\Gamma(c+2)}{\Gamma^2\left(c+\frac{1}{2}\right)} + 6\frac{\Gamma^2(c+1)\left(c+\frac{1}{2}\right)}{\Gamma^2\left(c+\frac{1}{2}\right)} - 3\frac{\Gamma^4(c+1)}{\Gamma^4\left(c+\frac{1}{2}\right)}\right]$$

Measure of skewness is

$$\gamma_1 = \frac{\Gamma(c+1)}{\Gamma\left(c+\frac{1}{2}\right)}\frac{\left[(c+1) - 3\left(c+\frac{1}{2}\right) + 2\frac{\Gamma^2(c+1)}{\Gamma^2\left(c+\frac{1}{2}\right)}\right]}{\left[\left(c+\frac{1}{2}\right) - \frac{\Gamma^2(c+1)}{\Gamma^2\left(c+\frac{1}{2}\right)}\right]^{\frac{3}{2}}}$$

Measure of kurtosis is

$$\gamma_2 = \frac{\left[\frac{\Gamma(c+2)}{\Gamma(c)} - 4\frac{\Gamma(c+1)\Gamma\left(c+\frac{3}{2}\right)}{\Gamma^2(c)} + 6\frac{\Gamma(c+1)\Gamma^2\left(c+\frac{1}{2}\right)}{\Gamma^3(c)} - 3\frac{\Gamma^4\left(c+\frac{1}{2}\right)}{\Gamma^4(c)}\right]}{\left[\left(c+\frac{1}{2}\right) - \frac{\Gamma^2(c+1)}{\Gamma^2\left(c+\frac{1}{2}\right)}\right]^2} - 3$$

The moment generating function is

$$M_X(t) = \frac{1}{\Gamma\left(c+\frac{1}{2}\right)}\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{\Gamma\left(c+\frac{r+1}{2}\right)}{(c\sigma^2)^{\frac{r}{2}}}$$

The cumulant generating function is

$$K_X^w(t) = \ln\left[\frac{1}{\Gamma\left(c+\frac{1}{2}\right)}\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{\Gamma\left(c+\frac{r+1}{2}\right)}{(c\sigma^2)^{\frac{r}{2}}}\right]$$

The first four cumulants are

$$\kappa_1 = \frac{\Gamma(c+1)}{\Gamma\left(c+\frac{1}{2}\right)(c\sigma^2)^{\frac{1}{2}}}$$

$$\kappa_2 = \frac{1}{(c\sigma^2)}\left[\left(c+\frac{1}{2}\right) - \frac{\Gamma^2(c+1)}{\Gamma^2\left(c+\frac{1}{2}\right)}\right]$$

$$\kappa_3 = \frac{\Gamma(c+1)}{\Gamma\left(c+\frac{1}{2}\right)(c\sigma^2)^{\frac{3}{2}}}\left[(c+1) - 3\left(c+\frac{1}{2}\right) + 2\frac{\Gamma^2(c+1)}{\Gamma^2\left(c+\frac{1}{2}\right)}\right]$$

$$\kappa_4 = \frac{1}{(c\sigma^2)^2}\left[\left(c+\frac{3}{2}\right)\left(c+\frac{1}{2}\right) - 4\frac{\Gamma(c+1)\Gamma(c+2)}{\Gamma^2\left(c+\frac{1}{2}\right)} + 6\frac{\Gamma^2(c+1)\left(c+\frac{1}{2}\right)}{\Gamma^2\left(c+\frac{1}{2}\right)} - 3\left(c+\frac{1}{2}\right)^2 - 6\frac{\Gamma^4(c+1)}{\Gamma^4\left(c+\frac{1}{2}\right)}\right]$$

The characteristic function is

$$\phi_X^w(t) = \frac{1}{\Gamma\left(c+\frac{1}{2}\right)}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\frac{\Gamma\left(c+\frac{r+1}{2}\right)}{(c\sigma^2)^{\frac{r}{2}}}$$

The harmonic mean is

$$H = \frac{\Gamma\left(c+\frac{1}{2}\right)}{\Gamma(c)(c\sigma^2)^{\frac{1}{2}}}$$

The recurrance relation is given by

$$\mu'_{r+2} = \frac{\Gamma\left(c+\frac{r+1}{2}\right)}{(c\sigma^2)}\mu'_r$$

Estimation

Method of moments is used to estimate the parameters of the length-biased WGRD. This method follows by equating the population moments

$$\mu'_r = \frac{\Gamma\left(c+\frac{r+1}{2}\right)}{\Gamma\left(c+\frac{1}{2}\right)(c\sigma^2)^{\frac{r}{2}}}$$

to the sample moments. Solving this will yield $\hat{c}$ and $\hat{\sigma}$ the moment estimates of c and σ respectively. The estimated parameters are

$\hat{c} = \frac{0.5\left(3-\frac{m_4'}{m_2'^2}\right)}{\left(\left(\frac{m_4'}{m_2'^2}\right)-1\right)}$ and $\hat{\sigma} = \sqrt{\left(\frac{(c+0.5)}{cm_2'}\right)}$ where m_2' and m_4' are the second and fourth sample moments.

Application

The monthly mean minimum temperature data has been fitted with length-biased WGRD (5). For the problem chosen, N=96, $\bar{x}$=19.3. Moment estimates for $\sigma$and c are $\hat{\sigma}$=0.053009 and $\hat{c}$=4.39.

Table 1 Monthly mean minimum temperature in degree Celsius

Temperature	Observed Frequency	Expected Frequency
9.8-13.8	8	8
13.8-17.8	17	16
17.8-21.8	20	23
21.8-25.8	33	33
$\geq$ 25.8	18	16
Total	96	96
χ^2		0.7038

CONCLUSION

An application to the real data showed that the length-biased WGRD is found to be a suitable distribution for modeling monthly mean minimum temperature pattern data. Thus length-biased WGRD is found to be suitable use for the practitioner in the environment sciences.

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