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Original Article

Andrews-Garvan-Liang's Spt-crank for Marked Overpartitions

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ABSTRACT

In 2009, Bingmann, Lovejoy and Osburn have shown the generating function for $\overline{spt}_2(n)$. In 2012, Andrews, Garvan, and Liang have defined the $\overline{sptcrank}$ in terms of partition pairs. In this article the number of smaller parts in the overpartitions of n with smallest part not overlined and even are discussed, and the vector partitions and $\overline{S}$-partitions with 4 components, each a partition with certain restrictions are also discussed. The generating function for $\overline{spt}_2(n)$, and the generating function for $M_{\overline{S}_2}(m,n)$ are shown with a result in terms of modulo 3. This paper shows how to prove the Theorem 1, in terms of $M_{\overline{S}_2}(m,n)$ with a numerical example, and shows how to prove the Theorem 2, with the help of $\overline{sptcrank}$ in terms of partition pairs. In 2014, Garvan and Jennings-Shaffer are capable to define the $\overline{sptcrank}$ for marked overpartitions. This paper also shows another result with the help of 15 $\overline{SP}_2$-partition pairs of 8 and shows how to prove the Corollary with the help of 15 marked overpartitions of 8.

INTRODUCTION

In this paper we give some related definitions of $\overline{spt}_2(n)$, various product notations, vector partitions and $\overline{S}$-partitions, $M_{\overline{S}_2}(m,n)$, $M_{\overline{S}_2}(m,t,n)$, $\overline{S}_2(z,x)$, marked partition and $\overline{sptcrank}$ for marked overpartitions. We discuss the generating function for $\overline{spt}_2(n)$ and prove the Corollary 1 with the help of generating function to prove the Result 1 with the help of 3 vector partitions from $\overline{S}_2$ of 4. We prove the Theorem 1 with the help of various generating functions and prove the Corollary 2 with a special series $\overline{S}_2(z,x)$, when $n=1$ and prove the Theorem 2 with the help of $\overline{sptcrank}$ in terms of partition pairs (λ_1,λ_2) when $0<s(\lambda_1)\le s(\lambda_2)$. We prove the Result 2 using the $\overline{crank}$ of partition pairs

$\vec{\lambda} = (\lambda_1, \lambda_2)$ and prove the Corollary 3 and 4 with the help of marked overpartition of $3n$ and of $3n+1$ (when n = 2) respectively. Finally we analyze the Corollary 5 with the help of marked overpartitions of $5n+3$ when n =1.

Some related definitions

In this section we have described some definitions related to the article following[7].

$\overline{spt_2}(n)$[4]: The number of smallest parts in the overpartitions of n with smallest part not overlined and even is denoted by $\overline{spt_2}(n)$ for example,

n			$\overline{spt_2}(n)$
1	:		0
2	:	$\dot{2}$	1
3	:		0
4	:	$\dot{4}$, $\dot{2}+\dot{2}$	3
5	:	$3+\dot{2}$, $\overline{3}+\dot{2}$	2
...	…	…	

From above we get;

$\overline{spt_2}(6) = 6,\ \overline{spt_2}(7) = 6, ...$

Product notations

$(x)_\infty = (1-x)(1-x^2)(1-x^3)...$

$(x^2;x^2)_\infty = (1-x^2)(1-x^4)...$

$(x)_k = (1-x)(1-x^2)(1-x^3)...(1-x^k)$

$(-x^5;x)_\infty = (1+x^5)(1+x^6)(1+x^7)...$

Vector partitions and $\overline{S}$-partitions

A vector partition can be done with 4 components each partition with certain restrictions [5]. Let, $\vec{V} = D \times P \times P \times D$, where D denote the set of all partitions into distinct parts, P denotes the set of all partitions. For a partition π, we let, $s(\pi)$ denotes the smallest part of π (with the convention that the empty partition has smallest part ∞), $\#(\pi)$ the number of parts in π, and $|\pi|$ the sum of the parts of π.

For $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4) \in \vec{V}$, we define the weight $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$, the crank c$(\vec{\pi})$ =$\#(\pi_2)-\#(\pi_3)$, the norm $|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4|$.

We say $\vec{\pi}$ is a vector partition of n if $\vec{\pi} = n$. Let $\overline{S}$ denotes the subset of $\overline{V}$ and it is given by;

$\overline{S} = \{(\pi_1, \pi_2, \pi_3, \pi_4) \in \vec{V}, 1 \le s(\pi_1) < \infty, s(\pi_1) \le s(\pi_2), s(\pi_1) \le s(\pi_3), s(\pi_1) < s(\pi_4)\}$.

Let $\overline{S_2}$ denotes the subset of $\overline{S}$ with $s(\pi_1)$ even.

$M_{\overline{S}_2}(m,n)$: The number of vector partitions of n in $\overline{S}_2$ with crank m are counted according to the weight ω is exactly $M_{\overline{S}_2}(m,n)$.

$M_{\overline{S}_2}(m,t,n)$: The number of vector partitions of n in $\overline{S}_2$ with crank congruent to m modulo t are counted according to the weight ω is exactly $M_{\overline{S}_2}(m,t,n)$.

$\overline{S}_2$(z, x): The series $\overline{S}_2$(z, x) is defined by the generating function for $M_{\overline{S}_2}(m,n)$.

i.e., $\overline{S}_2$(z, x)

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S}_2}(m,n) z^m x^n .$$

Marked Partition[1]: We define a marked partition as a pair (λ, k) where λ is a partition and k is an integer identifying one of its smallest parts i.e., k =1, 2, ..., $\nu(\lambda)$, where $\nu(\lambda)$ is the number of smallest parts of λ.

$\overline{sptcrank}$ for Marked overpartitions[6]: We define a marked overpartitions of n as a pair (π, j) where π is an overpartition of n in which the smallest part is not overlined and even. It is clear that $\overline{spt_2}$ (n) = # of marked overpartitions (π, j) of n. For example, there are 3 marked overpartitions of 4, like:
(4,1), $(\overline{2}+2,1)$, and $(\overline{2}+2,2)$.
Then, $\overline{spt}_2(4) = 3$.

The generating function for $\overline{spt_2}$ (n)

The generating function (Bringmann *et al*. have shown in[5]) for $\overline{spt_2}$ (n) is given by;

$$\sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$$

$$= \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(x^3;x)_\infty} + \frac{x^4(-x^5;x)_\infty}{(1-x^4)^2(x^5;x)_\infty} + \ldots$$

$$= o.x + 1.x^2 + o.x^3 + 3.x^4 + 2.x^5 + 6.x^6 + \ldots$$

$$= \overline{spt_2}(1)x + \overline{spt_2}(2)x^2 + \overline{spt_2}(3)x^3 + \overline{spt_2}(4)x^4 + \overline{spt_2}(5)x^5 + \ldots.$$

$$= \sum_{n=1}^{\infty} \overline{spt_2}(n)x^n.$$

For convenience, define $\overline{spt_2}$ (1) = 0.

From above we get $\overline{spt_2}(3) = 0$, $\overline{spt_2}(6) = 6, \ldots$

i.e., $\overline{spt_2}(3.1) = 0 \equiv 0$ (mod 3),
$\overline{spt_2}(3.2) = 6 \equiv 0$ (mod 3), ...

We can conclude that $\overline{spt_2}(3n) \equiv 0 \pmod 3$.

We also get $\overline{spt_2}(4) = 3$, $\overline{spt_2}(7) = 6, \ldots$

i.e., $\overline{spt_2}(3+1) = 3 \equiv 0 \pmod 3$,
$\overline{spt_2}(3.2+1) = 6 \equiv 0 \pmod 3$, ...

We can conclude that $\overline{spt_2}(3n+1) \equiv 0 \pmod 3$[4]. Again from above we get;
$\overline{spt_2}(3) = 0$, $\overline{spt_2}(8) = 15, \ldots$
i.e., $\overline{spt_2}(3) = 0 \equiv 0 \pmod 5$,
$\overline{spt_2}(5+3) = 15 \equiv 0 \pmod 5$, ...

We can conclude that $\overline{spt_2}(5n+3) \equiv 0 \pmod 5$.

Corollary 1

$$\overline{spt_2}(n) = \sum_{m=-\infty}^{\infty} M_{\bar{S}_2}(m,n).$$

Proof

The generating function for $M_{\bar{S}_2}(m,n)$ is given by;

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\bar{S}_2}(m,n)\, z^m x^n$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.$$

If z = 1, then,

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\bar{S}_2}(m,n)\, x^n$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(x^{2n};x)_\infty(x^{2n};x)_\infty}$$

$$= \frac{x^2(-x^3;x)_\infty(x^3;x)_\infty}{(x^2;x)_\infty^2} + \frac{x^4(-x^5;x)_\infty(x^5;x)_\infty}{(x^4;x)^2_\infty} + \ldots$$

$$= \frac{x^2(-x^3;x)_\infty(1-x^3)(1-x^4)\ldots}{(1-x^2)^2(1-x^3)^2\ldots} + \frac{x^4(-x^5;x)_\infty(1-x^5)(1-x^6)\ldots}{(1-x^4)^2(1-x^5)^2\ldots} + \ldots$$

$$= \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(1-x^3)(1-x^4)\ldots} + \frac{x^4(-x^5;x)_\infty}{(1-x^4)^2(1-x^5)(1-x^6)\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \overline{spt}_2(n)x^n .$$

i.e., $\sum_{n=1}^{\infty} \overline{spt}_2(n)x^n = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S}_2}(m,n)\, x^n$.

Now equating the co-efficient of x^n from both sides we get;

$$\overline{spt}_2(n) = \sum_{m=-\infty}^{\infty} M_{\overline{S}_2}(m,n).$$

Hence the Corollary.

RESULT 1

$$M_{\overline{S}_2}(0,3,4) = M_{\overline{S}_2}(1,3,4) = M_{\overline{S}_2}(2,3,4) = \frac{1}{3}\overline{spt}_2(4).$$

Proof

We prove the result with the help of examples. We see the vector partitions from $\overline{S}_2$ of 4 along with their weights and cranks and are given as follows: (See table 1.)

Here we have used ϕ to indicate the empty partition. Thus we have,

$M_{\overline{S}_2}(0,3,4) = 1,\quad M_{\overline{S}_2}(1,3,4) = 1$,

$M_{\overline{S}_2}(2,3,4) = M_{\overline{S}_2}(-1,3,4) = 1$

$\therefore M_{\overline{S}_2}(0,3,4) = M_{\overline{S}_2}(1,3,4)$

$= M_{\overline{S}_2}(2,3,4) = 1 = \frac{1}{3}.3 = \frac{1}{3}\overline{spt}_2(3)$.

Hence the Result.

Now from above table we get; $\sum \omega(\vec{\pi}) = 3$, i.e., $\sum_{k=0}^{2} M_{\overline{S}_2}(k,3,4) = 3$.

$$\therefore \overline{spt}_2(4) = \sum_{k=0}^{2} M_{\overline{S}_2}(k,3,4) = \sum \omega(\vec{\pi}).$$

Now we can define;

$$M_{\overline{S}_2}(k,t,n) = \sum_{m \equiv k \pmod t} M_{\overline{S}_2}(m,n)$$

and

$$\overline{spt}_2(n) = \sum_{m=-\infty}^{\infty} M_{\overline{S}_2}(m,n) = \sum_{k=0}^{t-1} M_{\overline{S}_2}(k,t,n).$$

Theorem 1

The number of vector partitions of n in $\overline{S}_2$ with crank m counted according to the weight ω is non-negative, i.e., $M_{\overline{S}_2}(m,n) \geq 0$.

Proof

The generating function for $M_{\overline{S}_2}(m,n)$ is given by;

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_{\overline{S}_2}(m,n) z^m x^n$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.(x^{4n+2};x^2)_\infty .$$

[Since $\sum_{n=1}^{\infty} (x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty$

$= (x^3;x)_\infty(-x^3;x)_\infty + (x^5;x)_\infty(-x^5;x)_\infty + \ldots$

$= (1-x^3)(1-x^4)\ldots(1+x^3)(1+x^4)\ldots +$
$(1-x^5)(1-x^6)\ldots(1+x^5)\ldots + \ldots$

$= (1-x^6)(1-x^8)\ldots + (1-x^{10})(1-x^{12})\ldots +$
$(1-x^{14})\ldots + \ldots$

$= (x^6;x^2)_\infty + (x^{10};x^2)_\infty + \ldots$

$= \sum_{n=1}^{\infty} (x^{4n+2};x^2)_\infty$]

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{4n};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.\frac{(x^{4n+2};x^2)_\infty}{(x^{4n};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(x^{4n};x)_\infty}{(zx^{2n};x)_\infty(z^{-1}x^{2n};x)_\infty}.\frac{1}{(1-x^{4n})(x^{4n+1};x^2)_\infty}$$

[Since,

$$\sum_{n=1}^{\infty} \frac{(x^{4n+2};x^2)_\infty}{(x^{4n};x)_\infty} = \frac{(x^6;x^2)_\infty}{(x^4;x)_\infty} + \frac{(x^{10};x^2)_\infty}{(x^8;x)_\infty} + \ldots$$

$$= \frac{(1-x^6)(1-x^8)\ldots}{(1-x^4)(1-x^5)(1-x^6)\ldots} + \frac{(1-x^{10})(1-x^{12})\ldots}{(1-x^8)(1-x^9)(1-x^{10})(1-x^{11})\ldots} + \ldots$$

$$= \frac{1}{(1-x^4)(1-x^5)(1-x^7)\ldots} + \frac{1}{(1-x^8)(1-x^9)(1-x^{11})\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{1-x^{4n}} \cdot \frac{1}{(x^{4n+1};x^2)_\infty}]$$

$$= \sum_{n=1}^{\infty} x^{2n} \sum_{k=0}^{\infty} \frac{(z^{-1}x^{2n})^k}{(zx^{2n+k};x)_\infty (x)_k} \cdot \frac{1}{(1-x^{4n})(x^{4n+1};x^2)_\infty}$$

[Since, $\sum_{n=1}^{\infty} \frac{x^{2n}(x^{4n};x)_\infty}{(zx^{2n};x)_\infty (z^{-1}x^{2n};x)_\infty}$

$$= \sum_{n=1}^{\infty} x^{2n} \sum_{k=0}^{\infty} \frac{(z^{-1}x^{2n})^k}{(zx^{2n+k};x)_\infty (x)_k}] \text{. (by}^3)$$

We see that the coefficient of any power x in the right hand side is non-negative so the coefficient $M_{\overline{S}_2}(m,n)$ of $z^m x^n$ is non-negative, i.e., $M_{\overline{S}_2}(m,n) \geq 0$. Hence the Theorem.

Numerical example 1

The vector partitions from $\overline{S_2}$ of 5 along with their weights and cranks are given as follows: (See table 2.)

Here we have used ϕ to indicate the empty partition. Thus we have;

$M_{\overline{S}_2}(0,5) = 1-1 = 0$, $M_{\overline{S}_2}(1,5) = 1$, and

$M_{\overline{S}_2}(-1,5) = 1$, i.e., $\sum_m M_{\overline{S}_2}(m,5) = 2$,

i.e., every term is non-negative, i.e., $M_{\overline{S}_2}(m,n) \geq 0$.

So we can conclude that, $M_{\overline{S}_2}(m,n) \geq 0$.

Corollary 2

$$\overline{S}_2(1,x) = \sum_{n=1}^{\infty} \overline{spt_2}(n) x^n.$$

Proof

We get;

$$\overline{S}_2(z,x) = \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(zx^{2n};x)_\infty (z^{-1}x^{2n};x)_\infty} \quad 2$$

If $z = 1$, then we get;

$$\overline{S}_2(1,x) = \sum_{n=1}^{\infty} \frac{x^{2n}(x^{2n+1};x)_\infty(-x^{2n+1};x)_\infty}{(x^{2n};x)_\infty (x^{2n};x)_\infty}$$

$$= \frac{x^2(x^3;x)_\infty(-x^3;x)_\infty}{(x^2;x)_\infty^2} + \frac{x^4(-x^5;x)_\infty(x^5;x)_\infty}{(x^4;x)_\infty^2} + \ldots$$

$$= \frac{x^2(-x^3;x)_\infty(1-x^3)(1-x^4)\ldots}{(1-x^2)^2(1-x^3)^2\ldots} + \frac{x^4(-x^5;x)_\infty(1-x^5)(1-x^6)\ldots}{(1-x^4)^2(1-x^5)^2\ldots} + \ldots$$

$$= \frac{x^2(-x^3;x)_\infty}{(1-x^2)^2(1-x^3)\ldots} + \frac{x^4(-x^5;x)_\infty}{(1-x^4)^2(1-x^5)\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \overline{spt_2}(n) x^n.$$

i.e., $\overline{S}_2(1,x) = \sum_{n=1}^{\infty} \overline{spt_2}(n) x^n$. Hence the Corollary.

Theorem 2

$$\overline{spt_2}(n) = \sum_{\substack{\overline{\lambda} \in \overline{SP}_2 \\ |\overline{\lambda}| = |\lambda_1| + |\lambda_2| = n}} 1$$

Proof

First we define the $\overline{sptcrank}$ in terms of partition pairs,

$$\overline{SP} = \{\overline{\lambda} = (\lambda_1, \lambda_2) \in P \times P : 0 < s(\lambda_1) \leq s(\lambda_2)$$

and all parts of λ_2 that are $\geq 2s(\lambda_1)+1$ are odd}.

Let $\overline{SP}_2$ be the set of $\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}$ with $s(\lambda_1)$ even. The generating function for $\overline{spt}_2(n)$ is given by;

$$\sum_{n=1}^{\infty} \overline{spt}_2(n)x^n = \sum_{n=1}^{\infty} \frac{x^{2n}(-x^{2n+1};x)_\infty}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty}(-x^{2n+1};x)_\infty$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty} \cdot \frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}$$

[Since,

$$\sum_{n=1}^{\infty}(-x^{2n+1};x)_\infty = (-x^3;x)_\infty + (-x^5;x)_\infty + \ldots$$

$$= (1+x^3)(1+x^4)\ldots + (1+x^5)(1+x^6)\ldots + (1+x^7)(1+x^8)\ldots + \ldots$$

$$= \frac{(1-x^6)(1-x^8)\ldots}{(1-x^3)(1-x^4)\ldots} + \frac{(1-x^{10})(1-x^{12})\ldots}{(1-x^5)(1-x^6)\ldots} + \frac{(1-x^{14})\ldots}{(1-x^7)\ldots} + \ldots$$

$$= \frac{(x^6;x^2)_\infty}{(x^3;x)_\infty} + \frac{(x^{10};x^2)_\infty}{(x^5;x)_\infty} + \ldots.$$

$$= \sum_{n=1}^{\infty} \frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}\,]$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty} \cdot \frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{(x^{2n};x)_\infty} \cdot \frac{1}{(1-x^{2n})} \cdot \frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty}$$

[Since, $\sum_{n=1}^{\infty} \frac{x^{2n}}{(1-x^{2n})^2(x^{2n+1};x)_\infty}$

$$= \frac{x^2}{(1-x^2)^2(x^3;x)_\infty} + \frac{x^4}{(1-x^4)^2(x^5;x)_\infty} + \ldots$$

$$= \frac{x^2}{(1-x^2)^2(1-x^3)(1-x^4)\ldots} + \frac{x^4}{(1-x^4)^2(1-x^5)(1-x^6)\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{(x^{2n};x)_\infty} \cdot \frac{1}{(1-x^{2n})}\,]$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{(x^{2n};x)_\infty} \cdot \frac{1}{(1-x^{2n})(1-x^{2n+1})\ldots(1-x^{4n})(x^{4n+1};x^2)_\infty}$$

[Since,

$$\sum_{n=1}^{\infty} \frac{(x^{4n+2};x^2)_\infty}{(x^{2n+1};x)_\infty} = \frac{(x^6;x^2)_\infty}{(x^3;x)_\infty} + \frac{(x^{10};x^2)_\infty}{(x^5;x)_\infty} + \ldots$$

$$= \frac{(1-x^6)(1-x^8)\ldots}{(1-x^3)(1-x^4)\ldots} + \frac{(1-x^{10})(1-x^{12})\ldots}{(1-x^5)(1-x^6)\ldots(1-x^{10})(1-x^{11})\ldots} + \ldots$$

$$= \frac{1}{(1-x^3)(1-x^4)(1-x^5)\ldots} + \frac{1}{(1-x^5)(1-x^6)\ldots(1-x^9)(1-x^{11})\ldots} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{(1-x^{2n+1})\ldots(1-x^{4n})(x^{4n+1};x^2)_\infty}\,]$$

$$= \sum_{n=1}^{\infty} \sum_{\substack{\lambda_1 \in P \\ S(\lambda_1)=n}} x^{|\lambda_1|} \sum_{\substack{\lambda_2 \in P \\ s(\lambda_2) \geq n}} x^{|\lambda_2|}$$

All parts in $\lambda_2 \geq 2n+1$ are odd

$$= \sum_{n=1}^{\infty} \sum_{\substack{\bar{\lambda} \in \overline{SP}_2 \\ |\bar{\lambda}| = |\lambda_1| + |\lambda_2| = n}} x^{|\bar{\lambda}|}.$$

Equating the co-efficient of x^n from both sides we get;

$$\overline{spt}_2(n) = \sum_{\substack{\bar{\lambda} \in \overline{SP}_2 \\ |\bar{\lambda}| = |\lambda_1| + |\lambda_2| = n}} 1.$$

Hence the Theorem.

Numerical example 2

The overpartitions of 6 with smallest parts not overlined and even are 6, 4+2, $\bar{4}+2$, and 2+2+2. Consequently, the number of

smallest parts in the overpartitions of 6 with smallest part not overlined and even is given by;
$\dot{6}$ $4+\dot{2}$, $\overline{4}+\dot{2}$, $\dot{2}+\dot{2}+\dot{2}$,
so that $\overline{spt}_2(6)=6$ i.e., there are $6\ \overline{SP}_2$-partition pairs of 6 like:
$(6,\phi)$, $(4+2,\phi)$, $(2,4)$, $(2+2+2,\phi)$, $(2+2,2)$ and (2, 2+2).

RESULT 2

$$M_{\overline{S}_2}(0,5,8)=M_{\overline{S}_2}(1,5,8,)=M_{\overline{S}_2}(2,5,8,)=M_{\overline{S}_2}(3,5,8)=M_{\overline{S}_2}(4,5,8)=3=\frac{1}{5}\overline{spt}_2(8).$$

Proof

We prove the result with the help of examples. We can define a $\overline{crank}$ of partition pairs $\vec{\lambda}=(\lambda_1,\lambda_2)\in\overline{SP}_2$.

For $\vec{\lambda}=(\lambda_1,\lambda_2)\in\overline{SP}_2$, we define, $k(\vec{\lambda})=$ # of pairs j in λ_2 such that $s(\lambda_1)\le j\le 2\ s(\lambda_1)-1$, and also define;

$$\overline{crank}(\vec{\lambda})=\begin{cases}(\#\text{of parts of } \lambda_1\ge s(\lambda_1)+k)-k; \\ \text{if } k>0 \\ (\#\text{of parts of } \lambda_1)-1; \text{if } k=0\end{cases}$$

where $k=k(\vec{\lambda})$.

We know that $\overline{spt}_2(8)=15$. There are 15 $\overline{SP}_2$-partition pairs of 8. (See table 3.)
From the table we get;

$$M_{\overline{S}_2}(0,5,8)=M_{\overline{S}_2}(1,5,8,)=M_{\overline{S}_2}(2,5,8,)=M_{\overline{S}_2}(3,5,8)=M_{\overline{S}_2}(4,5,8)=3=\frac{1}{5}\overline{spt}_2(8)$$

. Hence the Result.

Now we will describe the $\overline{sptcrank}$ of a marked overpartition[6]. To define the $\overline{sptcrank}$ of a marked overpartition we first need to define a function $k(m,n)$ for positive integers *m*, *n* such that $m\ge n+1$, we write $m=b2^j$, where *b* is odd and $j\ge o$. For a given odd integer *b* and a positive integer *n* we define $j_0=j_0(b,n)$ to be the smallest non-negative integer j_0 such that $b2^{j_0}\ge n+1$.

We define;

$$k(m,n)=\begin{cases}0, \text{if } b\ge 2n \\ 2^{j-j_0} \text{ if } b2^{j_0}<2n \\ 0, \text{if } b2^{j_0}=2n.\end{cases}$$

For a marked overpartitions (π,j) we let π_1 be the partition formed by the non-overlined parts of π, π_2 be the partition (into distinct parts) formed by the overlined parts of π so that $s(\pi_2)>s(\pi_1)$, we define $\overline{k}(\pi,i)=\nu(\pi_1)-j+k(\pi_2,s(\pi_1))$, where $\nu(\pi_1)$ is the number of smallest parts of π_1.

Now we can define;

$$\overline{sptcrank}(\pi,j)=\begin{cases}(\#\text{of parts of } \pi_1\ge s(\pi_1)-\overline{k}), \\ \text{if } \overline{k}=\overline{k}(\pi,j)>0 \\ (\#\text{of parts of } \pi_1)-1; \\ \text{if } \overline{k}=\overline{k}(\pi,j)=0.\end{cases}$$

Corollary 3[8]

The residue of the $\overline{sptcrank}(\text{mod}\,3)$ divides the marked overpartitions of 3*n* with the smallest part not overlined and even into 3 equal classes.

Proof

We prove the Corollary with the help of an example when *n* = 2. There are 6 marked overpartitions of 3*n* (when *n* = 2) with the smallest part not overlined and even so that, $\overline{spt}_2(6)=6$.

We see that the residue of the $\overline{sptcrank}(\mathrm{mod}\,3)$ divides the marked overpartitions of 3*n* (when *n* = 2) with smallest part not overlined and even into 3 equal classes. Hence the Corollary. (See table 4.)

Corollary 4

The residue of the $\overline{sptcrank}(\mathrm{mod}\,3)$ divides the marked overpartitions of 3*n*+1 with smallest part not overlined and even into 3 equal classes.

Proof

We prove the Corollary with the help of an example when *n* = 2. There are 6 marked overpartitions of 7 with the smallest part not overlined and even, so that $\overline{spt}_2(7) = 6$. We see that the residue of the $\overline{sptcrank}$ (mod 3) divides the marked overpartitions of 3*n*+1 (when *n* = 2) with smallest part not overlined and even. Hence the Corollary. (See table 5.)

Corollary 5

The residue of the $\overline{sptcrank}(\mathrm{mod}\,5)$ divides the marked overpartitions of 5*n*+3 with smallest part not overlined and even into 5 equal classes.

Proof

We prove the Corollary with the help of example when *n* = 1. There are 15 marked overpartitions of 5*n* + 3 (when *n* =1) with the smallest part not overlined and even so that $\overline{spt}_2(8) = 15$.

We see that the residue of the $\overline{sptcrank}(\mathrm{mod}\,5)$ divides the marked overpartitions of 8 with the smallest part not overlined and even into 5 equal classes. Hence the corollary. (See table 6.)

CONCLUSION

In this study we have found the number of smallest parts in the overpartitions of *n* with the smallest part not overlined and even for *n*=1, 2, 3, 4 and 5. We have shown various relations $\overline{spt}_2(3n) \equiv 0(\mathrm{mod}\,3)$,

$\overline{spt}_2(3n+1) \equiv 0(\mathrm{mod}\,3)$,

$\overline{spt}_2(5n+3) \equiv 0$ (mod 5),

$$M_{\overline{S_2}}(0,3,4) = M_{\overline{S_2}}(1,3,4) = M_{\overline{S_2}}(2,3,4) = \frac{1}{3}\overline{spt}_2(4)$$

and

$$M_{\overline{S_2}}(0,5,8) = M_{\overline{S_2}}(1,5,8) = M_{\overline{S_2}}(2,5,8) = M_{\overline{S_2}}(3,5,8) = M_{\overline{S_2}}(4,5,8) = 3 = \frac{1}{5}\,\overline{spt}_2(8)$$

With numerical examples respectively. We have verified the Theorem 1 when *n* = 5 and have verified the Theorem 2 when *n* = 6. We have verified the Corollary 3 with 6 marked overpartitions of 6 and have verified the Corollary 4 with 6 marked overpartitions of 7 and also have established the Corollary 5 with 15 marked overpartition of 8.

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Table 1. Vector partitions their weights and cranks ($\overline{S_2}$ of 4)

$\overline{S_2}$ -vector partition $(\vec{\pi})$ of 4	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$	Mod 3
$\vec{\pi}_1 = (4, \phi, \phi, \phi)$	1	0	0
$\vec{\pi}_2 = (2+2, \phi, \phi)$	1	1	1
$\vec{\pi}_3 = (2, \phi, 2, \phi)$	1	−1	2
	$\sum \omega(\vec{\pi}) = 3$		

Table 2. Vector partitions their weights and cranks ($\overline{S_2}$ of 5)

$\overline{S_2}$ -vector partition $(\vec{\pi})$ of 5	Weight $\omega(\vec{\pi})$	Crank c$(\vec{\pi})$
$\vec{\pi}_1 = (3+2, \phi, \phi, \phi)$	−1	0
$\vec{\pi}_2 = (2, \phi, \phi, 3)$	1	0
$\vec{\pi}_3 = (2, 3, \phi, \phi)$	1	1
$\vec{\pi}_4 = (2, \phi, 3, \phi)$	1	−1
	$\sum \omega(\vec{\pi}) = 2$	

Table 3. 15 $\overline{SP}_2$ -partition pairs of 8

$\overline{SP}_2$ -partition pair of 8	*k*	$\overline{crank}$	(Mod 5)
(3+2, 3)	1	0	0
(4+2, 2)	1	0	0
$(8,\phi)$	0	0	0
(2+2, 4)	0	1	1
(4+4, ϕ)	0	1	1
(6+2, ϕ)	0	1	1
(2, 2+2+2)	3	–3	2
(3+3+2, ϕ)	0	2	2
(4+2+2, ϕ)	0	2	2
(2, 3+3)	2	–2	3
(2+2, 2+2)	2	–2	3
(2+2+2+2, ϕ)	0	3	3
(2, 4+2)	1	–1	4
(4, 4)	1	–1	4
(2+2+2, 2)	1	–1	4

Table 4. Marked overpartition (π, j) of 6

Marked overpartition (π, j) of 6	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1)))$	$\overline{k}$	$\overline{sptcrank}$	(Mod 3)
(6 ,1)	6	ϕ	1	0	0	0	0
(4+2 ,1)	4+2	ϕ	1	0	0	1	1
$(\overline{4}+2,1)$	2	4	1	0	0	0	0
(2+2+2, 1)	2+2+2	ϕ	3	0	2	–2	1
(2+2+2, 2)	2+2+2	ϕ	3	0	1	–1	2
(2+2+2, 3)	2+2+2	ϕ	3	0	0	2	2

Table 5. Marked overpartition (π, j) of 7

Marked overpartition (π, j) of 7	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(Mod 3)
(5+2, 1)	5+2	ϕ	1	0	0	1	1
$(\bar{5}+2,1)$	2	5	1	0	0	0	0
(3+2+2, 1)	3+2+2	ϕ	2	0	1	0	0
(3+2+2, 2)	3+2+2	ϕ	2	0	0	2	2
($\bar{3}$+2+2, 1)	2+2	3	2	1	2	−2	1
($\bar{3}$+2+2, 2)	2+2	3	2	1	1	−1	2

Table 6. Marked overpartition (π, j) of 8

Marked overpartition (π, j) of 8	π_1	π_2	$\nu(\pi_1)$	$k((\pi_2, s(\pi_1))$	$\bar{k}$	$\overline{sptcrank}$	(Mod 5)
$(\bar{6}+2,1)$	2	6	1	2	2	−2	3
$(\bar{4}+2+2,1)$	2+2	4	2	0	1	−1	4
$(\bar{4}+2+2,2)$	2+2	4	2	0	0	1	1
$(\bar{3}+3+2,1)$	3+2	3	1	1	1	0	0
(2+2+2+2, 1)	2+2+2+2	ϕ	4	0	3	−3	2
(2+2+2+2, 2)	2+2+2+2	ϕ	4	0	2	−2	3
(2+2+2+2, 3)	2+2+2+2	ϕ	4	0	1	−1	4
(2+2+2+2, 4)	2+2+2+2	ϕ	4	0	0	3	3
(3+3+2, 1)	3+3+2	ϕ	1	0	0	2	2
(4+2+2, 1)	4+2+2	ϕ	1	0	1	0	0
(4+2+2, 2)	4+2+2	ϕ	2	0	0	2	2
(6+2, 1)	6+2	ϕ	1	0	0	1	1
(4+4, 1)	4+4	ϕ	2	0	1	−1	4
(4+4, 2)	4+4	ϕ	2	0	0	1	1
(8, 1)	8	ϕ	1	0	0	0	0