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An optimal fourth order weighted-Newton method for computing multiple roots and basin attractors for various methods

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ABSTRACT

In this paper, we present an optimal fourth order method for finding multiple roots of a nonlinear equation f(x)=0. In terms of computational cost, the method uses one evaluation of the function and two evaluations of its first derivative per iteration. Therefore, the method has optimal order with efficiency index 1.587 which is better than efficiency indices 1.414 of Newton method, 1.442 of Halley's method and 1.414 of Neta-Johnson method. Numerical examples are given to support that the method thus obtained is competitive with other similar robust methods. The basins of attraction of the proposed method are presented and compared with other existing methods.

Keywords: Rootfinding, Newton method, Multiple root, Order of convergence, Efficiency, Basins of attraction
2010 Mathematics Subject Classification: *65H05, 65B99*

INTRODUCTION

In this study, we apply iterative methods to find a multiple root α of multiplicity $m>1$, i.e. $f^{(j)}(\alpha)=0$, $j=0,1,....m-1$ and $f^{(m)}(\alpha)\neq 0$, of a nonlinear equation $f(x)=0$, where $f(x)$ be the continuously differentiable real or complex function. Modified Newton method sch is an important and basic method for finding multiple roots

$$x_{k+1}=x_k-m\,\frac{f(x_k)}{f'(x_k)}, \tag{1}$$

which converges quadratically and requires the knowledge of multiplicity m of root α.

In order to improve the order of convergence of (1), several higher-order methods have been proposed in the literature with known multiplicity m, for example, [2-19].

Through this work, we contribute a little more in the theory of iterative methods by developing a method of optimal order [20] four for computing multiple roots. The algorithm is based on the composition of two weighted-Newton steps and uses three function evaluations, namely one f and two f' per iteration. The paper is organized in five sections. In section 2, the method is developed and its convergence behavior is analyzed. In section 3, the presented method is compared with closest competitors in a series of numerical examples. The basins of attraction of the presented optimal method and other existing methods are given in section 4. Section 5 contains the concluding remarks.

2 Development of the Method

Let us consider the two-step weighted-Newton method of the type

$$\begin{cases} y_k = x_k - \theta \dfrac{f(x_k)}{f'(x_k)}; \\ x_{k+1} = x_k - \left(A + B\dfrac{f'(x_k)}{f'(y_k)} + C\dfrac{f'(y_k)}{f'(x_k)} \right) \dfrac{f(x_k)}{f'(x_k)}; \end{cases} \tag{2}$$

where A, B, C and θ are some constants which are to be determined. A natural question arises: Is it possible to find A,B,C and θ such that the iterative method (2) has maximum order of convergence? The answer to this question is affirmative and is proved in the following theorem.

Theorem 1 *Let α be a multiple root of multiplicity m of a sufficiently smooth real or complex function f(x) in some neighborhood I. If $\alpha \in I$ and x_0 is sufficiently close to α, then the scheme defined by (2) has fourth order convergence provided*

$$A = -\frac{1}{4}m(m^3+3m^2+2m-4), \qquad B = \frac{1}{8}m\left(\frac{m}{m+2}\right)^m (m+2)^3, \; C = \frac{1}{8}m^4\left(\frac{m}{m+2}\right)^{-m} \text{ and } \theta = \frac{2m}{m+2}.$$

Proof Let e_k be the error at k^{th} iteration, then $e_k = x_k - \alpha$. Expanding $f(x_k)$ and $f'(x_k)$ in a Taylor series about α, we have

$$f(x_k) = \frac{f^m(\alpha)}{m!} e_k^m [1 + A_1 e_k + A_2 e_k^2 + A_3 e_k^3 + A_4 e_k^4 + O(e_k^5)] \tag{3}$$

and

$$f'(x_k) = \frac{f^m(\alpha)}{(m-1)!} e_k^{m-1} [1 + B_1 e_k + B_2 e_k^2 + B_3 e_k^3 + B_4 e_k^4 + O(e_k^5)], \tag{4}$$

where

$$A_i = \frac{m! f^{(m+i)}(\alpha)}{(m+i)! f^{(m)}(\alpha)}, B_i = \frac{(m-1)! f^{(m+i)}(\alpha)}{(m+i-1)! f^{(m)}(\alpha)}, \qquad i = 1,2,3,\ldots$$

From (3) and (4), we get

$$\frac{f(x_k)}{f'(x_k)} = \frac{e_k}{m} - \frac{A_1 e_k^2}{m^2} + \frac{1}{m^3}\left[(m+1)A_1^2 - 2mA_2\right] e_k^3 + O(e_k^4). \tag{5}$$

Taking $\hat{e}_k = y_k - \alpha$, where $y_k = x_k - \theta f(x_k)/f'(x_k)$ and using (5), we can get

$$\hat{e_k} = e_k \left[d_0 + d_1 e_k + d_2 e_k^2 + d_3 e_k^3 + O(e_k^4)\right], \tag{6}$$

where $d_0 = 1 - \dfrac{\theta}{m}$, $d_1 = \dfrac{\theta}{m^2} A_1$, $d_2 = \dfrac{-\theta}{m^3}\left[(m+1)A_1^2 - 2mA_2\right]$

and $d_3 = \dfrac{\theta}{m^4}\left[(m+1)^2 A_1^3 - m(3m+4)A_1 A_2 + 3m^2 A_3\right]$.

Expansion of $f'(y_k)$ about α yields

$$f'(y_k) = \frac{f^m(\alpha)}{(m-1)!} \hat{e}_k^{m-1} [1 + C_1 \hat{e}_k + C_2 \hat{e}_k^2 + C_3 \hat{e}_k^3 + C_4 \hat{e}_k^4 + O(\hat{e}_k^5)]. \tag{7}$$

Using (4), (5) and (7) in (2), one gets the error equation

$$e_{k+1} = D_1 e_k + D_2 e_k^2 + D_3 e_k^3 + D_4 e_k^4 + O(e_k^5), \tag{8}$$

where

$D_1 = 1 - \frac{\alpha}{m} - \frac{B\mu^{1-m}}{m} - \frac{C\mu^{-1+m}}{m},$

$D_2 = \frac{\mu^{-m}}{m^4(m-\theta)^2}[mB(-5+\theta)\theta^3 + B\theta^4 - m^2\theta^2(2B(-4+\theta)+\mu^m(-A+C\mu^m)) + m^4(B$
$+\mu^m(A+C\mu^m)) + m^3\theta(B(-5+\theta) - \mu^m(2A+C(-1+\theta)\mu^m))]A_1,$

$$D_3 = m_1 A_1^2 + m_2 A_2, \tag{9}$$

$m_1 = \frac{\mu^{-m}}{2m^6(m-\theta)^3}[-4mB(-4+\theta)\theta^5 - 2B\theta^6 - 2m^2B\theta^4(24-12\theta+\theta^2) + m^3\theta^3(B(70$
$-55\theta+8\theta^2) + 2\mu^m(A-2C\mu^m)) - 2m^7(B+\mu^m(A+C\mu^m)) - m^4\theta^2(B$
$(52-64\theta+13\theta^2) + 2\mu^m(-A(-3+\theta)+C(-5+4\theta)\mu^m)) - m^6(B$
$(2-14\theta+3\theta^2) - \mu^m(A(-2+6\theta)+C(-2-2\theta+3\theta^2)\mu^m)) + m^5\theta(B$
$(18-41\theta+10\theta^2) - \mu^m(6A(-1+\theta)+C(6-15\theta+4\theta^2)\mu^m))],$

$m_2 = \frac{\mu^{-m}}{m^5(m-\theta)^2}[-mB(-10+\theta)\theta^4 - 2B\theta^5 + m^2\theta^3(B(-22+5\theta)+2C\mu^{2m})$
$+2m^5(B+\mu^m(A+C\mu^m)) - m^3\theta^2(B(-24+7\theta) - \mu^m(2A+C(-6$
$+\theta)\mu^m)) + m^4\theta(3B(-4+\theta) - \mu^m(4A+C(-4+3\theta)\mu^m))],$

and $\mu = \frac{m-\theta}{m}.$

For fourth order convergence, the coefficients D_1, D_2 and D_3 must vanish. Therefore, $D_1 = 0$ yields

$$A = -m\left(-1 + \frac{B\mu^{1-m}}{m} + \frac{C\mu^{-1+m}}{m}\right), \tag{10}$$

and $D_2 = 0$ with the use of (10) implies

$$B = \frac{m^2\mu^m(-m^3+2m^2\theta+C\theta^2\mu^m+m\theta(-2C\mu^m+\theta(-1+C\mu^m)))}{(m-\theta)^2\theta(m(-2+\theta)+\theta)} \tag{11}$$

On substituting the value of B in (10), we get

$$A = \frac{1}{(m-\theta)\theta(m(-2+\theta)+\theta)}[m(m^3+m^2(-4+\theta)\theta-\theta^2(\theta+2C\mu^m) - m\theta(\theta^2-4C\mu^m+2\theta(-2+C\mu^m)))]. \tag{12}$$

Using the values of A and B in the expressions of m_1 and m_2 and simplifying, we obtain

$m_1 = \frac{1}{2m^4(m-\theta)^3(m(-2+\theta)+\theta)}[m^7(-2+\theta) - 2C\theta^5\mu^m + m^4\theta^2(-44+37\theta-6\theta^2)$
$+m^6(-6+13\theta-4\theta^2) + m^5\theta(26-32\theta+7\theta^2) - 2m^3\theta^2(-8C\mu^m+\theta^2(10$
$-6C\mu^m) + \theta^3(-1+C\mu^m) + 6\theta(-3+2C\mu^m)) - 2m^2\theta^3(12C\mu^m+\theta(7$
$-12C\mu^m) + \theta^2(-2+3C\mu^m)) - 2m\theta^4(-6C\mu^m+\theta(-1+3C\mu^m))],$

$m_2 = -\frac{(m-\theta)(m(-2+\theta)+2\theta)}{m^2(m(-2+\theta)+\theta)}.$

For D_3, to vanish, both m_1 and m_2 should vanish. On taking $m_2 = 0$, we find

$$\theta=\frac{2m}{m+2},(\theta\neq m) \tag{13}$$

On using the value of θ obtained in (13), $m_1=0$ implies

$$C=\frac{1}{8}m^4\left(\frac{m}{m+2}\right)^{-m} \tag{14}$$

Also, using the values of θ and C in (11) and (12), the parameters A and B are finally given by

$$A=\frac{-1}{4}m(m^3+3m^2+2m-4), \tag{15}$$

$$B=\frac{1}{8}m\left(\frac{m}{m+2}\right)^{m}(m+2)^3. \tag{16}$$

With these values, the error equation (8) turns out to be

$$e_{k+1}=D_4e_k^4+O(e_k^5), \tag{17}$$

where

$$D_4=\frac{(m^5+6m^4+14m^3+14m^2+12m+16)A_1^3}{3m^4(m+2)^2}-\frac{A_1A_2}{m}+\frac{mA_3}{(m+2)^2}. \tag{18}$$

Thus equation (17) establishes the fourth order convergence for the iterative method (2). This completes proof of the theorem 1.

□

Hence, the method (2) for obtaining a multiple root of multiplicity m is given by

$$x_{k+1}=x_k-\frac{m}{8}\Big[-2(m^3+3m^2+2m-4)+\left(\frac{m}{m+2}\right)^{m}(m+2)^3\frac{f'(x_k)}{f'(y_k)}$$
$$+m^3\left(\frac{m}{m+2}\right)^{-m}\frac{f'(y_k)}{f'(x_k)}\Big]\frac{f(x_k)}{f'(x_k)}, \tag{19}$$

where $y_k=x_k-\frac{2m}{m+2}\frac{f(x_k)}{f'(x_k)}$.

It is clear that the presented method requires three evaluations per iteration and therefore, it is of optimal order. Since, the scheme is based on weighted Newton steps, we call the method (19) as the weighted Newton method (WNM).

Remark Obviously, the proposed iterative method defined by (19) requires one evaluation of the function and two evaluations of its first derivative per iteration and achieves fourth order convergence. We consider the definition of efficiency index [21] as $p^{1/n}$ where p is the order of the method and n is the number of function evaluations per iteration required by the method. Thus the presented method has the efficiency index equal to $\sqrt[3]{4}\approx1.587$, which is better than $\sqrt{2}\approx1.414$ of modified Newton method, $\sqrt[3]{3}\approx1.442$ of third order methods [5,7,10,22] and $\sqrt[4]{4}\approx1.414$ of fourth order method [11]. This value, however, is same as that of the fourth order method introduced in [23] by Li et al.

3 Numerical Results and Discussions

We employ the present method (WNM) to solve some nonlinear equations, which not only illustrate the method practically but also serve to check the validity of theoretical results we have derived. To check the theoretical order of convergence, we obtain the computational order of convergence (ρ) using the formula [24]

$$\rho \approx \frac{\ln|(x_{k+1}-\alpha)/(x_k-\alpha)|}{\ln|(x_k-\alpha)/(x_{k-1}-\alpha)|}.$$

The performance is compared with modified Newton method (MNM) defined by equation (1), Osada's [10] third order method (OM) expressed as

$$x_{k+1}=x_k-\frac{1}{2}m(m+1)\frac{f(x_k)}{f'(x_k)}+\frac{1}{2}(m-1)^2\frac{f'(x_k)}{f''(x_k)},$$

one of third order methods due to Homeier [25] denoted by HM and defined by

$$x_{k+1}=x_k-m^2\left(\frac{m}{m+1}\right)^{m-1}\frac{f(x_k)}{f'\left(x_k-\frac{m}{m+1}\frac{f(x_k)}{f'(x_k)}\right)}+m(m-1)\frac{f(x_k)}{f'(x_k)},$$

the third order method due to Victory and Neta [7] denoted by VN and which is given by

$$y_k=x_k-\frac{f(x_k)}{f'(x_k)},$$

$$x_{k+1}=y_k-\frac{f(x_k)+Af(y_k)}{f(x_k)+Bf(y_k)}\frac{f(y_k)}{f'(x_k)},$$

where

$$A=\left(\frac{m}{m-1}\right)^{2m}-\left(\frac{m}{m-1}\right)^{m+1},B=-\frac{\left(\frac{m}{m-1}\right)^{m}(m-2)(m-1)+1}{(m-1)^2},\ m\neq 1$$

and fourth order method due to Li et al. [23] denoted by LM, which is given by

$$y_k=x_k-\frac{2m}{m+2}\frac{f(x_k)}{f'(x_k)},$$

$$x_{k+1}=x_k-\frac{\frac{1}{2}m(m-2)(\frac{m}{m+2})^{-m}f'(y_k)-\frac{m^2}{2}f'(x_k)}{f'(x_k)-(\frac{m}{m+2})^{-m}f'(y_k)}\frac{f(x_k)}{f'(x_k)}.$$

Table 1. Test functions

$f(x)$	α	m
$f_1(x)=x^5-8x^4+24x^3-34x^2+23x-6$	1.0	3
$f_2(x)=x^2e^x-\sin x+x$	0.0	2
$f_3(x)=(x^3-1)^2$	1.0, $-1/2\pm\sqrt{3}/2i$	2
$f_4(x)=(x^2-e^x-3x+2)^5$	0.25753028543986075	
$f_5(x)=(1+\cos x)(e^x-2)^2$	0.69314718055994532	
$f_6(x)=\ln^2(x-2)(e^{x-3}-1)\sin\frac{\pi x}{3}$	3.0	4
$f_7(x)=(\sin x-\frac{\sqrt{2}}{2})^2(x+1)$	0.78539816339744832	
$f_8(x)=\ln(x^2+x+2)-x+1$	4.15259073675715831	

Test functions along with root α correct up to 16 decimal places and multiplicity m are displayed in Table 1. Table 2 shows the values of initial approximation (x_0) chosen from both sides to the root, values of the error $|e_k|=|x_k-\alpha|$ calculated by costing the same total number of function evaluations (NFE) for each method and the computational order of convergence (ρ). The NFE is counted as sum of the number of evaluations of the function plus the number of evaluations of the derivatives. In calculations, the NFE used for all the methods is 12. That means for MNM, the error $|e_k|$ is calculated in the sixth iteration, whereas for the remaining methods this is calculated at the fourth iteration. It is quite understood that increasing the order of the method leads us to obtain more precision widening the mantissa. For this reason and for better comparison as well, in Table 2 all computations are done with multiprecision arithmetic using 600 significant digits.

Results in table 2 show that the computational order of convergence is in accordance with the theoretical order of convergence. Moreover, it is quite clear that in all the considered problems accuracy of WNM is higher than MNM, HM, OM and VN. However, the accuracy is almost same when compared with LM as expected from the methods of similar character. It should be noted that Osada's method reduces to Newton method when m=1, therefore, in example 8 computational order of convergence is 2. The Victory-Neta method is not defined in last example because here m=1.

In the next section, we give the comparison of these iterative methods in the complex plane.

Table 2. Performance of the methods

$f(x)$		$\|e_k\|=\|x_k-r\|$						ρ					
	x_0	MNM	HM	OM	VN	LM	WNM	MNM	HM	OM	VN	LM	WNM
f_1	0	4.16(−33)	2.97(−36)	6.31(−29)	7.24(−39)	2.43(−101)	4.12(−101)	2	3	3	3	4	4
	1.4	1.02(−35)	1.54(−32)	div	4.07(−41)	1.16(−69)	6.01(−69)	2	3	-	3	4	4
f_2	−0.5	6.68(−27)	6.59(−29)	4.88(−6)	7.03(−41)	2.34(−56)	7.70(−56)	2	3	3	3	4	4
	1	3.64(−25)	1.04(−30)	1.03(−19)	1.23(−32)	5.62(−94)	2.55(−89)	2	3	2.994	3	4	4
f_3	2	1.55(−16)	5.00(−18)	7.77(−12)	3.71(−21)	1.08(−59)	5.17(−53)	2	2.997	3	3	4	4
	−0.4±0.5i	5.85(−21)	2.43(−14)	3.37(−6)	5.11(−32)	4.32(−45)	9.30(−60)	2	3	3	3	4	4
f_4	−2	1.55(−37)	1.30(−42)	1.45(−40)	3.44(−43)	1.36(−112)	1.32(−111)	2	3	3	3	4	4
	1.5	1.06(−57)	2.44(−50)	5.15(−50)	4.14(−41)	4.81(−114)	1.06(−114)	2	3	3	3	4	4
f_5	0	9.75(−35)	3.73(−35)	nconv	1.66(−43)	9.60(−85)	3.49(−83)	2	3	-	3	4	4
	1.5	1.91(−55)	7.77(−50)	2.41(−34)	5.61(−49)	2.95(−192)	9.38(−189)	2	3	3	3	4	4
f_6	2.25	3.53(−47)	7.81(−68)	2.05(−70)	1.09(−41)	1.70(−144)	6.17(−154)	2	3	3	3	4	4
	4.5	3.97(−40)	5.04(−43)	1.06(−22)	2.38(−42)	3.19(−81)	1.13(−82)	2	3	3	3	4	4
f_7	0	8.81(−32)	1.22(−32)	nconv	3.56(−35)	1.45(−41)	5.51(−41)	2	3	-	3	3.999	3.998
	1.25	4.87(−51)	3.62(−38)	nconv	1.60(−43)	1.10(−85)	3.03(−85)	2	3	-	3	4	4
f_8	6	4.47(−68)	3.88(−59)	4.47(−68)	ndef	1.58(−229)	7.81(−221)	2	3	2	-	4	4
	2+4i	3.91(−34)	1.15(−17)	3.91(−34)	ndef	6.21(−92)	4.51(−91)	2	3	2	-	4	4

Here a(-b)=a×10^-b^,div=divergent, nconv=not converges to required root, ndef=not defined

5. Dynamical Aspects. The study of the rational functions associated to an iterative method, using the theory of complex dynamics gives important information about numerical features of the method as its convergence and stability [26].

The idea of analysing complex dynamics of iteration schemes was first introduced by Vrscay and Gilbert [27]. Later many researchers followed this idea in their works, for example, (see [28-32] and references therein).

To start with, let us recall some basic ideas and terminology of iteration.

Let $S:C\to C$ be a rational map on the complex plane. For $z\in C$, we define its orbit as the set $orb(z)=\{z,S(z),S^2(z),\cdots\}$.

A point $z_0\in C$ is called periodic point with minimal period m if $S^m(z_0)=z_0$, where m is the smallest integer with this property. A periodic point with minimal period 1 is called fixed point. Moreover, a fixed point z_0 is called attracting if $|S'(z_0)|<1$, repelling if $|S'(z_0)|>1$, and neutral otherwise. In addition, if $|S'(z_0)|=0$, the fixed point is super attracting.

The Julia set of a nonlinear map $S(z)$, denoted by $J(S)$, is the closure of the set of its repelling periodic points. The complement of $J(S)$ is the Fatou set $F(S)$, where the basins of attraction of the different roots lie.

Various researchers have used basins of attraction to compare iteration schemes, for example, [33-35]. Following [36], we generate basins of attraction for the complex cube roots of unity with multiplicity 2, to study the dynamical behaviour. We take a square $[-2,2]\times[-2,2]$ ($\subseteq C$) of 1024×1024 points, which contains all roots of concerned nonlinear equation and we apply the iterative method starting in every z_0 in the square. We assign a color to each point z_0 according to the multiple root to which the corresponding orbit of the iterative method, starting from z_0, converges. If the corresponding orbit does not reach any root of the polynomial, with tolerance $\varepsilon=10^{-3}$ in a maximum of 25 iterations, we mark those points z_0 with black color.

For the test problem $p(z)=(z^3-1)^2$, Fig.1 clearly shows that the proposed method WNM (Fig. 1(f)) seems to produce larger basins of attraction than LM (Fig. 1(e)), almost competitive basins of attraction as HM (Fig. 1(b)) and smaller basins of attraction than MNM (Fig. 1(a)), OM (Fig. 1(c)) and VN (Fig. 1(d)).

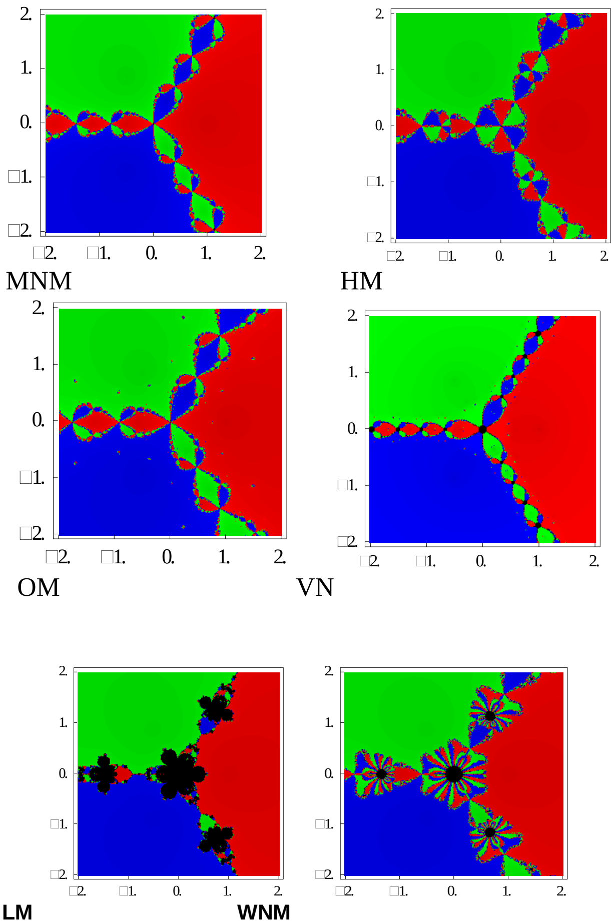

Figure 1: Basins of attraction for $f(z)=(z^3-1)^2$ for various methods

CONCLUSION

In this work, we have proposed a fourth order method for finding multiple roots. The algorithm is based on the composition of two weighted-Newton steps. Hence the name weighted Newton method. An important characteristic of the WNM is that it does not use of second derivative f''. This feature makes the present method more useful in the problems where f'' is difficult to evaluate. The superiority is also corroborated by numerical results displayed in the table 2. A reasonably close starting value is necessary for the method to converge. This condition, however, practically applies to all iterative methods for solving equations. Since the present approach utilizes one f and two f' per iteration, therefore, the newly developed method is very useful in the applications in which the derivative f' can be rapidly evaluated compared to f itself. Example of this kind occurs when f is defined by a polynomial or an integral.

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