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A Prey-Predator Model with an Alternative Food for the Predator and Optimal Harvesting of the Prey

K. Madhusudhan Reddy and *K. Lakshmi Narayan

Vardhaman College of Engineering, Hyderabad, India
**SLC'S Institute of Engineering & Technology, Piglipur, Hyderabad, India*

ABSTRACT

The present paper deals with a prey - predator model incorporating i) the predator is provided with an alternative food in addition to the prey, ii) the prey is harvested under optimal conditions. The model is characterized by a pair of first order non-linear differential equations. All the four equilibrium points of the model are identified and the criteria for the stability are discussed .The possibility of existence of bioeconomic equilibrium is discussed. The optimal harvest policy is studied with the help of Pontryagin's maximum principle. Finally, some numerical examples are discussed.

Key words: Prey, Predator, equilibrium points, stability, bionomic equilibrium point, optimal harvesting, threshold results, normal steady state, catch-per-unit-effort.

INTRODUCTION

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [6] and by Volterra [7].Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Paul Colinvaux [13], Freedman [14], Kapur [2, 3] etc. Harvesting of multispecies fisheries is an important area of study in fishery modeling. The issues and techniques related to this field of study and the problem of combined harvesting of two ecologically independent populations obeying the logistic law of growth are discussed in detail by Clark [11,12].. Chaudhuri [9, 10] formulated an optimal control problem for the combined harvesting of two competing species. Models on the combined harvesting of a two-species prey-predator fishery have been discussed by Chaudhuri and Saha Ray [8] Biological and bionomic equilibria of a multispecies fishery model with optimal harvesting policy is discussed in detail by Kar and Chaudhari [15]. Recently Archana Reddy [1] discussed the stability analysis of two interacting species with harvesting of both species. Lakshmi Narayan and Pattabhiramacharyulu [4, 5] and Shiva Reddy et al. [16,17] have discussed different prey-predator models in detail. Srilatha et al [19,20] discussed a four species model with different combination of interactionsbetween the them. Most of the mathematical models on the harvesting of a multispecies fishery have so far assumed that the species are affected by harvesting only.

A population model proposed by Kar and Chaudhuri, (c.f. Harvesting in a two-prey one-predator fishery: Bioeconomic model, ANZIAM J.45 (2004), 443-456) and this model motivated the present investigation. In the present investigation, we discussed a prey-predator model by taking an alternative food for the predator and harvesting of the prey under optimal conditions. The model is characterized by a pair of first order non-linear differential equations**.** The existence of the possible steady states along with their local stability is discussed. We derive the conditions for global stability of the system using a Liapunov function .The possibility of existence of bioeconomic equilibrium is discussed. The optimal harvest policy is studied and the solution is derived in the equilibrium case by using Pontryagin's maximum principle [18]. Finally, some numerical examples are discussed.

2. Mathematical Model.

The model equations for a two species prey-predator system are given by the following system of non-linear ordinary differential equations employing the following notation:

N_1 and N_2 are the populations of the prey and predator with natural growth rates a_1 and a_2 respectively, α_{11} is rate of decrease of the prey due to insufficient food, α_{12} is rate of decrease of the prey due to inhibition by the predator, α_{21} is rate of increase of the predator due to successful attacks on the prey, α_{22} is rate of decrease of the predator due to insufficient food other than the prey; q_1 is the catch ability co-efficient of the prey, E is the harvesting effort and $q_1 EN_1$ is the catch-rate function based on the CPUE (catch-per-unit-effort) hypothesis]. Further both the variables N_1 and N_2 are non-negative and the model parameters a_1, a_2, α_{11}, α_{12}, α_{21}, α_{22}, q_1, E and $(a_1 - q_1 E)$ are assumed to be non-negative constants.

$$\frac{dN_1}{dt} = (a_1 - q_1 E) N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 \tag{2.1}$$

$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^2 + \alpha_{21} N_1 N_2 \tag{2.2}$$

3. Equilibrium States.

The system under investigation has four equilibrium states defined by

I. The fully washed out state with the equilibrium point $\overline{N}_1 = 0; \overline{N}_2 = 0$ (3.1)

II. The state in which, only the predator survives given by $\overline{N}_1 = 0 ; \overline{N}_2 = \dfrac{a_2}{\alpha_{22}}$ (3.2)

III**.** The state in which, only the prey survives given by $\overline{N}_1 = \dfrac{(a_1 - q_1 E)}{\alpha_{11}} ; \overline{N}_2 = 0$ (3.3)

IV**.** The co-existent state (**normal steady state**) given by

$$\overline{N}_1 = \frac{\alpha_{22}(a_1 - q_1E) - a_2\alpha_{12}}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}; \quad \overline{N}_2 = \frac{a_2\alpha_{11} + \alpha_{21}(a_1 - q_1E)}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}} \tag{3.4}$$

This state would exit only when $\alpha_{22}\left(a_1 - q_1E\right) > a_2\alpha_{12}$ (3.5)

4. Stability of the Equilibrium States

To investigate the stability of the equilibrium states we consider small perturbations u_1, u_2 in $N_1$and N_2 over $\overline{N}_1$ and $\overline{N}_2$ respectively, so that

$$N_1 = \overline{N}_1 + u_1 ; N_2 = \overline{N}_2 + u_2 \tag{4.1}$$

By substituting (4.1) in (2.1) & (2.2) and neglecting second and higher powers of the perturbations u_1, u_2 we get the equations of the perturbed state

$$\frac{dU}{dt} = AU \tag{4.2}$$

where

$$A = \begin{bmatrix} (a_1 - q_1E) - 2\alpha_{11}\overline{N}_1 - \alpha_{12}\overline{N}_2 & -\alpha_{12}\overline{N}_1 \\ \alpha_{21}\overline{N}_2 & a_2 + \alpha_{21}\overline{N}_1 - 2\alpha_{22}\overline{N}_2 \end{bmatrix} \tag{4.3}$$

The characteristic equation for the system is $det\left[A - \lambda I\right] = 0$ (4.4)

The equilibrium state is **stable** only when the roots of the equation (4.4) are negative in case they are real or have negative real parts in case they are complex.

The equilibrium points I, II, and III are found to unstable, so we restricted our study to the normal steady state only.

4.1 Stability of the normal steady state:

In this case the characteristic equation is

$$\lambda^2 + (\alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2)\lambda + [\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}]\,\overline{N}_1\overline{N}_2 = 0 \tag{4.5}$$

Since the sum of the roots of (4.5) is negative and the product of the roots is positive, the roots of which can be noted to be negative .Hence the co-existent equilibrium state is **stable.**
The solutions curves are:

$$u_1 = \left[\frac{u_{10}(\lambda_1 + \alpha_{22}\overline{N}_2) - u_{20}\alpha_{12}\overline{N}_1}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{10}(\lambda_2 + \alpha_{22}\overline{N}_2) - u_{20}\alpha_{12}\overline{N}_1}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t} \tag{4.6}$$

$$u_2 = \left[\frac{u_{20}(\lambda_1 + \alpha_{11}\overline{N}_1) + u_{10}\alpha_{21}\overline{N}_2}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{20}(\lambda_2 + \alpha_{11}\overline{N}_1) + u_{10}\alpha_{21}\overline{N}_2}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t} \quad (4.7)$$

Where λ_1, λ_2 are the roots of the equation (4.5)

5. Global stability

Let us consider the following Liapunov's function

$$V(N_1, N_2) = N_1 - \overline{N}_1 - \overline{N}_1 \ln\left[\frac{N_1}{\overline{N}_1}\right] + l\left\{N_2 - \overline{N}_2 - \overline{N}_2 \ln\left[\frac{N_2}{\overline{N}_2}\right]\right\} \quad (5.1)$$

where '*l*' is positive constant ,to be chosen later

Differentiating V w.r.to 't' we get

$$\frac{dV}{dt} = \left(\frac{N_1 - \overline{N}_1}{N_1}\right)\frac{dN_1}{dt} + l\left(\frac{N_2 - \overline{N}_2}{N_2}\right)\frac{dN_2}{dt} \quad (5.2)$$

Substituting (2.1) and (2.2) in (5.2), we get

$$\frac{dV}{dt} = \left\{-\alpha_{11}\left(N_1 - \overline{N}_1\right)^2 - \alpha_{12}\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right)\right\} + l\left\{\alpha_{21}\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right) - \alpha_{22}\left(N_2 - \overline{N}_2\right)^2\right\}$$

Choosing $l = \frac{\alpha_{12}}{\alpha_{21}}$ and with some algebraic manipulation yields

$$\frac{dV}{dt} = -\alpha_{11}(N_1 - \overline{N}_1)^2 - \frac{\alpha_{12}}{\alpha_{21}}\alpha_{22}(N_2 - \overline{N}_2)^2 < 0. \quad (5.3)$$

Therefore , the equilibrium point $(\overline{N}_1, \overline{N}_2)$ is globally **asymptotically stable.**

6. Bionomic equilibrium

The term bionomic equilibrium is an amalgamation of the concepts of biological equilibrium as well as economic equilibrium. The economic equilibrium is said to be achieved when the total revenue obtained by selling the harvested biomass equals the total cost for the effort devoted to harvesting.

Let $c_1 =$ fishing cost per unit effort of the prey, $p_1 =$ price per unit biomass of the prey. The net economic revenue for the prey at any time *t* is given by

$$R_1 = \left(p_1 q_1 N_1 - c_1\right) E \quad (6.1)$$

The biological equilibrium is $\left((N_1)_\infty, (N_2)_\infty, (E)_\infty\right)$, where $(N_1)_\infty, (N_2)_\infty, (E)_\infty$ are the positive solutions of

$$\left(a_1 - q_1 E\right) N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 = 0 \quad (6.2)$$

$$a_2 N_2 - \alpha_{22} N_2^2 + \alpha_{21} N_1 N_2 = 0 \quad (6.3)$$

and $\left(p_1 q_1 N_1 - c_1\right) E = 0 \quad (6.4)$

From (6.4), we have

$$\{p_1q_1(N_1)_\infty - c_1\}(E)_\infty = 0 \Rightarrow (N_1)_\infty = \frac{c_1}{p_1q_1} \quad (6.5)$$

From (6.3) and (6.5), we get $(N_2)_\infty = \frac{1}{\alpha_{22}}\left(a_2 + \alpha_{21}\frac{c_1}{p_1q_1}\right)$ (6.6)

From (6.2), (6.5) & (6.6), we get $(E)_\infty = \frac{1}{q_1}\left(a_1 - \alpha_{11}\frac{c_1}{p_1q_1} - \alpha_{12}(N_2)_\infty\right)$ (6.7)

It is clear that $(E)_\infty > 0$ if $\left(a_1 - \alpha_{11}\frac{c_1}{p_1q_1}\right) > \alpha_{12}(N_2)_\infty$ (6.8)

Thus the bionomic equilibrium $((N_1)_\infty, (N_2)_\infty, (E)_\infty)$ exists, if inequality (6.8) holds.

7. Optimal harvesting policy

The present value J of a continuous time-stream of revenues is given by

$$J = \int_0^\infty e^{-\delta t}(p_1q_1N_1 - c_1)E\,dt \quad (7.1)$$

Where δ denotes the instantaneous annual rate of discount. Our problem is to maximize J subject to the state equations (2.1) & (2.2) and control constraints $0 \le E \le (E)_{\max}$ by invoking Pontryagin's maximum principle.

The Hamiltonian for the problem is given by

$$H = e^{-\delta t}(p_1q_1N_1 - c_1)E + \lambda_1(a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1N_2 - q_1EN_1) + \lambda_2(a_2N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_1N_2) \quad (7.2)$$

Where λ_1, λ_2 are the adjoint variables.

Let us assume that the control constraints are not binding i.e. the optimal solution does not occur at $(E)_{\max}$. At $(E)_{\max}$ we have a singular control.

By Pontryagin's maximum principle,

$$\frac{\partial H}{\partial E} = 0 \;;\quad \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_1} \;;\quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial N_2}$$

$$\frac{\partial H}{\partial E} = 0 \Rightarrow e^{-\delta t}(p_1q_1N_1 - c_1) - \lambda_1q_1N_1 = 0 \Rightarrow \lambda_1 = e^{-\delta t}\left(p_1 - \frac{c_1}{q_1N_1}\right) \quad (7.3)$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_1} = -\left\{e^{-\delta t}p_1q_1E + \lambda_1\left[(a_1 - q_1E) - 2\alpha_{11}N_1 - \alpha_{12}N_2\right] + \lambda_2(\alpha_{21}N_2)\right\}$$

$$\Rightarrow \frac{d\lambda_1}{dt} = (\lambda_1\alpha_{11}N_1 - \lambda_2\alpha_{21}N_2 - e^{-\delta t}p_1q_1E) \quad (7.4)$$

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_2} = -\left\{\lambda_1\left(-\alpha_{12}N_1\right) + \lambda_2\left(a_2 + \alpha_{21}N_1 - 2\alpha_{22}N_2\right)\right\}$$

$$\Rightarrow \frac{d\lambda_2}{dt} = \left(\lambda_1\alpha_{12}N_1 + \lambda_2\alpha_{22}N_2\right) \quad (7.5)$$

From (7.3) & (7.5), we get $\frac{d\lambda_2}{dt} - \lambda_2\alpha_{22}N_2 = A_1 e^{-\delta t}$

Where $A_1 = \alpha_{12}\overline{N}_1\left(p_1 - \frac{c_1}{q_1\overline{N}_1}\right)$

Whose solution is given by $\lambda_2 = -\frac{A_1}{\left(\alpha_{22}\overline{N}_2 + \delta\right)} e^{-\delta t}$ (7.6)

From (7.4) & (7.6), we get

$$\frac{d\lambda_1}{dt} - \lambda_1\alpha_{11}N_1 = -A_2 e^{-\delta t}$$

Where $A_2 = \left[p_1 q_1 E - \frac{A_1\alpha_{21}\overline{N}_2}{\left(\alpha_{22}\overline{N}_2 + \delta\right)}\right]$

Whose solution is given by $\lambda_1 = \frac{A_2}{\left(\alpha_{11}\overline{N}_1 + \delta\right)} e^{-\delta t}$ (7.7)

From (7.3) & (7.7), we get a singular path

$$\left(p_1 - \frac{c_1}{q_1\overline{N}_1}\right) = \frac{A_2}{\left(\alpha_{11}\overline{N}_1 + \delta\right)} \quad (7.8)$$

Thus (7.8) can be written as

$$F(\overline{N}_1) = \left(p_1 - \frac{c_1}{q_1\overline{N}_1}\right) - \frac{A_2}{\left(\alpha_{11}\overline{N}_1 + \delta\right)}$$

There exist a unique positive root $\overline{N}_1 = \left(N_1\right)_\delta$ of $F(\overline{N}_1) = 0$ in the interval $0 < \overline{N}_1 < k_1$, if the following hold $F(0) < 0$, $F(k_1) > 0$, $F'(\overline{N}_1)) > 0$ for $\overline{N}_1 > 0$.

For $\overline{N}_1 = \left(N_1\right)_\delta$

we get $\left(N_2\right)_\delta = \frac{1}{\alpha_{22}}\left(a_2 + \alpha_{21}\frac{c_1}{p_1 q_1}\right)$ (7.9)

and $\left(E\right)_\delta = \frac{1}{q_1}\left[a_1 - \alpha_{11}\frac{c_1}{p_1 q_1} - \alpha_{12}\left(N_2\right)_\delta\right]$ (7.10)

Hence once the optimal equilibrium $\left(\left(N_1\right)_\delta, \left(N_2\right)_\delta\right)$ is determined, the optimal harvesting effort $\left(E\right)_\delta$ can be determined.

From (7.3), (7.6) and (7.7), we found that λ_1, λ_2 do not vary with time in optimal equilibrium. Hence they remain bounded as $t \to \infty$.

From (7.8), we also note that

$$\left(p_1 - \frac{c_1}{q_1 \overline{N}_1} \right) = \frac{A_2}{\left(\alpha_{11} \overline{N}_1 + \delta \right)} \rightarrow 0 \text{ as } \delta \rightarrow \infty$$

Thus, the net economic revenue of the prey $R_1 = 0$.

This implies that if the discount rate increases, then the net economic revenue decreases and even may tend to zero if the discount rate tend to infinity. Thus it has been concluded that high interest rate will cause high inflation rate. This conclusion was also drawn by Clark [11] in the combined harvesting of two ecologically independent populations and by Chaudhuri [8] in the combined harvesting of two competing species.

9. Numerical Examples

Let $a_1 = 3$; $\alpha_{11} = 0.1$; $\alpha_{12} = 0.12$; $a_2 = 1.5$; $\alpha_{21} = 0.03$; $\alpha_{22} = 0.4$; $q_1 = 0.02$ & $E = 20$.

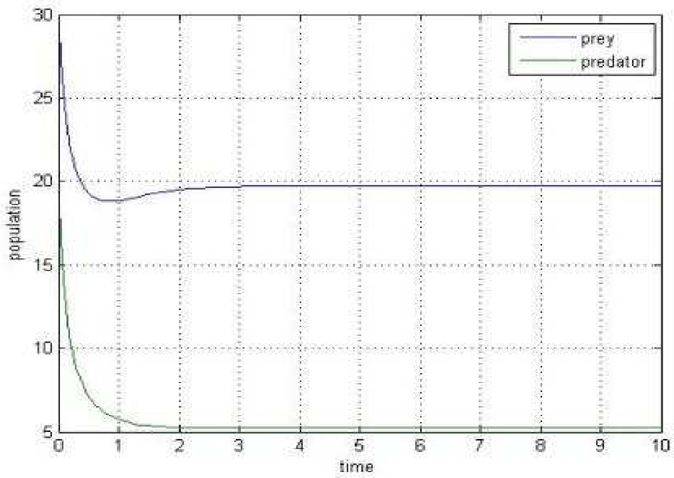

Fig. 4

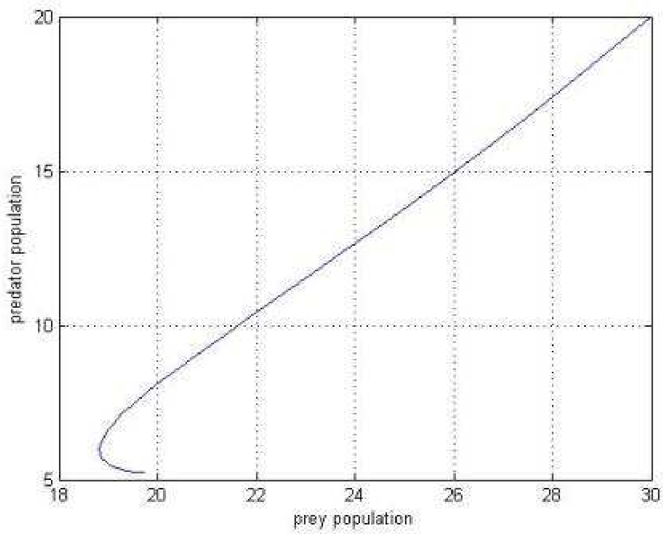

Fig. 5

(i) Fig.4 shows the variation of the populations against the time and **(ii)** Fig.5 shows the trajectory corresponding to the prey and predator populations beginning with N_1=30 and N_2=20.

(2) Let $a_1 = 8;\ \alpha_{11} = 0.01;\ \alpha_{12} = 0.3;\ a_2 = 1.5;\ \alpha_{21} = 0.12;\ \alpha_{22} = 0.14;\ q_1 = 0.04\ \&\ E = 5.$

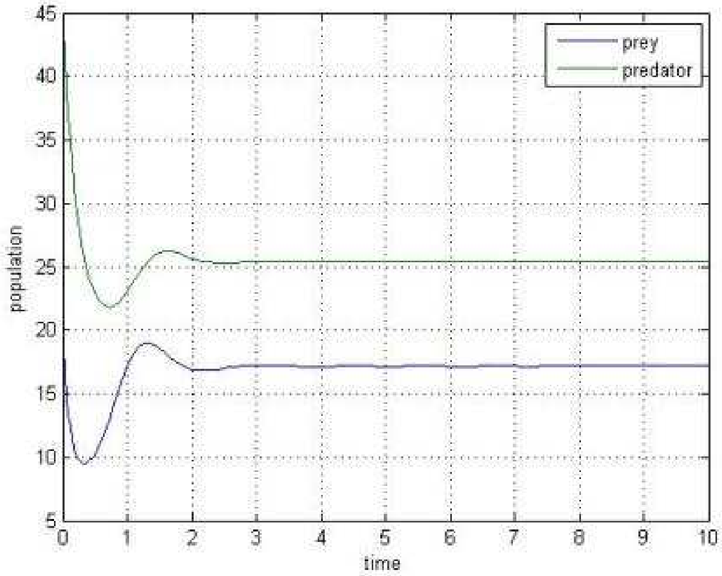

Fig. 6

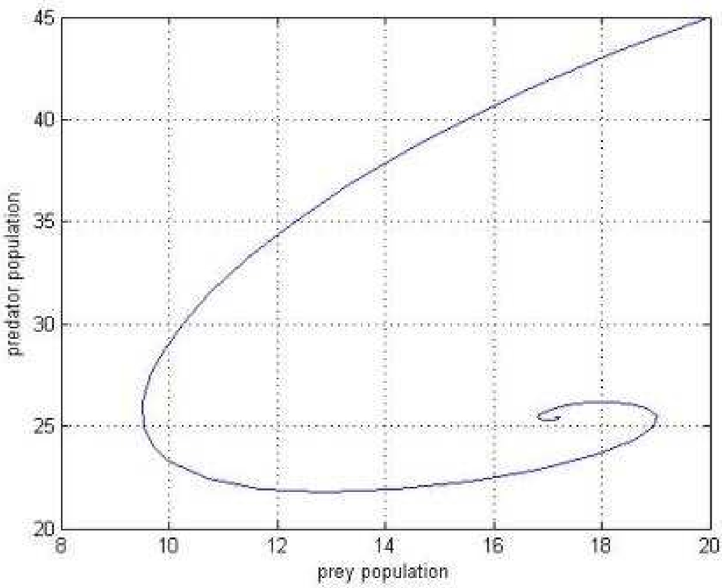

Fig. 7

(i) Fig.6 shows the variation of the populations against the time and **(ii)** Fig.7 shows the trajectory corresponding to the prey and predator populations beginning with N_1=20 and N_2=45.

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