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# A mathematical theorem in magnetothermohaline convection in porous medium

## Jyoti Prakash<sup>\*</sup>, Sanjay Kumar Gupta and Vinod Kumar<sup>\*\*</sup>

Department of Mathematics, Himachal Pradesh University, Summer Hill, Shimla -171005, India \*\* MLSM College Sunder Nagar, District Mandi, H.P., India

#### ABSTRACT

The present paper mathematically establishes that magnetothermohaline convection of the Veronis type in porous medium cannot manifest as oscillatory motion of growing amplitude in an initially bottom heavy configuration if the thermohaline Rayleigh number  $R_s$ , the Lewis number  $\tau$ , the Prandtl number  $p_1$ , the porosity  $\epsilon$ , satisfy the inequality  $R_s \leq \frac{4\pi^2}{p_1} + \frac{27\pi^4}{4} \frac{\tau}{E'\epsilon p_1}$ , where  $P_l$  and E' are constants which depend upon porosity of the medium. It further establishes that this result is uniformly valid for the quite general nature of the bounding surfaces. A similar characterization theorem is also proved for magnetothermohaline convection of the Stern type.

#### INTRODUCTION

The thermohaline convection problem has been extensively studied in the recent past on account of its interesting complexities as a double diffusive phenomenon as well as its direct relevance in many problems of practical interest in the fields of oceanography, astrophysics, limnology and chemical engineering etc. [1]. Two fundamental configurations have been studied in the context of thermohaline convection problems, one by Veronis [2], wherein the temperature gradient is destabilizing and the concentration gradient is stabilizing; and another by Stern [3], wherein the temperature gradient is stabilizing and the concentration gradient is destabilizing. The main results of Veronis and Stern for their respective configuration are that both allow the occurrence of a steady motion or an oscillatory motion of growing amplitude, provided the destabilizing temperature gradient or the concentration gradient is sufficiently large. In case of Veronis' configuration, oscillatory motions of growing amplitude are preferred mode of onset of instability whereas in case of Stern's configuration, stationary convection is the preferred mode of onset of instability and these results are independent of the initially gravitationally stable or unstable character of the two configurations. Thus thermohaline configurations of Veronis and Stern type can further be classified into the following two classes:

(i) the first class, in which thermohaline instability manifests itself when the total density field is initially bottom heavy, and

(ii) the second class, in which thermohaline instability manifests itself when the total density field is initially top heavy.

Banerjee et al [4] derived a characterization theorem for the nonexistence of oscillatory motions of growing amplitude in an initially bottom heavy configuration of Veronis type. The essence of Banrjee et al's theorem lies in that it provides a classification of the neutral or unstable thermohaline convection configuration of the Veronis and

Stern types into two classes, the bottom heavy class and the top heavy class, and then strikes a distinction between them by means of characterization theorems which disallow the existence of oscillatory motions in the former class.

In recent years, many researchers have shown their keen interest in analyzing the onset of convection in a fluid layer subjected to a vertical temperature gradient in a porous medium [5,6,7,8,9,10,11]. The extension of these two important hydrodynamical theorems to the domains of convection in porous medium, due to its importance in the prediction of ground water movement in equifers, in the energy extraction process from the geothermal reservoirs, in assessing the effectiveness of fibrous insulations, drying of foods or other natural minerals and in nuclear engineering, is very much sought after in the present context. This paper, which mathematically analyses the hydrodynamic thermohaline convection-configuration of the Veronis and the Stern types in porous medium wherein a uniform magnetic field parallel to gravity is superimposed, may be regarded as a first step in this scheme of extended investigations.

The present paper mathematically establishes that magnetothermohaline convection of the Veronis type in porous medium cannot manifest as oscillatory motion of growing amplitude in an initially bottom heavy configuration if the thermohaline Rayleigh number R<sub>s</sub>, the Lewis number  $\tau$ , the Prandtl number p<sub>1</sub>, the porosity  $\epsilon$ , satisfy the inequality  $R_s \leq \frac{4\pi^2}{P_l} + \frac{27\pi^4}{4} \frac{\tau}{E'\epsilon p_1}$ , where P<sub>l</sub> and E' are constants which depend upon porosity of the medium. It further establishes that this result is uniformly valid for the quite general nature of the bounding surfaces. A similar characterization theorem is also proved for magnetothermohaline convection of the Stern type.

#### **1. FORMULATION OF THE PROBLEM**

An infinite horizontal porous layer filled with a viscous fluid is statically confined between two horizontal boundaries z = 0 and z = d, maintained at constant temperatures  $T_0$  and  $T_1$  (<  $T_0$ ) and solute concentrations  $S_0$  and  $S_1$ (<  $S_0$ ) at the lower and upper boundaries respectively in the presence of a uniform vertical magnetic field acting opposite to the direction of gravity. It is further assumed that the saturating fluid and the porous layer are incompressible and that the porous medium is a constant porosity medium. The problem is to investigate the stability of this initial stationary state.

Let the origin be taken on the lower boundary z = 0 with the positive direction of the z-axis along the vertically upward direction. Then the basic hydrodynamic equations that govern the problem are given by:

Equation of Continuity  

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$
(1)

Equations of Motion

$$\frac{1}{\epsilon}\frac{\partial u}{\partial t} + \frac{1}{\epsilon^{2}}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \frac{\mu_{e}}{4\pi\rho_{0}}\left(H_{1}\frac{\partial H_{1}}{\partial x} + H_{2}\frac{\partial H_{1}}{\partial y} + H_{3}\frac{\partial H_{1}}{\partial z}\right)$$

$$= -\frac{\partial}{\partial x}\left(\frac{p}{\rho_{0}} + \frac{\mu_{e}|H|^{2}}{8\pi\rho_{0}}\right) - \frac{v}{k_{1}}u$$

$$\frac{1}{\epsilon}\frac{\partial v}{\partial t} + \frac{1}{\epsilon^{2}}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) - \frac{\mu_{e}}{4\pi\rho_{0}}\left(H_{1}\frac{\partial H_{2}}{\partial x} + H_{2}\frac{\partial H_{2}}{\partial y} + H_{3}\frac{\partial H_{2}}{\partial z}\right)$$

$$= -\frac{\partial}{\partial y}\left(\frac{p}{\rho_{0}} + \frac{\mu_{e}|H|^{2}}{8\pi\rho_{0}}\right) - \frac{v}{k_{1}}v$$
(2)

$$\frac{1}{\epsilon} \frac{\partial w}{\partial t} + \frac{1}{\epsilon^2} \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) - \frac{\mu_e}{4\pi\rho_0} \left( H_1 \frac{\partial H_3}{\partial x} + H_2 \frac{\partial H_3}{\partial y} + H_3 \frac{\partial H_3}{\partial z} \right) \\ = -\frac{\partial}{\partial z} \left( \frac{p}{\rho_0} + \frac{\mu_e |\mathbf{H}|^2}{8\pi\rho_0} \right) - \frac{v}{k_1} w - \frac{\rho}{\rho_0} g$$
(4)

Equation of Heat Conduction  

$$E \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \kappa_T \nabla^2 T$$
(5)

Equation of Mass Diffusion  $E'\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z} = \kappa_S \nabla^2 S$ (6)

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Equation of Magnetic Induction

$$\epsilon \frac{\partial H_1}{\partial t} + u \frac{\partial H_1}{\partial x} + v \frac{\partial H_1}{\partial y} + w \frac{\partial H_1}{\partial z} = H_1 \frac{\partial u}{\partial x} + H_2 \frac{\partial u}{\partial y} + H_3 \frac{\partial u}{\partial z} + \epsilon \eta \nabla^2 H_1$$
(7)

$$\epsilon \frac{\partial H_2}{\partial t} + u \frac{\partial H_2}{\partial x} + v \frac{\partial H_2}{\partial y} + w \frac{\partial H_2}{\partial z} = H_1 \frac{\partial v}{\partial x} + H_2 \frac{\partial v}{\partial y} + H_3 \frac{\partial v}{\partial z} + \epsilon \eta \nabla^2 H_2$$
(8)

$$\epsilon \frac{\partial H_3}{\partial t} + u \frac{\partial H_3}{\partial x} + v \frac{\partial H_3}{\partial y} + w \frac{\partial H_3}{\partial z} = H_1 \frac{\partial w}{\partial x} + H_2 \frac{\partial w}{\partial y} + H_3 \frac{\partial w}{\partial z} + \epsilon \eta \nabla^2 H_3$$
(9)

Equation of Solenoidal character of the Magnetic Field

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} + \frac{\partial H_3}{\partial z} = 0$$
(10)

Equation of State

$$\rho = \rho_0 [1 + \alpha (T_0 - T) - \gamma (S_0 - S)], \tag{11}$$

where u, v, w are the components of velocity in the x, y, z-directions respectively,  $H_1$ ,  $H_2$ ,  $H_3$  are components of magnetic field **H** in x, y, z-directions and  $\frac{p}{\rho_0} + \frac{\mu_e |\mathbf{H}|^2}{8\pi\rho_0}$  is the modified magnetohydrodynamic pressure. Further t,  $\rho$ , T, S,  $\epsilon$ ,  $k_1$ ,  $\mu_e$ , v,  $\kappa_T$ ,  $\kappa_S$  and  $\eta$ , are, respectively, the time, the density, the temperature, the concentration, the porosity of the porous medium, the permeability of the porous medium, the magnetic permeability, the kinematic viscosity, the thermal diffusivity, the mass diffusivity and the resistivity; and  $\alpha$  and  $\gamma$  are respectively the coefficients of volume expansion due to temperature and concentration variation . Here  $E = \epsilon + (1 - \epsilon) \frac{\rho_s C_s}{\rho_0 C_f}$  is a constant and E' is also a constant analogous to E but corresponding to concentration rather than heat, where  $\rho_s$ ,  $C_s$  and  $\rho_0$ ,  $C_f$  stand for density and heat capacity of the solid (porous matrix) material and fluid respectively. The suffix '0' denotes the values of the various parameters at some suitably chosen reference temperature  $T_0$  and concentration  $S_0$ .

The basic state is assumed to be quiescent state and is given by  $(u \ y \ w) = (0 \ 0 \ 0)$ 

$$\begin{array}{l} (u, v, w) = (0, 0, 0) \\ p \equiv p(z) \\ T \equiv T(z) \\ S \equiv S(z) \\ (H_1, H_2, H_3) \equiv (0, 0, H) \\ \rho \equiv \rho(z) \end{array}$$
(12)

Thus the basic state solution on the basis of the basic state is given by

$$\begin{array}{c} (u, v, w) = (0, 0, 0) \\ p + \frac{\mu_{e} |H|^{2}}{8\pi\rho_{0}} = P = p_{0} - g\rho_{0} \left(z + \frac{\alpha\beta z^{2}}{2} - \frac{\gamma\delta z^{2}}{2}\right) \\ T = T_{0} - \beta z \\ S = S_{0} - \delta z \\ (H_{1}, H_{2}, H_{3}) = (0, 0, H) \\ \rho = \rho_{0} [1 + \alpha \left(T_{0} - T\right) - \gamma (S_{0} - S)] \end{array} \right\}$$

$$(13)$$

Assume small perturbations around the basic state and let u', v', w', P',  $\theta', \phi'$ ,  $h'_x$ ,  $h'_y$ ,  $h'_z$  denote, respectively, the perturbations in three components of velocity, pressure, temperature, concentration and three components of magnetic field intensity. Then the linearized perturbation equations are given by

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$rac{\partial \mathbf{u}^{'}}{\partial \mathbf{x}}+rac{\partial \mathbf{v}^{'}}{\partial \mathbf{y}}+rac{\partial \mathbf{w}^{'}}{\partial \mathbf{z}}=0,$	(14)
$\frac{1}{\epsilon}\frac{\partial u^{'}}{\partial t} - \frac{\mu_{e}H}{4\pi\rho_{e}}\frac{\partial h^{'}_{x}}{\partial z} = -\frac{1}{\rho_{e}}\frac{\partial p^{'}}{\partial x} - \frac{v}{k_{1}}u^{'},$	(15)
$\frac{1}{\epsilon} \frac{\partial \mathbf{v}'}{\partial t} - \frac{\mu_e H}{4\pi\rho_o} \frac{\partial \mathbf{h}'_y}{\partial z} = -\frac{1}{\rho_o} \frac{\partial \mathbf{p}'}{\partial y} - \frac{\mathbf{v}}{\mathbf{k}_1} \mathbf{V}',$	(16)
$\frac{1}{\varepsilon}\frac{\partial w^{'}}{\partial t} - \frac{\mu_{e}H}{4\pi\rho_{0}}\frac{\partial h^{'}_{z}}{\partial z} = -\frac{1}{\rho_{0}}\frac{\partial p^{'}}{\partial z} + g\alpha\theta^{'} - g\gamma\varphi^{'} - \frac{\nu}{k_{1}}w^{'},$	(17)
$\mathrm{E}rac{\partial \theta^{'}}{\partial t} - eta \mathbf{w}^{'} = \kappa_{\mathrm{T}}  \nabla^{2} \theta^{'}$ ,	(18)
$\mathbf{E}' \frac{\partial \mathbf{\Phi}'}{\partial t} - \delta \mathbf{w}' = \kappa_{\mathbf{S}} \nabla^2 \mathbf{\Phi}'$ ,	(19)
$\epsilon rac{\partial h_x^{'}}{\partial t} = H rac{\partial u^{'}}{\partial z} + \epsilon \eta \nabla^2 h_x^{'},$	(20)
$\epsilon rac{\partial \dot{h_y}}{\partial t} = H rac{\partial v'}{\partial z} + \epsilon \eta \nabla^2 \dot{h_y}$	(21)
$\epsilon rac{\partial \dot{h_z}}{\partial t} = H rac{\partial w'}{\partial z} + \epsilon \eta \nabla^2 \dot{h_z}$	(22)
and	

$$\frac{\partial h'_x}{\partial x} + \frac{\partial h'_y}{\partial y} + \frac{\partial h'_z}{\partial z} = 0.$$
(23)

Now we analyze the perturbations u', v', w', P',  $\theta'$ ,  $\varphi'$ ,  $h'_x$ ,  $h'_y$  and  $h'_z$  into two-dimensional periodic waves. We assume, to all quantities describing the perturbation, a dependence on x, y, and t of the form

$$F'(x,y,z,t) = F''(z)exp[i(k_x x+k_y y) + nt],$$
(24)

where  $k_x$  and  $k_y$  are the wave numbers along the x- and y- directions, respectively, and  $k = \sqrt{(k_x^2 + k_y^2)}$  is the resultant wave number. Following the normal mode analysis, Eqs. (14) - (23) thus, becomes

$$ik_x u'' + ik_y v'' + \frac{dw''}{dz} = 0,$$
 (25)

$$\frac{1}{\epsilon} n u'' - \frac{\mu_e H}{4\pi\rho_0} \frac{dh''_x}{dz} = -i \frac{k_x P''}{\rho_0} - \frac{v}{k_1} u'',$$
(26)

$$\frac{1}{\epsilon} nv'' - \frac{\mu_e H}{4\pi\rho_0} \frac{dh''_y}{dz} = -i \frac{k_y P''}{\rho_0} - \frac{v}{k_1} v'',$$
(27)

$$\frac{1}{\epsilon} \mathbf{n}\mathbf{w}^{"} - \frac{\mu_{\mathbf{e}}\mathbf{H}}{4\pi\rho_{0}} \frac{\mathbf{d}\mathbf{h}_{z}^{"}}{\mathbf{d}z} = -\frac{1}{\rho_{0}} \frac{\mathbf{d}\mathbf{P}^{"}}{\mathbf{d}z} + g\alpha\boldsymbol{\theta}^{"} - g\gamma\boldsymbol{\varphi}^{"} - \frac{\nu}{k_{1}}\mathbf{w}^{"},$$
(28)

$$\operatorname{En}\theta^{''} - \beta w^{''} = \kappa_{\mathrm{T}} \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \mathrm{k}^2\right) \theta^{''},\tag{29}$$

$$\mathbf{E}'\mathbf{n}\boldsymbol{\varphi}'' - \delta \mathbf{w}'' = \kappa_{\mathrm{S}} \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \mathbf{k}^2\right) \boldsymbol{\varphi}'',\tag{30}$$

$$n\epsilon h_{x}'' = H\frac{du''}{dz} + \epsilon \eta \left(\frac{d^{2}}{dz^{2}} - k^{2}\right) h_{x}'', \qquad (31)$$

$$\mathbf{n}\boldsymbol{\epsilon} \,\mathbf{h}_{\mathbf{y}}^{''} = \mathbf{H}_{\mathbf{d}\mathbf{z}}^{\mathbf{d}\mathbf{y}} + \boldsymbol{\epsilon}\boldsymbol{\eta} \Big(\frac{\mathbf{d}^{2}}{\mathbf{d}\mathbf{z}^{2}} - \mathbf{k}^{2}\Big) \mathbf{h}_{\mathbf{y}}^{''}, \tag{32}$$

$$n \epsilon h_{z}^{"} = H \frac{dw^{"}}{dz} + \epsilon \eta \left( \frac{d^{2}}{dz^{2}} - k^{2} \right) h_{z}^{"},$$
(33)  
and  $i k_{x} h_{x}^{"} + i k_{y} h_{y}^{"} + \frac{dh_{z}^{"}}{dz} = 0,$ (34)

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where 
$$\frac{\partial}{\partial t} = n$$
,  $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2$  and  $\nabla^2 = \frac{d^2}{dz^2} - k^2$ . (35)

Now multiplying equations (26) and (27) by  $ik_x$  and  $ik_y$  respectively; adding the resulting equations and using Eqs. (25) and (34), we obtain

$$-\frac{1}{\epsilon}n\frac{dw^{''}}{dz} + \frac{\mu_{e}H}{4\pi\rho_{0}}\frac{d^{2}h_{z}^{''}}{dz^{2}} = \frac{P^{''}}{\rho_{0}}k^{2} + \frac{v}{k_{1}}\frac{dw^{''}}{dz}.$$
(36)

Now eliminating P" between (28) and (36), we get  $\left(\frac{n}{\epsilon} + \frac{v}{k_1}\right) \left(\frac{d^2}{dz^2} - k^2\right) w'' = \frac{\mu_e H}{4\pi\rho_0} \frac{d}{dz} \left(\frac{d^2}{dz^2} - k^2\right) h''_z - k^2 \left(g\alpha\theta'' - g\gamma\varphi''\right)$ (37)

Also equations (29), (30), and (33) can be written as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\mathrm{z}^2} - \mathrm{k}^2 - \frac{\mathrm{En}}{\mathrm{\kappa}_{\mathrm{T}}}\right) \theta^{''} - \frac{\beta \, \mathrm{w}^{''}}{\mathrm{\kappa}_{\mathrm{T}}} \tag{38}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \mathrm{k}^2 - \frac{\mathrm{E'n}}{\kappa_{\mathrm{S}}}\right) \varphi'' = -\frac{\delta w''}{\kappa_{\mathrm{S}}} \tag{39}$$

and 
$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\eta}\right)h_z'' = -\frac{H}{\epsilon\eta}\frac{dw''}{dz}$$
 (40)

Now using the following non-dimensional parameters

$$a = kd, \ z_* = \frac{z}{d}, \ \tau_* = \frac{\kappa_S}{\kappa_T}, \ p_1 = \frac{v}{\kappa_T}, \ p_2 = \frac{v}{\eta}, \ p_l = \frac{k_1}{d^2}, \ D_* = d\frac{d}{dz}, \ \sigma_* = \frac{nd^2}{v}, \ R_* = \frac{g\alpha\beta d^4}{\kappa_T v}, \ R_{s*} = \frac{g\gamma\delta d^4}{\kappa_T v}, \quad Q_* = \frac{\mu_e H^2 d^2}{4\pi\rho_0 v \eta}, \ w_* = \frac{\beta d^2}{\kappa_T}, \ w'', \ \theta_* = \theta'', \ \varphi_* = \frac{\beta}{\delta} \varphi'' \text{ and } h_{z*} = \frac{\eta\beta d}{H\kappa_T} h_z'',$$

we can write Eqs. (37) - (40) in the following non-dimensional form(dropping the asterisks for simplicity)

$$\left(\frac{\sigma}{\epsilon} + \frac{1}{P_l}\right)(D^2 - a^2)w = -R a^2\theta + R_s a^2 \varphi + Q D(D^2 - a^2)h_z, \qquad (41)$$

$$(D^2 - a^2 - E \sigma p_1)\theta = -w, \qquad (42)$$

$$\left(D^2 - a^2 - \frac{E'\sigma p_1}{\tau}\right)\phi = -\frac{w}{\tau},\tag{43}$$

$$(D^2 - a^2 - \sigma p_2)h_z = -\frac{1}{\epsilon}Dw.$$
 (44)

The Eqs. (41)-(44) are to be solved by using the following boundary conditions:

$$w = \theta = \varphi = Dw = h_z = 0 \text{ at } z = 0 \text{ and at } z = 1,$$
(45)

(when both the boundaries are rigid and perfectly conducting)

$$w = \theta = \varphi = D^2 w = h_z = 0 \text{ at } z = 0 \text{ and at } z = 1.$$
(46)

(when both the boundaries are free and perfectly conducting)

#### MATHEMATICAL ANALYSIS

**Theorem1.** If R > 0,  $R_s > 0$ , Q > 0,  $\frac{\tau p_2}{E' p_1} \le 1$ ,  $p_r \ge 0$ ,  $p_i \ne 0$  and  $R_s \le \frac{4\pi^2}{P_l} + \frac{27\pi^4}{4} \frac{\tau}{E' \in p_1}$ , then a necessary condition for the existence of nontrivial solution (w,  $\theta$ ,  $\varphi$ ,  $h_z$ ,  $\sigma$ ) of Eqs. (41) – (44) with boundary conditions (45) or (46) is that  $R_s < R$ .

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**Proof:** Multiplying Eq. (41) by  $w^*$  (the superscript \* here denotes the complex conjugation) and integrating the resulting equation over vertical range of z, we obtain

$$\left(\frac{\sigma}{\epsilon} + \frac{1}{P_l}\right) \int_0^1 w^* \left(D^2 - a^2\right) w \, dz = -Ra^2 \int_0^1 w^* \theta \, dz + R_s a^2 \int_0^1 w^* \phi \, dz + Q \int_0^1 w^* D(D^2 - a^2) h_z \, dz.$$
(47)

Making use of Eqs. (42) – (44) and the fact that w(0) = 0 = w(1), we can write

$$-R a^{2} \int_{0}^{1} w^{*} \theta dz = R a^{2} \int_{0}^{1} \theta (D^{2} - a^{2} - E\sigma^{*}p_{1}) \theta^{*} dz , \qquad (48)$$

$$R_{s}a^{2}\int_{0}^{1}w^{*} \phi \, dz = -R_{s}a^{2}\tau \int_{0}^{1} \phi \, (D^{2} - a^{2} - \frac{E'\sigma^{*}p_{1}}{\tau}) \, \phi^{*} \, dz \,, \tag{49}$$

$$\begin{aligned} Q \int_0^1 w^* D (D^2 - a^2) h_z \, dz = & Q |w^* (D^2 - a^2) h_z |_0^1 - Q \int_0^1 D w^* (D^2 - a^2) h_z \, dz \\ & = -Q \int_0^1 D w^* (D^2 - a^2) h_z \, dz \end{aligned}$$

$$= Q\epsilon \int_0^1 (D^2 - a^2) h_z (D^2 - a^2 - \sigma^* p_2) h_z^* dz.$$
(50)

Combining Eqs. (47) - (50), we get

$$\left(\frac{\sigma}{\epsilon} + \frac{1}{p_l}\right) \int_0^1 w^* (D^2 - a^2) w \, dz = Ra^2 \int_0^1 \theta \left(D^2 - a^2 - E\sigma^* p_1\right) \theta^* \, dz$$

$$- R_s a^2 \tau \int_0^1 \phi \left(D^2 - a^2 - \frac{E'\sigma^* p_1}{\tau}\right) \phi^* \, dz$$

$$+ Q\epsilon \int_0^1 (D^2 - a^2) h_z \left(D^2 - a^2 - \sigma^* p_2\right) h_z^* \, dz.$$
(51)

Integrating the various terms of Eq. (51), by parts, for an appropriate number of times and making use of either of the boundary conditions (45) or (46), it follows that

$$\begin{pmatrix} \sigma \\ \epsilon \end{pmatrix} \int_{0}^{1} (|Dw|^{2} + a^{2}|w|^{2}) dz = Ra^{2} \int_{0}^{1} (|D\theta|^{2} + a^{2}|\theta|^{2} + Ep_{1} \sigma^{*}|\theta|^{2}) dz - R_{s}a^{2} \tau \int_{0}^{1} (|D\phi|^{2} + a^{2}|\phi|^{2} + \frac{E'\sigma^{*}p_{1}}{\tau}|\phi|^{2}) dz - Q\epsilon [\int_{0}^{1} |(D^{2} - a^{2}) hz|^{2} dz + p_{2}\sigma^{*} \int_{0}^{1} (|Dh_{z}|^{2} + a^{2}|h_{z}|^{2}) dz].$$
(52)

Equating the real and imaginary parts of both sides of Eq.(52) and cancelling  $\sigma_i$  ( $\neq 0$ ) throughout from the imaginary part, we get

and

$$\frac{1}{\varepsilon} \int_{0}^{1} (|Dw|^{2} + a^{2}|w|^{2}) dz = -R a^{2} E p_{1} \int_{0}^{1} |\theta|^{2} dz + R_{s} a^{2} E' p_{1} \int_{0}^{1} |\varphi|^{2} dz$$

$$+ Q\varepsilon p_{2} \int_{0}^{1} (|Dh_{z}|^{2} + a^{2}|h_{z}|^{2}) dz .$$
(54)

We write Eq. (53) in the alternative form

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$$\begin{pmatrix} \frac{\sigma_{r}}{\varepsilon} + \frac{1}{P_{l}} \end{pmatrix} \int_{0}^{1} (|Dw|^{2} + a^{2}|w|^{2}) dz = Ra^{2} \int_{0}^{1} (|D\theta|^{2} + a^{2}|\theta|^{2}) dz - R_{s} a^{2} \tau dz + \sigma_{r} [Ra^{2} E p_{1} \int_{0}^{1} |\theta|^{2} dz - R_{s} a^{2} E p_{1} \int_{0}^{1} |\theta|^{2} dz - R_{s} a^{2}$$

and derive the validity of the theorem from the resulting inequality obtained by replacing each one of the terms of this equation by its appropriate estimate.

We first note that since w,  $\theta$ ,  $\phi$  and  $h_z$  satisfy w(0)=0=w(1),  $\theta(0)=0=\theta(1)$ ,  $\phi(0)=0=\phi(1)$  and  $h_z(0)=0=h_z(1)$ , we have by the Rayleigh-Ritz inequality [12]

$$\int_{0}^{1} |\mathbf{D}w|^{2} \, \mathrm{d}z \geq \pi^{2} \int_{0}^{1} |w|^{2} \, \mathrm{d}z \tag{56}$$

 $\int_{0}^{1} |D\theta|^{2} dz \geq \pi^{2} \int_{0}^{1} |\theta|^{2} dz$ (57)

$$\int_0^1 |\mathbf{D}\boldsymbol{\varphi}|^2 \, \mathrm{d}z \ge -\pi^2 \int_0^1 |\boldsymbol{\varphi}|^2 \, \mathrm{d}z \tag{58}$$

$$\int_{0}^{1} |\mathrm{Dh}_{z}|^{2} \, \mathrm{d}z \ge -\pi^{2} \int_{0}^{1} |\mathrm{h}_{z}|^{2} \, \mathrm{d}z \tag{59}$$

Utilizing inequality (56), we have

$$\int_0^1 (|\mathsf{D}w|^2 + a^2 |w|^2) \, \mathrm{d}z \geq (\pi^2 + a^2) \int_0^1 |w|^2 \, \mathrm{d}z.$$
(60)

Since  $\sigma_r \ge 0$ , we have

$$\frac{\sigma_{\rm r}}{\epsilon} \int_0^1 (|\mathrm{D}\mathbf{w}|^2 + a^2 |\mathbf{w}|^2) \,\mathrm{d}\mathbf{z} \ge 0 \tag{61}$$

Multiplying equation (42) by  $\theta^*$  throughout and integrating the various terms on the left hand side of the resulting equation, by parts, for an appropriate number of times by making use of the boundary conditions on  $\theta$ , we have from the real part of the final equation

$$\begin{split} &\int_{0}^{1} (|D\theta|^{2} + a^{2}|\theta|^{2})dz + \sigma_{r} E p_{1} \int_{0}^{1} |\theta|^{2}dz = \text{Real part of } \int_{0}^{1} w \ \theta^{*} \ dz \\ &\leq \left|\int_{0}^{1} w \ \theta^{*} \ dz\right| \\ &\leq \int_{0}^{1} |w| \ |\theta^{*}| \ dz \\ &\leq \int_{0}^{1} |w| \ |\theta| \ dz \\ &\leq \left[\int_{0}^{1} |w|^{2} \ dz\right]^{1/2} \left[\int_{0}^{1} |\theta|^{2} \ dz\right]^{1/2}, \\ (Using Cauchy- Schwartz inequality) \end{split}$$

and combining this inequality with the inequality (57) and the fact that  $\sigma_r \ge 0$ , we get  $(\pi^2 + \alpha^2) \int_0^1 |\theta|^2 dz \le \left[\int_0^1 |w|^2 dz\right]^{1/2} \left[\int_0^1 |\theta|^2 dz\right]^{1/2}$ 

which implies that

$$\left[\int_{0}^{1} |\theta|^{2} dz\right]^{1/2} \leq \frac{1}{(\pi^{2} + a^{2})} \left[\int_{0}^{1} |w|^{2} dz\right]^{1/2},$$
(62)

and thus  $\int_0^1 (|D\theta|^2 + a^2|\theta|^2) dz \le \frac{1}{(\pi^2 + a^2)} \int_0^1 |w|^2 dz$  (63)

Utilizing inequality (58), we have

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$$\begin{split} &R_{s}a^{2}\tau\int_{0}^{1}(|D\varphi|^{2}+a^{2}|\varphi|^{2})\,dz \geq R_{s}a^{2}\tau(\pi^{2}+a^{2})\int_{0}^{1}|\varphi|^{2}dz \\ &\geq \frac{\tau(\pi^{2}+a^{2})}{E^{'}p_{1}\varepsilon}[\frac{1}{\varepsilon}\int_{0}^{1}(|Dw|^{2}+a^{2}|w|^{2})\,dz - Q\varepsilon p_{2}\int_{0}^{1}(|Dh_{z}|^{2}+a^{2}|h_{z}|^{2})\,dz] \text{ (using Eq. (54))} \\ &\geq \frac{\tau(\pi^{2}+a^{2})^{2}}{E^{'}p_{1}\varepsilon}\int_{0}^{1}|w|^{2}\,dz - Q\varepsilon p_{2}\int_{0}^{1}(|Dh_{z}|^{2}+a^{2}|h_{z}|^{2})\,dz \quad \text{ (using Eq. (56))} \end{split}$$

Further, since  $h_z(0) = 0 = h_z(1)$ , we have

so that we have

$$\int_{0}^{1} |D^{2}h_{z}|^{2} dz \ge \pi^{2} \int_{0}^{1} |Dh_{z}|^{2} dz , \qquad (65)$$

and thus we can write

$$\begin{aligned} &Q \epsilon \int_{0}^{1} |(D^{2} - a^{2})h_{z}|^{2} dz = Q \epsilon \int_{0}^{1} (|D^{2}h_{z}|^{2} + 2 a^{2}|Dh_{z}|^{2} + a^{4}|h_{z}|^{2}) dz \\ &\geq Q \epsilon [\pi^{2} \int_{0}^{1} (|D h_{z}|^{2} + a^{2}|Dh_{z}|^{2} + a^{2}|Dh_{z}|^{2} + a^{4}|h_{z}|^{2}) dz] \\ &\geq Q \epsilon [(\pi^{2} + a^{2}) \int_{0}^{1} |Dh_{z}|^{2} dz + \pi^{2} a^{2} \int_{0}^{1} |h_{z}|^{2} dz + a^{4} \int_{0}^{1} |h_{z}|^{2} dz] \\ &\geq Q \epsilon (\pi^{2} + a^{2}) \int_{0}^{1} (|Dh_{z}|^{2} + a^{2}|h_{z}|^{2}) dz. \end{aligned}$$
(66)

Hence we can write

$$- R_{s} a^{2} \tau \int_{0}^{1} (|D\varphi|^{2} + a^{2}|\varphi|^{2}) dz - Q\epsilon \int_{0}^{1} |(D^{2} - a^{2})h_{z}|^{2} dz \leq - \frac{\tau (\pi^{2} + a^{2})^{2}}{E p_{1}\epsilon} \int_{0}^{1} |w|^{2} dz + Q\epsilon (\pi^{2} + a^{2}) [\frac{\tau p_{2}}{E p_{1}} - 1] \int_{0}^{1} (|Dh_{z}|^{2} + a^{2}|h_{z}|^{2}) dz .$$
(67)

Also, from Eq. (54) and the fact that  $\sigma_{r} \!\geq\! 0,$  we obtain

$$\sigma_{\rm r} \left[ R \ a^2 E \ p_1 \int_0^1 |\theta|^2 \ dz - R_{\rm s} \ a^2 E' p_1 \int_0^1 |\varphi|^2 dz - Q \varepsilon \ p_2 \int_0^1 (|Dh_z|^2 + \ a^2 |h_z|^2) dz \right] \le 0 \tag{68}$$

Now, if permissible, let  $R_s \ge R$ . Then in that case, we derive from Eq. (55) and inequalities (60), (61), (63), (67) and (68) that

$$\begin{bmatrix} \frac{(\pi^{2}+a^{2})}{P_{l}} + \frac{\tau(\pi^{2}+a^{2})^{2}}{E'p_{1}\varepsilon} - \frac{R_{s}a^{2}}{(\pi^{2}+a^{2})} \end{bmatrix} \int_{0}^{1} |w|^{2} dz - Q\varepsilon(\pi^{2}+a^{2}) \begin{bmatrix} \frac{\tau p_{2}}{E'p_{1}} - 1 \end{bmatrix} \int_{0}^{1} (|Dh_{z}|^{2} + a^{2}|h_{z}|^{2}) dz < 0$$

$$(69)$$

$$\left[\frac{(\pi^{2}+a^{2})^{2}}{a^{2}P_{l}} + \frac{\tau(\pi^{2}+a^{2})^{3}}{E'p_{1}\in a^{2}} - R_{s}\right] \int_{0}^{1} |w|^{2} dz$$
  
-Q\epsilon \frac{(\pi^{2}+a^{2})^{2}}{a^{2}} \left[ \frac{\tau p\_{2}}{E'p\_{1}} - 1 \right] \int\_{0}^{1} \left( |Dh\_{z}|^{2} + a^{2}|h\_{z}|^{2} \right) dz < 0 \text{ (70)}

Since  $\frac{\tau p_2}{E p_1} \le 1$ , Eq. (70) clearly implies that

$$R_{s} > \frac{(\pi^{2} + a^{2})^{2}}{a^{2}P_{l}} + \frac{\tau (\pi^{2} + a^{2})^{3}}{E'p_{1}\epsilon a^{2}},$$
(71)

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So that we necessarily have

 $R_{s} > \frac{4\pi^{2}}{P_{l}} + \frac{27\pi^{4}}{4} \frac{\tau}{E' \epsilon p_{1}},$   $(\tau^{2} + \epsilon^{2})^{2} = 0, \quad z =$ 

Since the minimum value of  $\frac{(\pi^2 + a^2)^2}{a^2}$  is  $4\pi^2$  (for  $a^2 = \pi^2$ ) and the minimum value of  $\frac{(\pi^2 + a^2)^3}{a^2}$  is  $\frac{27\pi^4}{4}$  (for  $a^2 = \frac{\pi^2}{2}$ ). Hence if

 $R_s \le \frac{4\pi^2}{P_l} + \frac{27\pi^4}{4} \frac{\tau}{E' \epsilon p_1}$ , then we must have  $R_s < R$ , and this completes the proof of the theorem.

Theorem 1 can be stated in an equivalent form as 'magnetothermohaline convection of the Veronis type in porous medium cannot manifest as oscillatory motion of growing amplitude in an initially bottom heavy configuration if  $R_s$ ,  $\tau$ ,  $p_1$ ,  $\epsilon$ ,  $P_l$  and E' satisfy the inequality

$$\mathbf{R}_{\mathrm{s}} \leq \frac{4\pi^2}{\mathbf{P}_l} + \frac{27\pi^4}{4} \frac{\tau}{\mathbf{E}' \epsilon \mathbf{p}_1}$$

Further, this result is uniformly valid for any combination of rigid and free boundaries.

A similar theorem can be proved for magnetothermohaline convection of Stern [3] type in the porous medium as follows:

Theorem 2. If  $R<0,\,R_s<0,\,Q>0,\,\frac{p_2}{Ep_1}\leq 1,\,\sigma_r\geq 0,\,\sigma_i\neq 0$  and

$$|\mathbf{R}| \leq \tau \left(\frac{4\pi^2}{P_l} + \frac{27\pi^4}{4} \frac{1}{E\varepsilon p_1}\right), \text{ then we must have } |\mathbf{R}| < |\mathbf{R}_s| \ .$$

Proof. Replacing R and  $R_s$  by -|R| and  $-|R_s|$ , respectively, in Eqs. (41) - (44) and proceeding exactly as in Theorem 1, we get the desired result.

Theorem 2 can be stated in an equivalent form as 'Magnetothermohaline convection of Stern type cannot manifest itself as oscillatory motions of growing amplitude in an initially bottom heavy configuration if R,  $\tau$ ,  $\epsilon$ ,  $p_1$ ,  $P_l$  and E satisfy the inequality  $|R| \le \tau \left(\frac{4\pi^2}{P_l} + \frac{27\pi^4}{4} \frac{1}{E\epsilon p_1}\right)$ '. Further, this result is uniformly valid for any combination of rigid and free boundaries.

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