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A Three Species Ecosystem Comprising of Two Predators competing for a Prey

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ABSTRACT

This paper deals with an investigation on a three species ecosystem consisting of a prey (S_1), two predators (S_2) and (S_3) surviving on the same prey (S_1). The mathematical model equations constitute a set of three first order non-linear simultaneous differential equations in N_1, N_2 and N_3, the population densities of S_1, S_2 and S_3 respectively. All possible equilibrium points of the model are identified. Local and global stabilities are discussed by using Routh-Hurwitz criteria and Lyapunov function respectively, and also the results are verified by numerical examples.

Key Words: Prey predator, equilibrium points, local and global stability.

INTRODUCTION

Ever since research in the discipline of theoretical ecology was initiated by Lotka [8] and by Volterra [15], several mathematicians and ecologists contributed in the growth of this area of knowledge, which has been extensively reported in the treatises of Meyer[9], Cushing [2], Paul Conlinvaux [10], Freedman[3], Kanpur[5,6]. The ecological interactions can be broadly classified as prey-predation, competitions, neutralism, and mutualism and so on. N.C.Srinivas [14] studied the competitive eco-system of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [7] has investigated the two species prey-predator models. Recently stability analysis of competitive species was investigated by Archana Reddy [1]. Local stability analysis for a two-species ecological mutualism model has been presented by B.Ravindra Reddy et al [11]. Recently, stability analysis of prey, two predators which are neutral to each other [12], prey, predator and super-predator [13] were carried out by the present authors.

The present investigation is an analytical study of three species system comprising a Prey (S_1) common to two predators (S_2) and (S_3) which are competing with each other. All the eight equilibrium points are identified based on the model equations and these are spread over three distinct classes: (i) Fully washed out (ii) Semi/partially washed out and (iii) Co-existent states. Criteria for the local and global asymptotic stability of the states have been derived.

Mathematical Model:
The model equations for a three species multi-system are given by a set of three non-linear ordinary differential equations as

(i) Equation for the growth rate of Prey species (S_1):

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 - \alpha_{13} N_1 N_3 \tag{2.1}$$

(ii) Equation for the growth rate of predator species (S_2):

$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^2 + \alpha_{21} N_1 N_2 - \alpha_{23} N_2 N_3 \tag{2.2}$$

(iii) Equation for the growth rate of predator species (S_3):

$$\frac{dN_3}{dt} = a_3 N_3 - \alpha_{33} N_3^2 + \alpha_{31} N_1 N_3 - \alpha_{32} N_2 N_3 \tag{2.3}$$

with the following notation.
N_i (t) : Population density of the species S_i at time t ,i=1,2,3.
a_i : The natural growth rates of S_i, i = 1,2,3
α_{ii} : The rate of decrease of S_i due to its own insufficient resources i = 1,2,3.
α_{1n} :The rate of decrease of the prey (S_1) due to inhibition by the predator (S_n),n=2,3
α_{m1} : The rate of increase of the predator (S_m) due to its successful attacks on the prey (S_1),m=2,3
$K_i=a_i / \alpha_{ii}$: Carrying capacities of S_i , i = 1, 2, 3.
$\beta_{1n} = \alpha_{1n}/ \alpha_{11}$: Co-efficient of inhibition of the species S_1 due to the species S_n , n=2,3
$\beta_{m1} = \alpha_{m1}/ \alpha_{mm}$: Co-efficient of predation of the species S_m, m=2,3 on the prey S_1.
Further the variables N_1, N_2 and N_3 are non-negative and the model parameters a_i, K_i, α_{ij}, i=1,2,3, j=1,2,3, β_{12}, β_{13}, β_{21}, β_{23}, β_{31} and β_{32} are assumed to be non-negative constants.

3. Equilibrium points:

The system under investigation has eight equilibrium states given by $\frac{dN_i}{dt}$ =0, *i* =1, 2, 3.

i. Fully washed out state:

$$(E_1) \quad \overline{N}_1 = 0, \overline{N}_2 = 0; \overline{N}_3 = 0 \tag{3.1}$$

ii. a. States in which two of the three species are washed out and third is surviving.

$$(E_2) \quad \overline{N}_1 = K_1; \overline{N}_2 = 0; \overline{N}_3 = 0 \tag{3.2}$$

$$(E_3) \quad \overline{N}_1 = 0; \overline{N}_2 = K_2; \overline{N}_3 = 0 \tag{3.3}$$

$$(E_4) \quad \overline{N}_1 = 0; \overline{N}_2 = 0; \overline{N}_3 = K_3 \tag{3.4}$$

ii. b. Only one of the three species is washed out while the other two are surviving.

$$(E_5) \quad \overline{N}_1 = \frac{K_1 - \beta_{12} K_2}{1 + \beta_{21}\beta_{12}}, \overline{N}_2 = \frac{K_2 + \beta_{21} K_1}{1 + \beta_{21}\beta_{12}}, \overline{N}_3 = 0 \tag{3.5}$$

This state would exist only when $K_1 > \beta_{12} K_2$

$$(E_6) \quad \overline{N}_1 = 0, \overline{N}_2 = \frac{K_2 - \beta_{23} K_3}{1 - \beta_{23}\beta_{32}}, \overline{N}_3 = \frac{K_3 - \beta_{32} K_2}{1 - \beta_{23}\beta_{32}} \tag{3.6}$$

This state would exist only when

$$1-\beta_{23}\beta_{32}>0, K_2>K_3\beta_{23} \text{ and } K_3>K_2\beta_{32}$$

$$(E_7) \quad \overline{N}_1=\frac{K_1-\beta_{13}K_3}{1+\beta_{13}\beta_{31}}, \overline{N}_2=0, \overline{N}_3=\frac{K_3+\beta_{31}K_1}{1+\beta_{13}\beta_{31}} \tag{3.7}$$

This state would exist only when $K_1>\beta_{31}K_3$

iii. The co-existent state or normal steady state

$$(E_8) \quad \overline{N}_1=\frac{K_1+\beta_{13}\beta_{32}K_2+\beta_{12}\beta_{23}K_3-(\beta_{23}\beta_{32}K_1+\beta_{12}K_2+\beta_{13}K_3)}{1+\beta_{12}\beta_{21}+\beta_{13}\beta_{31}-(\beta_{23}\beta_{32}+\beta_{13}\beta_{21}\beta_{32}+\beta_{31}\beta_{12}\beta_{23})},$$

$$\overline{N}_2=\frac{\beta_{21}K_1+(1+\beta_{13}\beta_{31})K_2-(\beta_{23}\beta_{31}K_1+(\beta_{23}+\beta_{13}\beta_{21})K_3)}{1+\beta_{12}\beta_{21}+\beta_{13}\beta_{31}-(\beta_{23}\beta_{32}+\beta_{13}\beta_{21}\beta_{32}+\beta_{31}\beta_{12}\beta_{23})}$$

$$\overline{N}_3=\frac{\beta_{31}K_1+(1+\beta_{21}\beta_{12})K_3-(\beta_{21}\beta_{32}K_1+(\beta_{32}+\beta_{31}\beta_{12})K_2)}{1+\beta_{12}\beta_{21}+\beta_{13}\beta_{31}-(\beta_{23}\beta_{32}+\beta_{13}\beta_{21}\beta_{32}+\beta_{31}\beta_{12}\beta_{23})} \tag{3.8}$$

This would exist only when

$$\begin{aligned} &1+\beta_{12}\beta_{21}+\beta_{13}\beta_{31}>(\beta_{23}\beta_{32}+\beta_{13}\beta_{21}\beta_{32}+\beta_{31}\beta_{12}\beta_{23}),\\ &K_1+\beta_{13}\beta_{32}K_2+\beta_{12}\beta_{23}K_3>(\beta_{23}\beta_{32}K_1+\beta_{12}K_2+\beta_{13}K_3)\\ &\beta_{21}K_1+(1+\beta_{13}\beta_{31})K_2>(\beta_{23}\beta_{31}K_1+(\beta_{23}+\beta_{13}\beta_{21})K_3)\\ &\text{and } \beta_{31}K_1+(1+\beta_{21}\beta_{12})K_3>(\beta_{21}\beta_{32}K_1+(\beta_{32}+\beta_{31}\beta_{12})K_2) \end{aligned} \tag{3.9}$$

4. Local Stability of the system at Equilibrium points:

Let $N=(N_1,N_2,N_3)=\overline{N}+U=(\overline{N}_1+u_1,\overline{N}_2+u_2,\overline{N}_3+u_3)$ where $U=(u_1,u_2,u_3)^T$ is a small perturbation over the equilibrium state $\overline{N}=(\overline{N_1},\overline{N_2},\overline{N_3})$ the basic equations (2.1), (2.2), and (2.3) are linearised to obtain the equations for the perturbed state as

$$\frac{dU}{dt}=AU \tag{4.1}$$

with

$$A=\begin{bmatrix} a_1-2\alpha_{11}\overline{N}_1-\alpha_{12}\overline{N}_2-\alpha_{13}\overline{N}_3 & -\alpha_{12}\overline{N}_1 & -\alpha_{13}\overline{N}_1 \\ \alpha_{21}\overline{N}_2 & a_2+\alpha_{21}\overline{N}_1-2\alpha_{22}\overline{N}_2-\alpha_{23}\overline{N}_3 & -\alpha_{23}\overline{N}_2 \\ \alpha_{31}\overline{N}_3 & -\alpha_{32}\overline{N}_3 & a_3+\alpha_{31}\overline{N}_1-\alpha_{32}\overline{N}_2-2\alpha_{33}\overline{N}_3 \end{bmatrix} \tag{4.2}$$

The characteristic equation for the system is $\det[A-\lambda I]=0$. (4.3)

The equilibrium state is stable only when the roots of the equation (4.3) are negative if they are real or have negative real parts if they are complex. The equilibrium state E_1,E_2,E_3,E_4 are noticed to be unstable (The detailed investigation of these states is not included in this paper). The stability criteria of states E_5,E_6,E_7,E_8 are discussed below.

4.1 Stability of Predator (S_3) Washed out Equilibrium State: (E_5)

One Eigen value of the variational matrix A, is $a_3+\alpha_{31}\overline{N}_1-\alpha_{32}\overline{N}_2$, and the other two are given by the quadratic equation $\lambda^2+(\alpha_{11}\overline{N}_1+\alpha_{22}\overline{N}_2)\lambda+(\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21})\overline{N}_1\overline{N}_2=0$
(4.1.1)

whose sum of the roots $-(\alpha_{11}\overline{N}_1+\alpha_{22}\overline{N}_2)$, is always negative and the product of the roots $(\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21})\overline{N}_1\overline{N}_2$, is always positive .Therefore the roots of (4.1.1) are real and negative or complex conjugates having negative real parts. Thus the state is asymptotically stable only if $a_3+\alpha_{31}\overline{N}_1<\alpha_{32}\overline{N}_2$.

The solution of the perturbation equations are:

$$u_1=u_{10}\left[A_0e^{\lambda_1 t}+B_0e^{\lambda_2 t}+C_0e^{\lambda_3 t}\right] \quad (4.1.2)$$

$$u_2=u_{20}\left[A_1e^{\lambda_1 t}+B_1e^{\lambda_2 t}+C_1e^{\lambda_3 t}\right] \quad (4.1.3)$$

$$u_3=u_{30}e^{\lambda_1 t} \quad (4.1.4)$$

where $\lambda_1=a_3+\alpha_{31}\overline{N}_1-\alpha_{32}\overline{N}_2$, λ_2 *and* λ_3 are roots of equation (4.1.1)

$$A_0=\frac{\lambda_1^2-P_1\lambda_3+Q_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},B_0=\frac{\lambda_2^2-P_1\lambda_2+Q_1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)},C_0=\frac{\lambda_3^2-P_1\lambda_3+Q_1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$$A_1=\frac{\lambda_1^2+P_2\lambda_1-Q_2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},\ B_1=\frac{\lambda_2^2+P_2\lambda_2-Q_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)},C_1=\frac{\lambda_3^2+P_2\lambda_3-Q_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$$P_1=\lambda_1-\alpha_{22}\overline{N}_2+(u_{30}\alpha_{13}\overline{N}_1+u_{20}\alpha_{12}\overline{N}_1)/u_{10}\,,P_2=\alpha_{11}\overline{N}_1-\lambda_1-(u_{30}\alpha_{23}\overline{N}_2-u_{10}\alpha_{12}\overline{N}_2)/u_{20}$$

$$Q_1=-\alpha_{22}\overline{N}_2\lambda_1+(u_{20}\alpha_{12}\overline{N}_1\lambda_1+u_{30}\alpha_{12}\alpha_{23}\overline{N}_1\overline{N}_2-u_{30}\alpha_{13}\alpha_{22}\overline{N}_1\overline{N}_2)/u_{10},$$

$$Q_2=\alpha_{11}\overline{N}_1\lambda_1+(u_{30}\alpha_{11}\alpha_{23}\overline{N}_1\overline{N}_2-u_{10}\alpha_{12}\overline{N}_2\lambda_1-u_{30}\alpha_{12}\alpha_{13}\overline{N}_1\overline{N}_2)/u_{20},$$

4.2. Stability of Prey (S_1) Washed out Equilibrium State: (E_6)

One of the Eigen value of variational matrix A is $a_1-\alpha_{12}\overline{N}_2-\alpha_{13}\overline{N}_3$ and the other two Eigen values are obtained from $\lambda^2+(\alpha_{22}\overline{N}_2+\alpha_{33}\overline{N}_3)\lambda+(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})\overline{N}_2\overline{N}_3=0$,
(4.2.1)

whose sum $-(\alpha_{22}\overline{N}_2+\alpha_{33}\overline{N}_3)$, is always negative and their product $(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})\overline{N}_2\overline{N}_3$, is always positive. Therefore the roots of (4.2.1) are real and negative or complex conjugates having negative real part. Thus the state is asymptotically stable only if $a_1<(\alpha_{12}\overline{N}_2+\alpha_{13}\overline{N}_3)$.

The solution of the perturbation equations are:

$$u_1=u_{10}e^{\lambda_1 t} \quad (4.2.2)$$

$$u_2=u_{20}\left[A_2e^{\lambda_1 t}+B_2e^{\lambda_2 t}+C_2e^{\lambda_3 t}\right] \quad (4.2.3)$$

$$u_3=u_{30}\left[A_3e^{\lambda_1 t}+B_3e^{\lambda_2 t}+C_3e^{\lambda_3 t}\right] \quad (4.2.4)$$

where $\lambda_1=a_1-\alpha_{12}\overline{N}_2-\alpha_{13}\overline{N}_3$, λ_2 *and* λ_3 are roots of equation (4.2.1)

$$A_2 = \frac{\lambda_1^2 + P_3\lambda_1 + Q_3}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \quad B_2 = \frac{\lambda_2^2 + P_3\lambda_2 + Q_3}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, \; C_2 = \frac{\lambda_3^2 + P_3\lambda_3 + Q_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}$$

$$A_3 = \frac{\lambda_1^2 + P_4\lambda_1 + Q_4}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \quad B_3 = \frac{\lambda_2^2 + P_4\lambda_2 + Q_4}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, \; C_3 = \frac{\lambda_3^2 + P_4\lambda_3 + Q_4}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}$$

$$P_3 = \alpha_{33}\overline{N}_3 - \lambda_1 + (u_{10}\alpha_{12}\overline{N}_2 - u_{30}\alpha_{23}\overline{N}_2)/u_{20}$$

$$Q_3 = -\alpha_{33}\overline{N}_3\lambda_1 + (u_{10}\alpha_{33}\alpha_{12}\overline{N}_2\overline{N}_3 + u_{30}\alpha_{23}\overline{N}_2\lambda_1 - u_{10}\alpha_{23}\alpha_{31}\overline{N}_2\overline{N}_3)/u_{20}.$$

$$P_4 = \alpha_{22}\overline{N}_2 - \lambda_1 + (u_{10}\alpha_{13}\overline{N}_3 - u_{20}\alpha_{32}\overline{N}_3)/u_{30}$$

$$Q_4 = -\alpha_{22}\overline{N}_2\lambda_1 + (u_{10}\alpha_{31}\alpha_{22}\overline{N}_2\overline{N}_3 - u_{10}\alpha_{12}\alpha_{32}\overline{N}_2\overline{N}_3 + u_{20}\alpha_{32}\overline{N}_3\lambda_1)/u_{30}$$

4.3. Stability of Predator (S_2) Washed out State: (E_7)

One Eigen value of variational matrix A is $a_2 + \alpha_{21}\overline{N}_1 - \alpha_{23}\overline{N}_3$, and the other two are roots of the quadratic equation

$$\lambda^2 + (\alpha_{11}\overline{N}_1 + \alpha_{33}\overline{N}_3)\lambda + (\alpha_{11}\alpha_{33} + \alpha_{13}\alpha_{31})\overline{N}_1\overline{N}_3 = 0, \tag{4.3.1}$$

whose sum $-(\alpha_{11}\overline{N}_1 + \alpha_{33}\overline{N}_3)$, is always negative and product $(\alpha_{11}\alpha_{33} + \alpha_{13}\alpha_{31})\overline{N}_1\overline{N}_3$ is always positive. Therefore the roots of (4.3.1) are real and negative or complex conjugates having negative real parts. Thus the state is asymptotically stable only if $a_2 + \alpha_{21}\overline{N}_1 < \alpha_{23}\overline{N}_3$.
The solution of the perturbation equations is:

$$u_1 = u_{10}\left[A_4 e^{\lambda_1 t} + B_4 e^{\lambda_2 t} + C_4 e^{\lambda_3 t}\right] \tag{4.3.2}$$

$$u_2 = u_{20} e^{\lambda_1 t} \tag{4.3.3}$$

$$u_3 = u_{30}\left[A_5 e^{\lambda_1 t} + B_5 e^{\lambda_2 t} + C_5 e^{\lambda_3 t}\right] \tag{4.3.4}$$

where $\lambda_1 = a_2 + \alpha_{21}\overline{N}_1 - \alpha_{23}\overline{N}_3$, λ_2 *and* λ_3 are roots of equation (4.3.1)

$$A_4 = \frac{\lambda_1^2 + P_5\lambda_1 + Q_5}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, B_4 = \frac{\lambda_2^2 + P_5\lambda_2 + Q_5}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, C_4 = \frac{\lambda_3^2 + P_5\lambda_3 + Q_5}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}$$

$$A_5 = \frac{\lambda_1^2 + P_6\lambda_1 - Q_6}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, B_5 = \frac{\lambda_2^2 + P_6\lambda_2 - Q_6}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, C_5 = \frac{\lambda_3^2 + P_6\lambda_3 - Q_6}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)},$$

$$P_5 = \alpha_{33}\overline{N}_3 - \lambda_1 - (u_{20}\alpha_{12}\overline{N}_1 + u_{30}\alpha_{13}\overline{N}_1)/u_{10}$$

$$Q_5 = -\alpha_{33}\overline{N}_3\lambda_1 + (u_{20}\alpha_{13}\alpha_{32}\overline{N}_1\overline{N}_3 + u_{30}\alpha_{13}\overline{N}_1\lambda_1 - u_{20}\alpha_{12}\alpha_{33}\overline{N}_1\overline{N}_3)/u_{10}$$

$$P_6 = \alpha_{11}\overline{N}_1 - \lambda_1 - (u_{20}\alpha_{32}\overline{N}_3 - u_{10}\alpha_{31}\overline{N}_3)/u_{30}$$

$$Q_6 = \alpha_{11}\overline{N}_1\lambda_1 + (u_{20}\alpha_{32}\alpha_{11}\overline{N}_1\overline{N}_3 - u_{10}\alpha_{31}\overline{N}_3\lambda_1 - u_{20}\alpha_{31}\alpha_{12}\overline{N}_1\overline{N}_3)/u_{30}$$

4.4. Stability of Co-Existing State: (E_8)

The characteristic equation of co-existing state is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{4.4.1}$$

where $b_1 = \alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2 + \alpha_{33}\overline{N}_3$

$b_2 = (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\overline{N}_2\overline{N}_3 + (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})\overline{N}_1\overline{N}_2 + (\alpha_{11}\alpha_{33} + \alpha_{13}\alpha_{31})\overline{N}_1\overline{N}_3$

$b_3 = (\alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{13}\alpha_{31}\alpha_{22} - \alpha_{11}\alpha_{23}\alpha_{32} - \alpha_{12}\alpha_{31}\alpha_{23} - \alpha_{13}\alpha_{21}\alpha_{32})\overline{N}_1\overline{N}_2\overline{N}_3$

According to Routh-Hurwitz's criteria, the necessary and sufficient conditions for local stability of co-existent points are $b_1 > 0, b_3 > 0$ and $b_3(b_1b_2 - b_3) > 0$
(4.4.2)

It is evident that $b_1 > 0$ and

$(\alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{13}\alpha_{31}\alpha_{22})\overline{N}_1\overline{N}_2\overline{N}_3 > (\alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{12}\alpha_{31}\alpha_{23} + \alpha_{13}\alpha_{21}\alpha_{32})\overline{N}_1\overline{N}_2\overline{N}_3$ Thus the stability of co-existent state is determined by the sign of $b_1b_2 - b_3$

By direct calculations we obtain

$$b_1b_2 - b_3 = (\alpha_{11}{}^2\alpha_{22} + \alpha_{11}\alpha_{12}\alpha_{21})\overline{N}_1{}^2\overline{N}_2 + (\alpha_{11}{}^2\alpha_{33} + \alpha_{11}\alpha_{13}\alpha_{31})\overline{N}_1{}^2\overline{N}_3 + (\alpha_{33}\alpha_{22}{}^2 - \alpha_{22}\alpha_{23}\alpha_{32})\overline{N}_3\overline{N}_2{}^2 + (\alpha_{11}\alpha_{22}{}^2 + \alpha_{12}\alpha_{21}\alpha_{22})\overline{N}_1\overline{N}_2{}^2$$
$$+ (\alpha_{22}\alpha_{33}{}^2 - \alpha_{23}\alpha_{32}\alpha_{33})\overline{N}_2\overline{N}_3{}^2 + (\alpha_{11}\alpha_{33}{}^2 + \alpha_{13}\alpha_{31}\alpha_{33})\overline{N}_1\overline{N}_3{}^2 + (2\alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{31}\alpha_{23} + \alpha_{13}\alpha_{21}\alpha_{32})\overline{N}_1\overline{N}_2\overline{N}_3 > 0$$

(4.4.3)

Hence the co-existent state is locally asymptotically stable.

The solution of the perturbation equations is:

$$u_1 = u_{10}\left[A_7 e^{\lambda_1 t} + B_7 e^{\lambda_2 t} + C_7 e^{\lambda_3 t}\right] \quad (4.4.4)$$

$$u_2 = u_{20}\left[A_8 e^{\lambda_1 t} + B_8 e^{\lambda_2 t} + C_8 e^{\lambda_3 t}\right] \quad (4.4.5)$$

$$u_3 = u_{30}\left[A_9 e^{\lambda_1 t} + B_9 e^{\lambda_2 t} + C_9 e^{\lambda_3 t}\right] \quad (4.4.6)$$

where

$$A_6 = \frac{\lambda_1{}^2 + T_1\lambda_1 + U_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)},\ B_6 = \frac{\lambda_2{}^2 + T_1\lambda_2 + U_1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)},\ C_6 = \frac{\lambda_3{}^2 + T_1\lambda_3 + U_1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}$$

$$A_7 = \frac{\lambda_1{}^2 + T_2\lambda_1 + U_2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)},\ B_7 = \frac{\lambda_2{}^2 + T_2\lambda_2 + U_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)},\ C_7 = \frac{\lambda_3{}^2 + T_2\lambda_3 + U_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}$$

$$A_8 = \frac{\lambda_1{}^2 + T_3\lambda_1 + U_3}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)},\ B_8 = \frac{\lambda_2{}^2 + T_3\lambda_2 + U_3}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)},\ C_8 = \frac{\lambda_3{}^2 + T_3\lambda_3 + U_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}$$

$T_1 = (\alpha_{22}\overline{N}_2 + \alpha_{33}\overline{N}_3) - (u_{20}\alpha_{12}\overline{N}_1 + u_{30}\alpha_{13}\overline{N}_1)/u_{10}$,

$T_2 = (\alpha_{33}\overline{N}_3 + \alpha_{11}\overline{N}_1) + (u_{10}\alpha_{21}\overline{N}_2 - u_{30}\alpha_{23}\overline{N}_2)/u_{20}$,

$T_3 = (\alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2) + (u_{10}\alpha_{31}\overline{N}_3 - u_{20}\alpha_{32}\overline{N}_3)/u_{30}$,

$U_1 = (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\overline{N}_2\overline{N}_3 + \left(u_{20}(\alpha_{13}\alpha_{32} - \alpha_{12}\alpha_{33})\overline{N}_1\overline{N}_3 + u_{30}(\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22})\overline{N}_1\overline{N}_2\right)/u_{10}$

$U_2 = (\alpha_{11}\alpha_{33} + \alpha_{31}\alpha_{13})\overline{N}_1\overline{N}_3 - \left(u_{30}(\alpha_{11}\alpha_{23} + \alpha_{13}\alpha_{21})\overline{N}_1\overline{N}_2 - u_{10}(\alpha_{21}\alpha_{33} - \alpha_{31}\alpha_{23})\overline{N}_2\overline{N}_3\right)/u_{20}$

$U_3 = (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})\overline{N}_1\overline{N}_2 - \left(u_{20}(\alpha_{11}\alpha_{32} + \alpha_{12}\alpha_{31})\overline{N}_1\overline{N}_3 - u_{10}(\alpha_{31}\alpha_{22} - \alpha_{21}\alpha_{32})\overline{N}_2\overline{N}_3\right)/u_{30}$

5. Global Stability:

Theorem (5.1) The Equilibrium point $E_5(\overline{N}_1, \overline{N}_2, 0)$ is globally asymptotically stable.

Proof: Let us consider the following Lyapunov function

$$V(N_1,N_2)=N_1-\overline{N}_1-\overline{N}_1\ln\left[\frac{N_1}{\overline{N}_1}\right]+d_1\left\{N_2-\overline{N}_2-\overline{N}_2\ln\left[\frac{N_2}{\overline{N}_2}\right]\right\} \qquad (5.1.1)$$

where ‘d_1 ‘ is positive constant ,to be chosen later

Differentiating V w.r.to ‘t’ we get

$$\frac{dV}{dt}=\left(\frac{N_1-\overline{N}_1}{N_1}\right)\frac{dN_1}{dt}+d_1\left(\frac{N_2-\overline{N}_2}{N_2}\right)\frac{dN_2}{dt} \qquad (5.1.2)$$

Choosing $d_1=\frac{\alpha_{12}}{\alpha_{21}}$, and with some algebraic manipulation yields

$$\frac{dV}{dt}=-\alpha_{11}(N_1-\overline{N}_1)^2-\frac{\alpha_{12}}{\alpha_{21}}\alpha_{22}(N_2-\overline{N}_2)^2<0\,. \qquad (5.1.3)$$

Therefore , $E_5(\overline{N}_1,\overline{N}_2,0)$ is globally asymptotically stable**.**

Theorem (5.2) The Equilibrium point $E_7(\overline{N}_1,0,\overline{N}_3)$ is globally asymptotically stable.
Proof: Let us consider the following Lyapunov function

$$V(N_1,N_3)=N_1-\overline{N}_1-\overline{N}_1\ln\left[\frac{N_1}{\overline{N}_1}\right]+d_2\left\{N_3-\overline{N}_3-\overline{N}_3\ln\left[\frac{N_3}{\overline{N}_3}\right]\right\} \qquad (5.2.1)$$

Where ‘d_2 ‘ is positive constant ,to be chosen later
Differentiating V w.r.to ‘t’ we get

$$\frac{dV}{dt}=\left(\frac{N_1-\overline{N}_1}{N_1}\right)\frac{dN_1}{dt}+d_2\left(\frac{N_3-\overline{N}_3}{N_3}\right)\frac{dN_3}{dt} \qquad (5.2.2)$$

Choosing $d_2=\frac{\alpha_{13}}{\alpha_{31}}$, and with some algebraic manipulation yields

$$\frac{dV}{dt}=-\alpha_{11}(N_1-\overline{N}_1)^2-\frac{\alpha_{13}}{\alpha_{31}}\alpha_{33}(N_3-\overline{N}_3)^2<0\,. \qquad (5.2.3)$$

Therefore , $E_7(\overline{N}_1,0,\overline{N}_3)$ is globally asymptotically stable**.**

Theorem (5.3) The Equilibrium point $E_8(\overline{N}_1,\overline{N}_2,\overline{N}_3)$ is globally asymptotically table.
Proof: Let us consider the following Lyapunov function

$$V(N_1,N_2,N_3)=N_1-\overline{N}_1-\overline{N}_1\ln\left[\frac{N_1}{\overline{N}_1}\right]+d_1^*\left\{N_2-\overline{N}_2-\overline{N}_2\ln\left[\frac{N_2}{\overline{N}_2}\right]\right\}+d_2^*\left\{N_3-\overline{N}_3-\overline{N}_3\ln\left[\frac{N_3}{\overline{N}_3}\right]\right\}$$

(5.3.1)

where d_1^*,d_2^* are positive constants ,to be chosen suitabily

Differentiating V w.r.to ‘t’ we get

$$\frac{dV}{dt}=\left(\frac{N_1-\overline{N}_1}{N_1}\right)\frac{dN_1}{dt}+d_1^*\left(\frac{N_2-\overline{N}_2}{N_2}\right)\frac{dN_2}{dt}+d_2^*\left(\frac{N_3-\overline{N}_3}{N_3}\right)\frac{dN_3}{dt} \qquad (5.3.2)$$

$$\frac{dV}{dt} = -\alpha_{11}(N_1-\overline{N}_1)^2 - \alpha_{12}(N_1-\overline{N}_1)(N_2-\overline{N}_2) - \alpha_{13}(N_1-\overline{N}_1)(N_3-\overline{N}_3)$$
$$+d_1^*\left\{\alpha_{21}(N_1-\overline{N}_1)(N_2-\overline{N}_2) - \alpha_{22}(N_2-\overline{N}_2)^2 - \alpha_{23}(N_2-\overline{N}_2)(N_3-\overline{N}_3)\right\}$$
$$+d_2^*\left\{\alpha_{31}(N_1-\overline{N}_1)(N_3-\overline{N}_3) - \alpha_{32}(N_2-\overline{N}_2)(N_3-\overline{N}_3) - \alpha_{33}(N_3-\overline{N}_3)^2\right\}$$

$$\frac{dV}{dt} < -\alpha_{11}(N_1-\overline{N}_1)^2 - (\alpha_{12}-\alpha_{21}d_1^*)\left[\frac{(N_1-\overline{N}_1)^2}{2} + \frac{(N_2-\overline{N}_2)^2}{2}\right]$$
$$-(\alpha_{13}-\alpha_{31}d_2^*)\left[\frac{(N_1-\overline{N}_1)^2}{2} + \frac{(N_3-\overline{N}_3)^2}{2}\right] - d_1^*\alpha_{22}(N_2-\overline{N}_2)^2 \qquad (5.3.3)$$
$$-(d_1^*\alpha_{23}+\alpha_{32}d_2^*)\left[\frac{(N_2-\overline{N}_2)^2}{2} + \frac{(N_3-\overline{N}_3)^2}{2}\right] - d_2^*\alpha_{33}(N_3-\overline{N}_3)^2 < 0$$

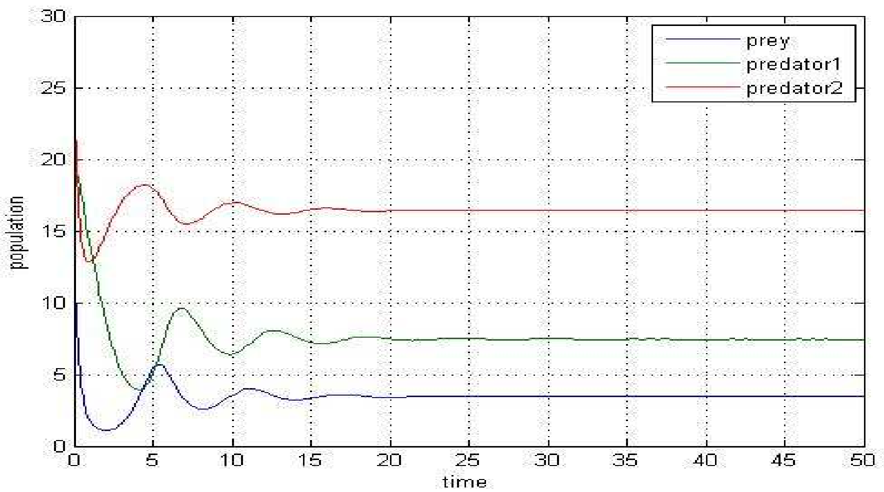
Choosing $d_1^* = \frac{\alpha_{12}}{\alpha_{21}}$, $d_2^* = \frac{\alpha_{13}}{\alpha_{31}}$ and with some algebraic manipulation yields

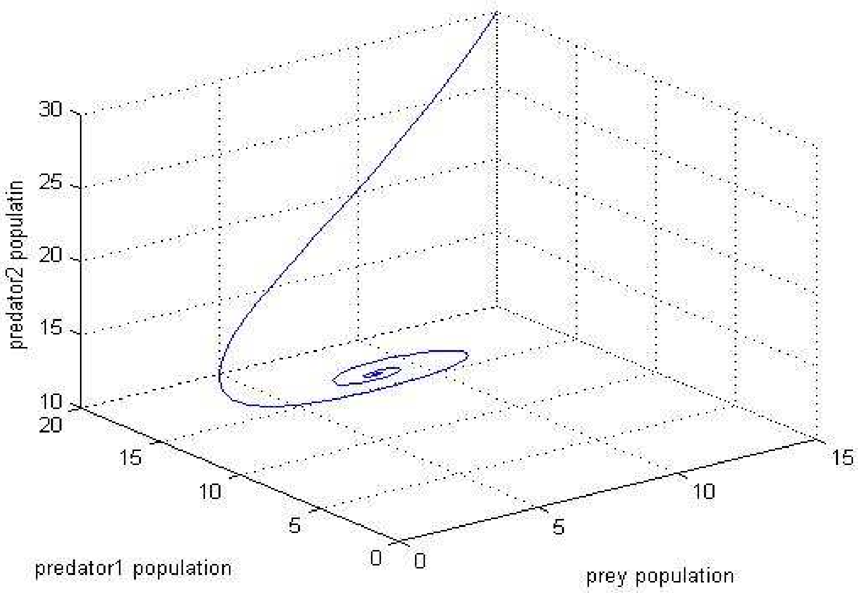
$$\frac{dV}{dt} < -\alpha_{11}(N_1-\overline{N}_1)^2 - \frac{\alpha_{12}}{\alpha_{21}}\alpha_{22}(N_2-\overline{N}_2)^2$$
$$-\left[\frac{\alpha_{12}}{\alpha_{21}}\alpha_{23} + \alpha_{32}\frac{\alpha_{13}}{\alpha_{31}}\right]\left[\frac{(N_2-\overline{N}_2)^2}{2} + \frac{(N_3-\overline{N}_3)^2}{2}\right] - \frac{\alpha_{13}}{\alpha_{31}}\alpha_{33}(N_3-\overline{N}_3)^2 < 0 \qquad (5.3.4)$$

Therefore , $E_8(\overline{N}_1, \overline{N}_2, \overline{N}_3)$ is globally asymptotically stable.

6. Numerical Examples:

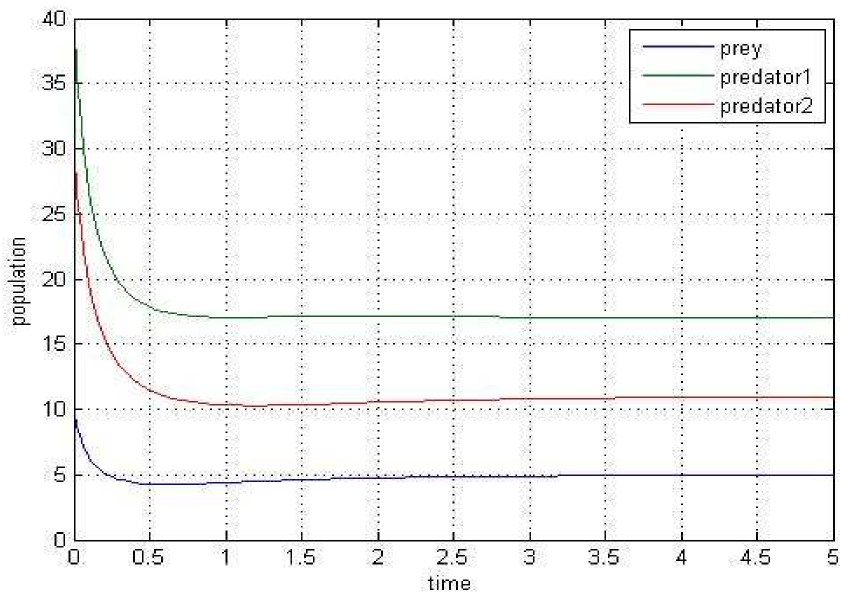
(1).Let $a_1=2$, $\alpha_{11}=0.1$, $\alpha_{12}=0.2$, $\alpha_{13}=0.01$, $a_2=3$, $\alpha_{21}=0.3$, $\alpha_{22}=0.1$, $\alpha_{23}=0.2$, $a_3=4$, $\alpha_{31}=0.01$, $\alpha_{32}=0.1$, $\alpha_{33}=0.2$

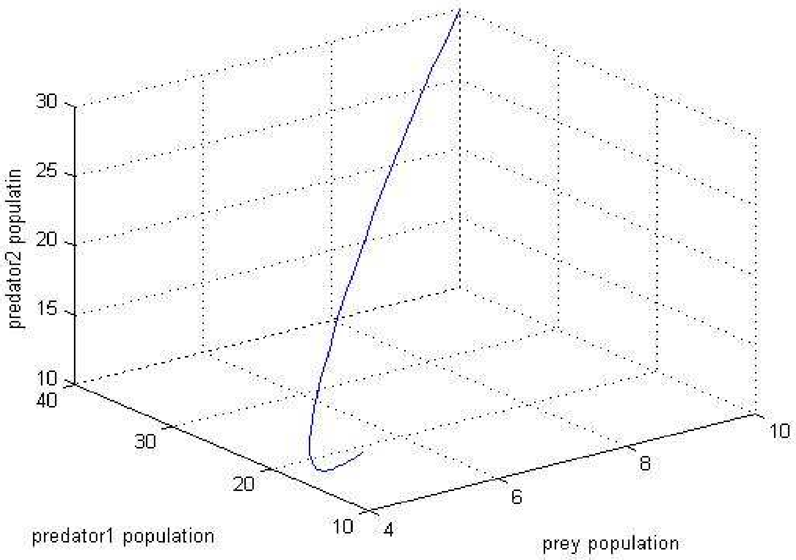




Figures shows the variation of the populations against the time .beginning with $N_1 = 15$, $N_2 = 20$ and $N_3 = 30$

(2). Let $a_1=5$, $\alpha_{11}=0.1$, $\alpha_{12}=0.2$, $\alpha_{13}=0.1$, $a_2=4$, $\alpha_{21}=0.1$, $\alpha_{22}=0.2$, $\alpha_{23}=0.1$, $a_3=3$, $\alpha_{31}=0.4$, $\alpha_{32}=0.1$, $\alpha_{33}=0.3$





Figures shows the variation of the populations against the time .beginning with N_1 =10, N_2 = 40 and N_3 =30

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REFERENCES

[1] Archana Reddy,R., On the stability of some mathematical models in biosciences interacting species, Ph. D thesis, JNTU **2009.**
[2] Cushing, J.M., Integro -differential equations and Delay models in Population Dynamics, Lecture Notes in Biomathematics,Vol.20,Springer -Verlag, Heidelberg**,1997**
[3] Freedman,H.I.,Deterministic Mathematical Models in Population Ecology, Marces Decker, New York, **1980.**
[4] George F. Simmons, Differential Equations with applications and historical notes, Tata McGraw-Hill, New Delhi**,1974.**
[5] Kapur,J.N., Mathematical Modelling,Weley-Eastern,**1998**
[6] Kapur,J.N., Mathematical Models in Biology and Medicine Affiliated East- West**,1985**.
[7] Lakshmi Narayan,K., A Mathematical study of Prey-Predator Ecological Models with a partial covers for the prey and alternative food for the predator, Ph. D thesis, , J.N.T. University **2004.**
[8] Lokta, A.J., Elements of Physical biology, Williams and Wilkins,Baltimore,**1925**.
[9] Meyer,W.J., Concepts of Mathematical Modelling, Mc Graw-Hill,**1985.**
[10] Paul Colinvaux, Ecology, John Wiley and Sons Inc., New York,**1986**.
[11] Ravindra Reddy,B.,LakshmiNarayan,K.,and Pattabhiramacharyulu,N.Ch., *International J. of Engg. Research and Indu.Appls*,2(II) **(2009)**,281-291.
[12] Shiva Reddy K.,Lakshmi Narayan K.,and Pattabhiramacharyulu,N.Ch., *The International J. of Math. Sci.& Engg*. Appls. Vol 4 (October,**2010**),pp. 129-145.
[13] Shiva Reddy K.,Lakshmi Narayan K.,and Pattabhiramacharyulu,N.Ch., *International J. of Mathematics Applied in Science and Technology*, Vol 2 ,No.1**(2010)**,pp.95-107.
[14] Srinivas,N.C., Some Mathematical aspects of modeling in Bio-Medical Sciences, Ph.D thesis, Kakatiya University**1991.**

[15] Volterra,V., Leconssen la theorie mathematique de la leitte pou lavie, Gauthier-Villars, Paris,**1931.**