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A numerical grid and grid less (Mesh less) techniques for the solution of 2D Laplace equation

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ABSTRACT

In this paper numerical technique has been used to solve two dimensional Laplace Equation with Dirichlet boundary conditions in a rectangular domain and focuses on certain numerical techniques for solving PDEs; in particular, the Grid technique viz Finite difference (FD), the Finite element (FE), and Grid less (GL) technique are presented by using Metlab and Spreadsheet. Finally the numerical solutions obtained by Numerical techniques are compared with exact solution to check the accuracy of the developed scheme.

Keywords: Finite Difference, Finite Element, Laplace Equation, Grid less technique.

INTRODUCTION

Partial differential equations occur very frequently in Science, Engineering and Applied Mathematics. Many partial differential equations cannot be solved by analytical methods in closed form solution. There are several different techniques to the construction of numerical methods for PDE solution. For problems involving complex geometries, generating a mesh is typically the most costly and time consuming part of the solution process. As a result, much effort has been devoted to the development of the so called grid based and grid less techniques. The most popular techniques are grid based-Finite difference (FD) and Finite element (FE) and without grid- Grid less techniques.

The remainder of the papers is organised as follows. In Section 2, a short review of gird based and without grid based methods is given. In Section 3, formulation of two different 2D Laplace equations with dirichlet boundary conditions. In Section 4, the numerical results obtained by using spreadsheet and metlab techniques. Finally, Section 5 concludes the paper.

MATERIALS AND METHODS

Finite Difference Grid Technique

The finite difference method (FDM) is conceptually simple. The problems to which the method applies are specified by a PDE, a solution region (geometry) and boundary conditions. For more detailed derivations the reader may consult [1] and [2]. The finite difference method entails three basic steps**.**

1. Divide the solution region into a grid of nodes. Grid points are typically arranged in a rectangular array of nodes.
2. Approximate the PDE and boundary conditions by a set of linear algebraic equations (the finite difference equations) on grid points within the solution region.
3. Solve this set of linear algebraic equations.

Finite Element Grid Technique

The finite element method (FEM) is a numerical technique for solving PDEs. FEM was originally applied to problems in Science and Engineering stream. For more detailed derivations the reader may consult [1] and [3]. The finite element analysis involves four basic steps.

1. Divide the solution region into a finite number of elements. The most common elements have triangular or quadrilateral shapes. The collection of all elements should resemble the original region as closely as possible.
2. Derive governing equations for a typical element. This step will determine the element coefficient matrix.
3. Assemble all elements in the solution region to obtain the global coefficient matrix.
4. Solve the resulting system of equations.

Grid less Technique

A mesh is a net that is formed by connecting nodes in a predefined manner. Mesh Free Methods use a set of nodes scattered within the problem domain as well as sets of nodes scattered on the boundaries of the domain to represent (not discretize) the problem domain and its boundaries. No mesh implies no information on the relationship between the nodes is required. For more detailed the reader may consult [4]. The Mesh less method involves four basic steps.

1. The problem domain is defined and a set of nodes is chosen to represent the problem domain and its boundary. Also, note that the density of the nodes is not uniform.
2. Define the support domain for a point x is a sphere of a certain radius that relates to the nodal spacing near the point x, then determines the number of nodes to be used to approximate the function value at x.
3. The field variable u at any point x within the problem domain is interpolated using the values of this field at all the nodes within the support domain of x. Mathematically,

$$u(x) = \sum_{i=1}^{n} \emptyset_i(x) u_i$$

4. The equations of a mesh less method can be formulated using the shape functions and a strong or weak form system equation.
5. Solve the resulting global Mesh less equations by using numerical method depend on the kind of equations.

PROBLEMS FORMULATION

Problem I

Consider, the two dimensional Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ,0 \leq x, y \leq 1$$

Subject to the Dirichlet boundary condition,

$$u(x,0) = 0\,, \quad u(1,y) = 0$$
$$u(x,1) = 1\,, \quad u(0,y) = 0$$

as shown in figure 1.

The exact solution of this problem is given by

$$u(x,y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)(\sin(n\pi x))\sinh(n\pi y)}{n(\sinh(n\pi y))}$$

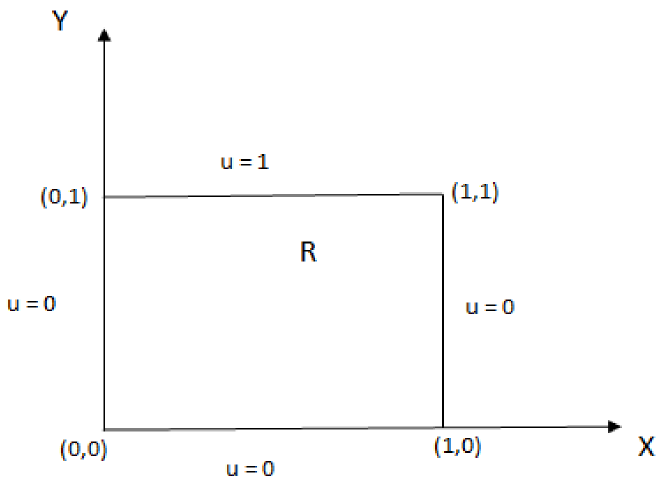

Figure 1:- Square region R with Dirichlet boundary conditions

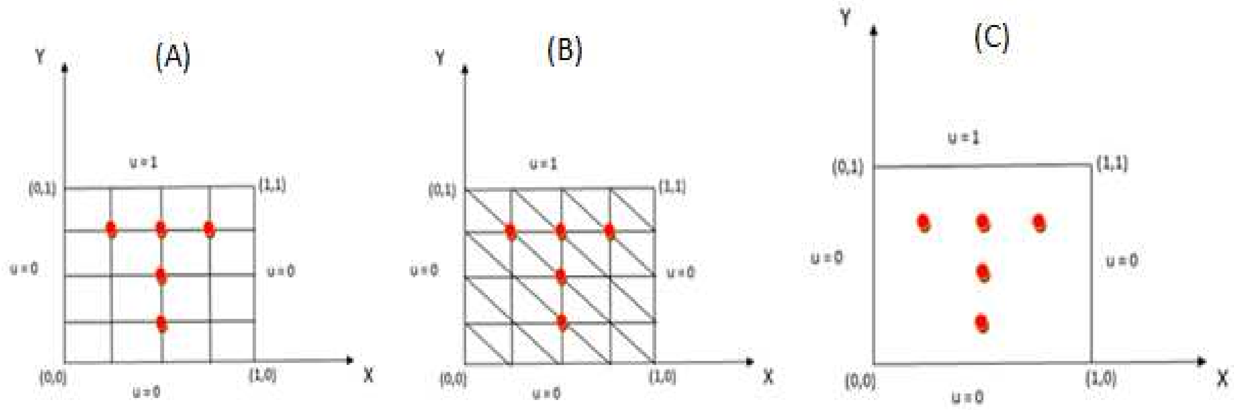

Figure2:- The region R showing prescribed potentials at the boundaries and the free nodes to illustrate (A) The Finite Difference Method, (B) The Finite Element Method and (C) The Grid less Method

Problem II

Consider, the two dimensional Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ,0 \leq x, y \leq 1$$

Subject to the Dirichlet boundary condition,

$$u(x,0) = -y^2 \quad , \qquad u(1,y) = 1 - y^2$$
$$u(x,1) = x^2 - 1, \qquad u(0,y) = x^2$$

as shown in figure 3.

The exact solution of this problem is given by $u(x,y) = x^2 - y^2$

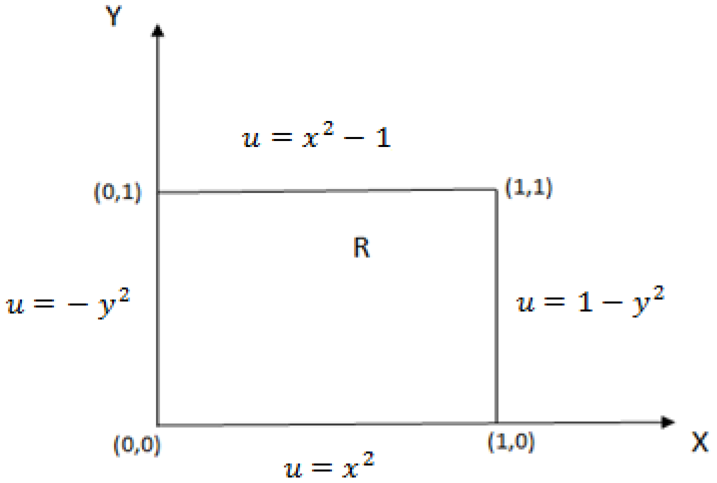

Figure 3:- Square region R with Dirichlet boundary conditions

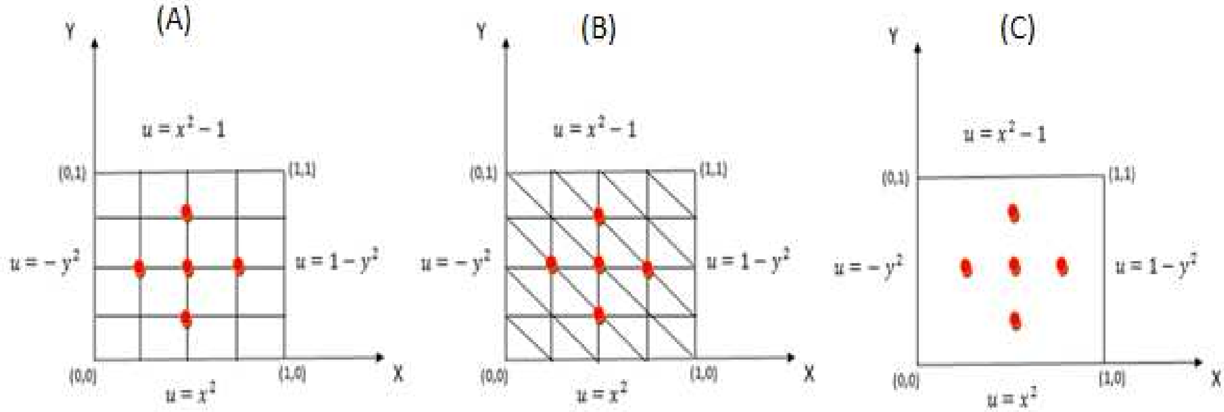

Figure 4:- The region R showing prescribed potentials at the boundaries and the free nodes to illustrate (A) The Finite Difference Method, (B) The Finite Element Method and (C) The Grid less Method

RESULTS AND DISCUSSION

As indicated in table 1 and 2, the potentials at the free nodes computed by FDM, FEM and GLM numerical solutions compared fairly well. The better agreement should be obtained between the finite Difference numerical solution results with rectangular grids, the finite element numerical solution results with a triangular grids and the mesh less method numerical solution results without grids.

Table 1: Comparison between Numerical solution with Exact for Problem I

Internal Points	Node	FDM	FEM	GLM	EXACT	ERROR		
						FDM	FEM	GLM
(0.25,0.75)	1	0.4227	0.4286	0.4304	0.4304	0.0093	0.0034	0.0016
(0.50,0.75)	2	0.5190	0.5268	0.5406	0.5406	0.0215	0.0137	-0.0001
(0.50,0.50)	3	0.2383	0.2500	0.2500	0.2500	0.0117	0.0000	0.0000
(0.75,0.75)	4	0.4227	0.4286	0.4337	0.4337	0.0093	0.0034	-0.0017
(0.50,0.25)	5	0.0904	0.0982	0.0954	0.0954	0.0050	-0.0028	0.0000

Table 2: Comparison between Numerical solution with Exact for Problem II

Internal Points	Node	FDM	FEM	GLM	EXACT	ERROR		
						FDM	FEM	GLM
(0.25,0.50)	1	-0.1875	-0.1875	-0.1875	-0.1875	0.0000	0.0000	0.0000
(0.50,0.25)	2	0.1875	0.1875	0.1875	0.1875	0.0000	0.0000	0.0000
(0.50,0.50)	3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
(0.50,0.75)	4	-0.3125	-0.3125	-0.3125	-0.3125	0.0000	0.0000	0.0000
(0.75,0.50)	5	0.3125	0.3125	0.3125	0.3125	0.0000	0.0000	0.0000

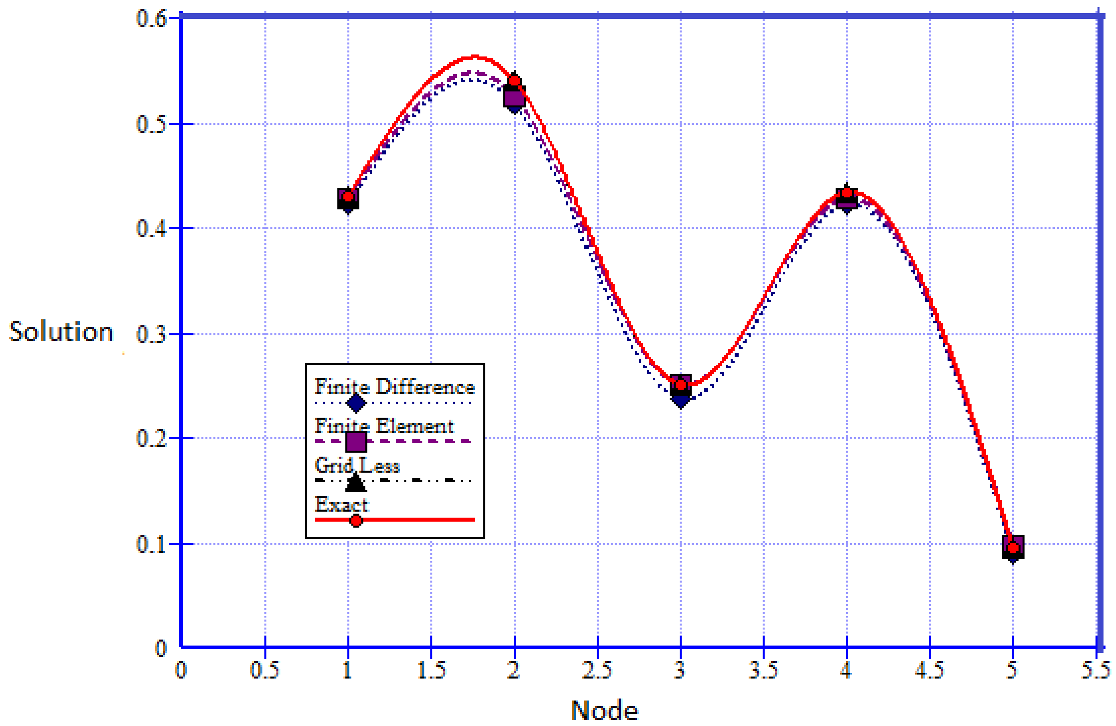

Figure 5: A comparison Between of FD, FE and GL with Exact Solution for Problem I

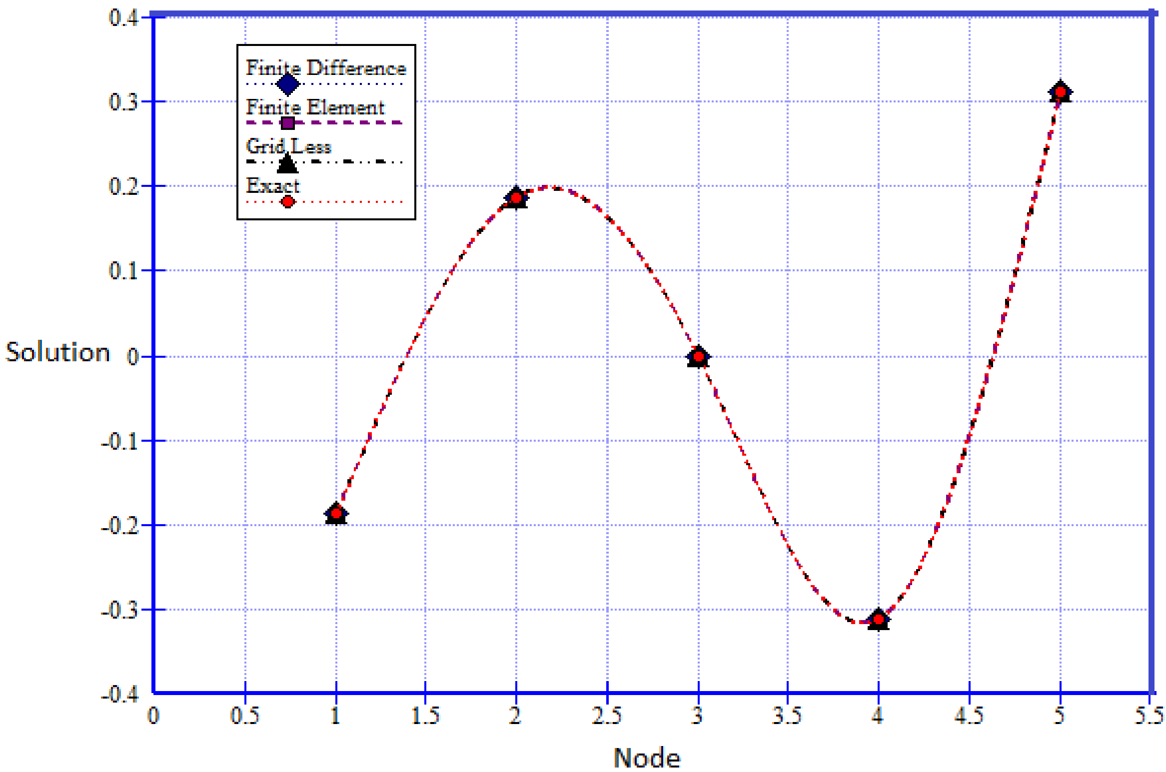

Figure 6: A comparison Between of FD, FE and GL with Exact Solution for Problem II

CONCLUSION

Laplace equation with Dirichlet boundary conditions are solved by Finite difference, Finite element and Grid less techniques. The spreadsheet implementation and MATLAB computer program were utilized for the calculation process of Finite difference, Finite element (Grid techniques) and Grid less technique respectively. The power of the Grid less technique becomes more evident, because the Grid technique will have much more difficulty in solving problems in a domain with complex geometries. It is observed that Finite difference, Finite element (Grid techniques) and Grid less technique Solution give near to the exact solution.

REFERENCES

[1] Parag V. Patil and Dr. J.S.V.R. Krishna Prasad, *International Organization of Scientific Research- Journal of Mathematics,6* (4), **2013**, 66-75.
[2] Mark A. Lau, and Sastry P. Kuruganty, *Spreadsheets in Education (eJSiE)*, 4(1), **2010**.
[3] M. N. O. Sadiku, *Elements of Electromagnetics* (New York: Oxford University Press, 4th edition, **2006**).
[4] Nissaya Chuathong and Wattana Toutip, Proceedings of the 16th Annual Meeting in Mathematics (AMM2011), 10-11 March.**2011**, Khon Kaen University, Khon Kaen, Thailand, pp. 29-42.
[5] Erwin kreyszig, Advanced Engineering Mathematics (New York: John Wiley & Sons, 10th edition, **2011**).