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A Note on Gravitational Radiation with Geometric Constraints

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ABSTRACT

Beginning with the Einstein Field equations, we characterize gravitational waves by expressing the one dimensional wave equation in terms of Lie derivatives for the space-time general relativity. A condition for a radiative gravitational field is obtained. This condition proves to be satisfied for the exact solution, including plane waves as well as approximate solutions.

Key words: Einstein Field equations, space-time, Lie derivative, gravitational radiation.

INTRODUCTION

Our starting point is the Einstein field equations of general relativity written as; where

$$G_{\mu\nu} = \gamma T_{\mu\nu} \qquad \text{(EFE)}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci tensor. $T_{\mu\nu}$ is the energy-momentum tensor. $\gamma = 8\pi G/c^4$, G being the Newtonian gravitational constant. For the moment we consider three basic important properties of the above equation.

1) The equation (EFE) is a tensor equation. This is necessarily so, since the principle of invariance under coordinate transformations must hold, in other words the equations of physics must look the same in any frame of reference.
2) We can interpret equation (EFE) more simply as;

Tensor representing geometry of space = *Tensor representing energy content of space*

i.e. it is the presence of matter in space that distorts the neighbouring geometry. Most equations of mathematical physics can be interpreted similarly.

3) The solution to equation (EFE) is a geometrical object, namely a line element given by

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \qquad (1)$$

where $g_{\mu\nu}$ is the metric tensor to be solved for in (EFE).
In this paper we discuss the notion of gravitational waves, the existence of which has now been generally accepted [Kenyon I.R (1990), Roos M. (2003)] In particular a gravitational wave is a solution of the time dependent Einstein

field equations (EFE). Such solutions do exist; however, there is need give a more precise definition of such waves. Consider for example the case of electromagnetic waves where all solutions of the time dependent Maxwell's equations do not always represent an electromagnetic wave. In the same manner in general relativity, not all time dependent solutions represent a wave.

The properties of gravitational waves are such that they travel through space-time at the speed of light, producing ripples of curvature, an oscillatory stretching and squeezing of space-time. Any matter passing through will feel this effect; however the waves act on an exceedingly weaker scale. In contrast to the electromagnetic field, which is a vector field, the gravitational field is a tensor field.

2.0 Weak field limit

In order to study gravitational waves in general relativity it is necessary that we place ourselves far from the source. This implies that we implement a linear approximation due to the weak nature of the field, in other words the metric tensor g with signature $(+\ -\ -\ -)$ is perturbed in Minkowski's space-time; i.e.

$$g_{\mu\eta} = \eta_{\mu\nu} + h_{\mu\nu}; \quad h_{\mu\nu} < 1 \qquad (2)$$

where $h_{\mu\nu}$ is a small increment to $\eta_{\mu\nu}$.

Far away from the gravitational source, the EFE is written;

$$\Box \bar{h}_{\mu\nu} \equiv \eta^{\alpha\beta} \bar{h}_{\mu\nu;\alpha\beta} = 0 \qquad (3.1)$$

where ";" denotes covariant derivative and

$$\Rightarrow \bar{h}^{\alpha\beta}_{;\alpha\beta} = 0 \qquad (3.2)$$

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta} h^{\varepsilon}{}_{\varepsilon} \qquad (4)$$

Wave solutions exist [Roos M. (2003), Peebles, P. J. E., Ratra, B. (2003)], for equations (3) and the line element (1) is written

$$ds^2 = c^2 dt - (1 + h_{xx}) dx^2 + (1 - h_{yy}) dy^2 + 2h_{xx} dxdy - dz^2 \qquad (5)$$

where

$$\begin{cases} h_{xx} = a\cos\left[\dfrac{\omega}{c}(z - ct) + \phi\right] \\ h_{yy} = b\cos\left[\dfrac{\omega}{c}(z - ct) + \varphi\right] \end{cases}, \qquad (6)$$

a, b, ϕ, φ being constants. We observe here that our coordinates are; $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$.

The metric tensor is thus;

$$g_{\mu\nu} = \begin{pmatrix} ct & 2h_{xx} & 0 & 0 & 0 \\ 0 & -(1 + h_{xx}) & 0 & 0 & 0 \\ 0 & 0 & (1 - h_{xx}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad (7)$$

It then follows that the metric g is a function of time, more precisely

$$g_{\mu\nu} = g_{\mu\nu}\left[(z - ct), xy\right] \qquad (8)$$

for waves propagating along the z-axis.

3.0 The Lie derivative fromulation

Our objective is to try to generalize this concept of gravitational wave.

Consider the one dimensional wave equation propagating along the z-axis;

$$\frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial z^2}=0; \quad \varphi=\varphi(t,z) \qquad (9)$$

v being the speed of propagation. On the other hand, a wave travelling in vacuum with the speed of light ($c=1$) can be written in the space time coordinates $\left(x^0,x^1,x^2,x^3\right)$ of general relativity as;

$$\frac{\partial^2\varphi}{\partial\left(x^0\right)^2}-\frac{\partial^2\varphi}{\partial\left(x^3\right)^2}=\partial_0^2\varphi-\partial_3^2\varphi=0 \qquad (10)$$

We will now reformulate the above equation in the space time $\left(R^4,g\right)$ of general relativity without employing d'Alembert's differential operator $\square\varphi_{\mu\nu}\equiv g^{\alpha\beta}\partial_\alpha\partial_\beta$ as is normally the approach [Gerssen , J. et al. (2002)].

To this end we define two vector fields $\left\{u^i\right\}$ (time-like) and $\left\{v^j\right\}$ (space-like) in the specified system of coordinates by;

$$u^i=\partial_{(0)}^{\ i}; \qquad v^j=\partial_{(3)}^{\ j} \qquad (11)$$

Hence the second order Lie derivatives of φ with respect to *u* and *v* are;

$$\ell_u^2\varphi=u^i\partial_i\left(u^j\partial_j\varphi\right)=\partial_0^2\varphi \qquad (12)$$

$$\ell_v^2\varphi=v^i\partial_i\left(v^j\partial_j\varphi\right)=\partial_3^2\varphi \qquad (13)$$

It then follows from the above that;

$$\partial_0^2\varphi-\partial_3^2\varphi=\left(\ell_u^2-\ell_v^2\right)\varphi \qquad (14)$$

Hence equation (10) can be written

$$\left(\ell_u^2-\ell_v^2\right)\varphi=0 \qquad (15)$$

where the vector field *u* and *v* commute from the construction, i.e. $[u,v]=uv-vu=0$.

Conversely if the space-time $\left(R^4,g\right)$ of general relativity possesses a space-like vector field *u* and a space-like vector field *v* such that they commute, then there exists a system of coordinates satisfying equation (11). It then follows that equation (15) reduces locally to a one dimensional wave equation (9). Since equation (15) involves lie derivatives and we seek an equation for the gravitational potential *g*, we propose the following definition.

Definition: The space-time $\left(R^4,g\right)$ of general relativity is said to be radiative, if there exists two vector fields *u* (time-like) and *v* (space-like) in R^4 such that

$$\left(\ell_u^2-\ell_v^2\right)g=0 \qquad (16)$$

It is obvious from the above definition that the two vector fields commute, moreover there exist a system of coordinates satisfying equation (11). In this system of coordinates equation (16) becomes;

$$\partial_0^2 g_{\mu\nu}-\partial_3^2 g_{\mu\nu}=0 \qquad (17)$$

The above equation has general solution

$$g_{\mu\nu}\left(x^0,x^1,x^2,x^3\right)$$
$$=h_{\mu\nu}\left(x^0-x^3,x^1,x^2\right)+f_{\mu\nu}\left(x^0+x^3,x^1,x^2\right) \quad (18)$$

Here h and f are two symmetric tensors of the same dimension as g, h representing the progressive radiating gravitational potential and f the receding counterpart. For an acceptable physical solution we set $f_{\mu\nu}=0$ in which case equation (18) becomes;

$$g_{\mu\nu}\left(x^0,x^1,x^2,x^3\right)=h_{\mu\nu}\left(x^0-x^3,x^1,x^2\right) \quad (19)$$

It is worthy of note that all known exact solutions including plane waves as well as approximate solutions satisfy the given definition.

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