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A Mathematical Model of Four Species Syn-Ecosymbiosis comprising of Prey-Predation, Mutualism and Commensalisms-IV (One of the Four Species are Washed out States)

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ABSTRACT

This investigation deals with a mathematical model of a four species (S_1, S_2, S_3 and S_4) Syn-Ecological system (one of the four species are washed out states). S_2 is a predator surviving on the prey S_1: the prey is a commensal to the host S_3 which itself is in mutualism with the fourth species S_4. Further S_2 and S_4 are neutral. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the stability of four of the sixteen equilibrium points: One of the four species are washed out states only are established in this paper. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories illustrated.

Key words: Equilibrium state, stability, Mutualism, Commensalisms.

INTRODUCTION

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [6] and by Volterra [12]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Meyer [7], Paul colinvaux [8], Freedman [2], Kapur [3, 4] etc. The ecological interactions can be broadly classified as prey-predation, competition, mutualism and so on. N.C. Srinivas [11] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [5] has investigated the two species prey-predator models. Recently, stability analysis of competitive species was investigated by Archana Reddy [1]. Local stability analysis for a two-species ecological mutualism model has been investigated by B. Ravindra Reddy et. al [9,10].

2. Basic equations:
Notation Adopted:

$N_1(t)$: The Population of the Prey (S_1)
$N_2(t)$: The Population of the Predator (S_2)
$N_3(t)$: The Population of the Host (S_3) of the Prey (S_1) and mutual to S_4
$N_4(t)$: The Population of S_4 mutual to S_3
t : Time instant
a_1,a_2,a_3,a_4 : Natural growth rates of S_1, S_2, S_3, S_4
$a_{11},a_{22},a_{33},a_{44}$: Self inhibition coefficients of S_1, S_2, S_3, S_4
a_{12},a_{21} : Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1
a_{13} : Coefficient for commensal for S_1 due to the Host S_3
a_{34}, a_{43} : Mutually interaction between S_3 and S_4

$\frac{a_1}{a_{11}},\frac{a_2}{a_{22}},\frac{a_3}{a_{33}},\frac{a_4}{a_{44}}$: Carrying capacities of S_1, S_2, S_3, S_4

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24} are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1N_1 - a_{11}N_1^2 - a_{12}N_1N_2 + a_{13}N_1N_3 \quad \ldots\ldots \quad (2.1)$$
$$\frac{dN_2}{dt} = a_2N_2 - a_{22}N_2^2 + a_{21}N_2N_1 \quad \ldots\ldots \quad (2.2)$$
$$\frac{dN_3}{dt} = a_3N_3 - a_{33}N_3^2 + a_{34}N_3N_4 \quad \ldots\ldots \quad (2.3)$$
$$\frac{dN_4}{dt} = a_4N_4 - a_{44}N_4^2 + a_{43}N_4N_3 \quad \ldots\ldots \quad (2.4)$$

3. Equilibrium states:
The system under investigation has sixteen equilibrium states are given by

$$\frac{dN_i}{dt} = 0,\ i = 1,2,3,4 \quad \ldots\ldots \quad (3.1)$$

I. Fully washed out state:

(1) $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$

II. States in which three of the four species are washed out and fourth is surviving

(2) $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$ (3) $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$

(4) $\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=0, \overline{N_4}=0$ (5) $\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=0$

III. States in which two of the four species are washed out while the other two are surviving

(6) $\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}}, \overline{N_4}=\frac{a_3a_{43}+a_4a_{33}}{a_{33}a_{44}-a_{34}a_{43}}$

(7) $\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$ (8) $\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$

(9) $\overline{N_1}=\frac{a_1}{a_{11}}, \overline{N_2}=0, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$

(10) $\overline{N_1}=\frac{a_1a_{33}+a_3a_{13}}{a_{11}a_{33}}, \overline{N_2}=0, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$

(11) $\overline{N_1}=\frac{a_1a_{22}-a_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_2}=\frac{a_1a_{21}+a_2a_{11}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_3}=0, \overline{N_4}=0$

III. States in which one of the four species is washed out while the other three are surviving

(12) $\overline{N_1}=0, \overline{N_2}=\frac{a_2}{a_{22}}, \overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}}, \overline{N_4}=\frac{a_4a_{33}+a_3a_{43}}{a_{33}a_{44}-a_{34}a_{43}}$

This state can exist only when $a_{33}a_{44}-a_{34}a_{43}>0$

(13) $\overline{N_1}=\frac{\alpha_1}{\alpha_2}, \overline{N_2}=0, \overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}}, \overline{N_4}=\frac{a_4a_{33}+a_3a_{43}}{a_{33}a_{44}-a_{34}a_{43}}$

Where
$\alpha_1=a_{13}(a_4a_{34}+a_3a_{44})+a_1(a_{33}a_{44}-a_{34}a_{43}), \alpha_2=a_{11}(a_{33}a_{44}-a_{34}a_{43})$

This state can exist only when $a_{33}a_{44}-a_{34}a_{43}>0$

(14) $\overline{N_1}=\frac{a_1a_{22}-a_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_2}=\frac{a_1a_{21}+a_2a_{11}}{a_{11}a_{22}+a_{12}a_{21}}, \overline{N_3}=0, \overline{N_4}=\frac{a_4}{a_{44}}$

This state can exist only when $a_1a_{22}-a_2a_{12}>0$

(15) $\overline{N_1}=\frac{\beta_2}{\beta_1}, \overline{N_2}=\frac{\beta_3}{\beta_1}, \overline{N_3}=\frac{a_3}{a_{33}}, \overline{N_4}=0$

Where
$\beta_1=a_{33}(a_{11}a_{22}+a_{12}a_{21}), \beta_2=a_{22}(a_1a_{33}+a_3a_{13})-a_2a_{12}a_{33}$
$\beta_3=a_{21}(a_1a_{33}+a_3a_{13})+a_2a_{11}a_{33}$
This state can exist only when β_2 >0

IV. The co-existent state (or) Normal steady state

(16) $\overline{N_1} = \dfrac{\gamma_1 + a_{13}a_{22}\gamma_2}{\gamma_3}, \overline{N_2} = \dfrac{\gamma_4 + a_{13}a_{21}\gamma_2}{\gamma_3}, \quad \overline{N_3} = \dfrac{a_4 a_{34} + a_3 a_{44}}{a_{33}a_{44} - a_{34}a_{43}}, \overline{N_4} = \dfrac{a_4 a_{33} + a_3 a_{43}}{a_{33}a_{44} - a_{34}a_{43}}$

Where
$\gamma_1 = (a_1 a_{22} + a_2 a_{12})(a_{33}a_{44} - a_{34}a_{43}),\ \gamma_2 = a_3 a_{44} + a_4 a_{34}$
$\gamma_3 = (a_{11}a_{22} + a_{12}a_{21})(a_{33}a_{44} - a_{34}a_{43}), \gamma_4 = (a_1 a_{21} - a_2 a_{11})(a_{33}a_{44} - a_{34}a_{43})$

The present paper deals with one of the four species are washed out states only. The stability of the other equilibrium states will be presented in the forth coming communications.

4. Stability of one of the four species washed out equilibrium states: (Sl. Nos 12,13,14,15 in the above Equilibrium States)

4.1 Stability of the Equilibrium State 12

To discuss the stability of the equilibrium state $\overline{N_1} = 0, \overline{N_2} = \dfrac{a_2}{a_{22}}, \overline{N_3} = \dfrac{a_4 a_{34} + a_3 a_{44}}{a_{33}a_{44} - a_{34}a_{43}}, \overline{N_4} = \dfrac{a_4 a_{33} + a_3 a_{43}}{a_{33}a_{44} - a_{34}a_{43}}$, we consider small deviations $u_1(t), u_2(t), u_3(t), u_4(t)$ from the steady state

i.e. $N_i(t) = \bar{N}_i + u_i(t),\ i = 1,2,3,4$ --- (4.1.1)

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$\dfrac{du_1}{dt} = Q_1 u_1$ … (4.1.2) $\quad \dfrac{du_2}{dt} = -a_2 u_2 + a_{21}\overline{N_2}u_1$ ... (4.1.3)

$\dfrac{du_3}{dt} = -a_{33}\overline{N_3}u_3 + a_{34}\overline{N_3}u_4$ … (4.1.4) $\quad \dfrac{du_4}{dt} = a_{43}\overline{N_4}u_3 - a_{44}\overline{N_4}u_4$ ... (4.1.5)

Here $Q_1 = a_1 - a_{12}\overline{N_2} + a_{13}\overline{N_3}$ ... (4.1.6)

The characteristic equation of which is
$(\lambda - Q_1)(\lambda + a_2)\left[\lambda^2 + (a_{33}\overline{N_3} + a_{44}\overline{N_4})\lambda + (a_{33}a_{44} - a_{34}a_{43})\overline{N_3}\ \overline{N_4}\right] = 0$ ... (4.1.7)

The characteristic roots of (4.1.7) are

$$\lambda = Q_1, \lambda = -a_2, \lambda = \frac{-(a_{33}\bar{N}_3 + a_{44}\bar{N}_4) \pm \sqrt{(a_{33}\bar{N}_3 - a_{44}\bar{N}_4)^2 + 4a_{34}a_{43}\bar{N}_3\bar{N}_4}}{2}$$

Case (A): If $Q_1 < 0$ [i.e. $a_1 < a_{12}\dfrac{a_2}{a_{22}} + a_{13}\overline{N_3}$]

Here Q_1 is negative and the other three roots are also negative.

Hence the equilibrium state is **stable**.

The solutions of the equations (4.1.2), (4.1.3), (4.1.4), (4.1.5) are

$$u_1 = u_{10} e^{Q_1 t} \quad \ldots (4.1.8)$$

$$u_2 = \left[u_{20} - \frac{a_2 a_{21} u_{10}}{a_{22}(Q_1 + a_2)} \right] e^{-a_2 t} + \frac{a_2 a_{21} u_{10}}{a_{22}(Q_1 + a_2)} e^{Q_1 t} \quad \ldots (4.1.9)$$

$$u_3 = \left[\frac{u_{30}\left(\lambda_3 + a_{44}\bar{N}_4\right) + u_{40} a_{34} \bar{N}_3}{\lambda_3 - \lambda_4} \right] e^{\lambda_3 t} + \left[\frac{u_{30}\left(\lambda_4 + a_{44}\bar{N}_4\right) + u_{40} a_{34} \bar{N}_3}{\lambda_4 - \lambda_3} \right] e^{\lambda_4 t} \quad \ldots (4.1.10)$$

$$u_4 = \left[\frac{u_{40}\left(\lambda_3 + a_{33}\bar{N}_3\right) + u_{30} a_{43} \bar{N}_4}{\lambda_3 - \lambda_4} \right] e^{\lambda_3 t} + \left[\frac{u_{40}\left(\lambda_4 + a_{33}\bar{N}_3\right) + u_{30} a_{43} \bar{N}_4}{\lambda_4 - \lambda_3} \right] e^{\lambda_4 t} \quad \ldots (4.1.11)$$

where u_{10}, u_{20}, u_{30}, u_{40} are the initial values of u_1, u_2, u_3, u_4 respectively.

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t)$, $u_{20}(t)$, $u_{30}(t)$, $u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.

The solutions are illustrated in figures 1 & 2.

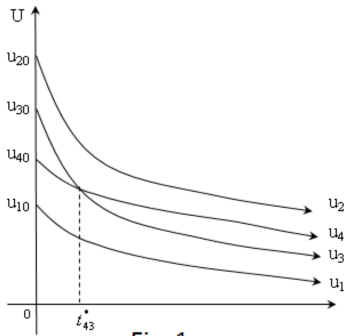


Fig. 1

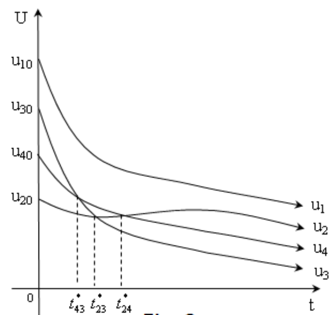


Fig. 2

<u>Case(i):</u> If $u_{10} < u_{40} < u_{30} < u_{20}$ and $a_3 < a_2 < Q_1 < a_4$

In this case initially the host (S_3) of S_1 dominates over the S_4 till the time instant t_{43}^* and there after the dominance is reversed. As $t \to \infty$, all the four species approach to the equilibrium point. Hence the equilibrium state is **stable.**

Case (ii): If $u_{20} < u_{40} < u_{30} < u_{10}$ and $a_2 < a_4 < a_3 < Q_1$

In this case initially the host (S_3) of S_1 dominates the predator (S_2) and S_4 till the time instant t_{43}^*, t_{23}^* respectively and there after the dominance is reversed. Also S_4 dominated over by the predator (S_2) till the time instant t_{24}^* and there after the dominance is reversed.

Case (B): If $Q_1 > 0$ [i.e. $a_1 > a_{12}\dfrac{a_2}{a_{22}} + a_{13}\overline{N_3}$]

Here the root Q_1 is positive and the other three roots are also negative.

Hence the equilibrium state is **unstable** and the solutions in this case are same as in Case (A). The solution curves are exhibited in figures 3 & 4.

Case (i): If $u_{20} < u_{10} < u_{40} < u_{30}$ and $a_2 < Q_1 < a_4 < a_3$

In this case initially the host (S_3) of S_1 dominates the predator (S_2), prey (S_1) and S_4 till the time instant $t_{43}^*, t_{13}^*, t_{23}^*$ respectively and there after the dominance is reversed. Also S_4 dominates over the prey (S_1) and predator (S_2) till the time instant t_{14}^*, t_{24}^* respectively and there after the dominance is reversed.

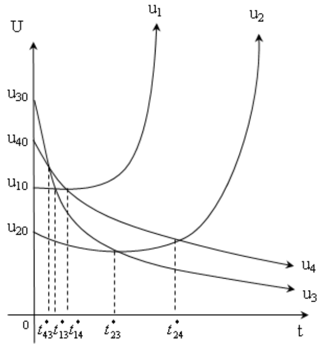


Fig. 3

Case (ii): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $a_3 < Q_1 < a_2 < a_4$

In this case initially S_4 dominates the host (S_3) of S_1 till the time instant t_{34}^* and there after the dominance is reversed. Also the predator (S_2) dominates the prey (S_1) till the time instant t_{12}^* and there after the dominance is reversed.

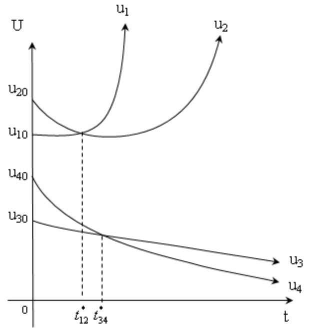


Fig. 4

4.2 Stability of the Equilibrium State 13

$$\overline{N_1} = \frac{\alpha_1}{\alpha_2}, \overline{N_2} = 0, \overline{N_3} = \frac{a_4 a_{34} + a_3 a_{44}}{a_{33} a_{44} - a_{34} a_{43}}, \overline{N_4} = \frac{a_4 a_{33} + a_3 a_{43}}{a_{33} a_{44} - a_{34} a_{43}}$$

Where

$$\alpha_1 = a_{13}(a_4 a_{34} + a_3 a_{44}) + a_1(a_{33} a_{44} - a_{34} a_{43}), \alpha_2 = a_{11}(a_{33} a_{44} - a_{34} a_{43})$$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = M_1 u_1 - a_{12} \overline{N_1} u_2 + a_{13} \overline{N_1} u_3 \quad \ldots (4.2.1) \qquad \frac{du_2}{dt} = r_2 u_2 \quad \ldots (4.2.2)$$

$$\frac{du_3}{dt} = -a_{33} \overline{N_3} u_3 + a_{34} \overline{N_3} u_4 \quad \ldots (4.2.3) \qquad \frac{du_4}{dt} = a_{43} \overline{N_4} u_3 - a_{44} \overline{N_4} u_4 \quad \ldots (4.2.4)$$

Here $M_1 = -(a_1 + a_{13} \overline{N_3})$, $r_2 = a_2 + a_{21} \overline{N_1}$ ... (4.2.5)

The characteristic equation of which is

$$(\lambda - M_1)(\lambda - r_2)\left[\lambda^2 + (a_{33} \overline{N_3} + a_{44} \overline{N_4})\lambda + (a_{33} a_{44} - a_{34} a_{43}) \overline{N_3}\, \overline{N_4}\right] = 0 \quad \ldots (4.2.6)$$

The characteristic roots of (4.2.6) are

$$\lambda = M_1, \lambda = r_2, \lambda = \frac{-(a_{33}\bar{N}_3 + a_{44}\bar{N}_4) \pm \sqrt{(a_{33}\bar{N}_3 - a_{44}\bar{N}_4)^2 + 4 a_{34} a_{43} \bar{N}_3 \bar{N}_4}}{2}$$

One root of the characteristic equation $\lambda = r_2$ is positive and the remaining three roots are negative. Hence the equilibrium state is unstable and the solutions are

$$u_1 = \left\{ u_{10} + \left(\frac{a_{12} \overline{N_1} u_{20} (\lambda_3 + M_1) - a_{13} \overline{N_1} (p_1 + p_2)(r_2 + M_1)}{(r_2 + M_1)(\lambda_3 + M_1)} \right) \right\} e^{-M_1 t} + \left(\frac{a_{13} \overline{N_1} (p_1 e^{\lambda_3 t} + p_2 e^{\lambda_4 t})(r_2 + M_1) - a_{12} \overline{N_1} u_{20} e^{r_2 t} (\lambda_3 + M_1)}{(r_2 + M_1)(\lambda_3 + M_1)} \right) \quad \ldots (4.2.7)$$

$$u_2 = u_{20} e^{r_2 t} \quad \ldots (4.2.8)$$

$$u_3 = p_1 e^{\lambda_3 t} + p_2 e^{\lambda_4 t} \quad \ldots (4.2.9)$$

$$u_4 = \left[\frac{u_{40}(\lambda_3 + a_{33}\bar{N}_3) + u_{30} a_{43} \bar{N}_4}{\lambda_3 - \lambda_4} \right] e^{\lambda_3 t} + \left[\frac{u_{40}(\lambda_4 + a_{33}\bar{N}_3) + u_{30} a_{43} \bar{N}_4}{\lambda_4 - \lambda_3} \right] e^{\lambda_4 t} \quad .. (4.2.10)$$

Where

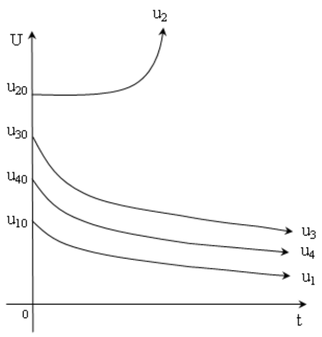
$$p_1 = \frac{u_{30}(\lambda_3 + a_{44}\bar{N}_4) + u_{40} a_{34} \bar{N}_3}{\lambda_3 - \lambda_4}, \; p_2 = \frac{u_{30}(\lambda_4 + a_{44}\bar{N}_4) + u_{40} a_{34} \bar{N}_3}{\lambda_4 - \lambda_3}$$

The solution curves are as shown in figures 5 & 6.

Case (i): If $u_{10} < u_{40} < u_{30} < u_{20}$ and $a_3 < a_1 < r_2 < a_4$

In this case initially the predator (S_2) dominates the host (S_3) of S_1 , S_4 and the prey (S_1) in natural growth rates well as in its initial population strength. Also we observe that the Predator species diverge from the equilibrium point while the other three species converge to the equilibrium point. Hence the equilibrium state is unstable as shown in figure 5.

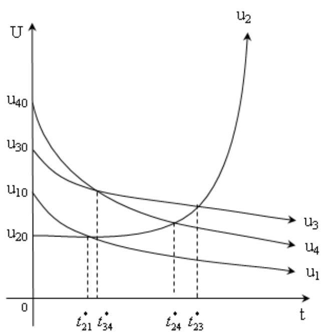
Fig. 5



Case (ii): If $u_{20} < u_{10} < u_{30} < u_{40}$ and $a_3 < a_4 < r_2 < a_1$

In this case initially S_4 dominates the host (S_3) of S_1 and the predator (S_2) till the time instant t_{34}^*, t_{24}^* respectively and there after the dominance is reversed. Also the host (S_3) of S_1 dominates the predator (S_2) till the time instant t_{23}^* and there after the dominance is reversed. Similarly the prey (S_1) dominates the predator (S_2) till the time instant t_{21}^* and there after the dominance is reversed.

Fig. 6



4.3 Stability of the Equilibrium State 14

$$\overline{N_1} = \frac{a_1 a_{22} - a_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_2} = \frac{a_1 a_{21} + a_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = -a_{11}\overline{N_1}u_1 - a_{12}\overline{N_1}u_2 + a_{13}\overline{N_1}u_3 \qquad \ldots (4.3.1)$$

$$\frac{du_2}{dt} = -a_{22}\overline{N_2}u_2 + a_{21}\overline{N_2}u_1 \qquad \ldots (4.3.2)$$

$$\frac{du_3}{dt} = l_3 u_3 \qquad \ldots (4.3.3) \qquad \frac{du_4}{dt} = -a_4 u_4 + a_{43}\frac{a_4}{a_{44}}u_3 \qquad \ldots (4.3.4)$$

Here $l_3 = a_3 + a_{34}\frac{a_4}{a_{44}}$ ... (4.3.5)

The characteristic equation of which is

$$(\lambda - l_3)(\lambda + a_4)\left[\lambda^2 + (a_{11}\overline{N_1} + a_{22}\overline{N_2})\lambda + a_{12}a_{21}\overline{N_1}\,\overline{N_2}\right] = 0 \qquad \text{... (4.3.6)}$$

The characteristic roots of (4.3.6) are

$$\lambda = l_3, \lambda = -a_4, \lambda = \frac{-(a_{11}\bar{N}_1 + a_{22}\bar{N}_2) \pm \sqrt{(a_{11}\bar{N}_1 + a_{22}\bar{N}_2)^2 - 4a_{12}a_{21}\bar{N}_1\bar{N}_2}}{2}$$

One root of the characteristic equation $\lambda = l_3$ is positive and the remaining three roots are negative. Hence the equilibrium state is unstable and the solutions are

$$u_1 = \left\{\frac{[(u_{10}+u_{20})a_{12} - u_{30}a_{13}]\overline{N_1} - \phi_1(\lambda_2 - l_3)}{\lambda_2 - \lambda_1}\right\}e^{\lambda_1 t}$$
$$+\left\{\frac{(u_{10}-\phi_1)(\lambda_2-\lambda_1) - [(u_{10}+u_{20})a_{12} - u_{30}a_{13}]\overline{N_1} + \phi_1(\lambda_2 - l_3)}{\lambda_2 - \lambda_1}\right\}e^{\lambda_2 t} + \phi_1 e^{l_3 t} \qquad \text{… (4.3.7)}$$

$$u_2 = \left\{\frac{[(u_{10}+u_{20})a_{12} - u_{30}a_{13}]\overline{N_1} - \phi_1(\lambda_2 - l_3)}{\lambda_2 - \lambda_1}\right\}e^{\lambda_1 t}\xi_1$$
$$+\left\{\frac{(u_{10}-\phi_1)(\lambda_2-\lambda_1) - [(u_{10}+u_{20})a_{12} - u_{30}a_{13}]\overline{N_1} + \phi_1(\lambda_2 - l_3)}{\lambda_2 - \lambda_1}\right\}e^{\lambda_2 t}\xi_2 + \phi_2 e^{l_3 t} \qquad \text{… (4.3.8)}$$

$$u_3 = u_{30}e^{l_3 t} \qquad \text{... (4.3.9)}$$

$$u_4 = \left[u_{40} - \frac{a_4 a_{43} u_{30}}{a_{44}(l_3 + a_4)}\right]e^{-a_4 t} + \frac{a_4 a_{43} u_{30}}{a_{44}(l_3 + a_4)}e^{l_3 t} \qquad \text{.. (4.3.10)}$$

Where

$$\phi_1 = \frac{\beta_2}{l_3^2 + \psi_1 l_3 + \beta_1}, \psi_1 = a_{11}\overline{N_1} + a_{22}\overline{N_2}, \phi_2 = \frac{a_{13}\overline{N_1}u_{30} - \phi_1(l_3 + p_3)}{a_{12}\overline{N_1}}$$

$$\beta_1 = (a_{11}a_{22} + a_{12}a_{21})\overline{N_1}\,\overline{N_2}, \beta_2 = u_{30}a_{13}(l_3 + a_{22}\overline{N_2})\overline{N_1}$$

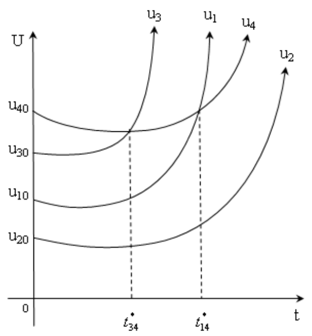
$$\xi_1 = \frac{-(\lambda_1 + p_3)}{a_{12}\overline{N_1}}, \xi_2 = \frac{-(\lambda_2 + p_3)}{a_{12}\overline{N_1}}, p_3 = a_{11}\overline{N_1}$$

The solutions are illustrated in figures 7 & 8.

Case (i): If $u_{20} < u_{10} < u_{30} < u_{40}$ and $a_4 < l_3 < a_2 < a_1$

In this case initially S_4 dominates the host (S_3) of S_1 and the prey (S_1) till the time instant t_{34}^*, t_{14}^* respectively and there after the dominance is reversed. It is evident that all the four species going away from the equilibrium point. Hence the equilibrium state is unstable as shown in figure 7.

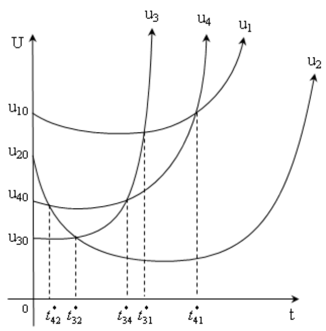
Fig. 7



<u>Case (ii):</u> If $u_{30} < u_{40} < u_{20} < u_{10}$ and $a_2 < a_1 < l_3 < a_4$

In this case initially the prey (S_1) dominates over S_4 and the host (S_3) of S_1 till the time instant t_{41}^*, t_{31}^* respectively and there after the dominance is reversed. Also the predator (S_2) dominates S_4 and the host (S_3) of S_1 and till the time instant t_{42}^*, t_{32}^* respectively and there after the dominance is reversed. Similarly S_4 dominates the host (S_3) of S_1 till the time instant t_{34}^* and there after the dominance is reversed.

Fig. 8



4.4 Stability of the Equilibrium State 15

$$\overline{N_1} = \frac{\beta_2}{\beta_1}, \overline{N_2} = \frac{\beta_3}{\beta_1}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$$

Where

$$\beta_1 = a_{33}(a_{11}a_{22} + a_{12}a_{21}), \beta_2 = a_{22}(a_1a_{33} + a_3a_{13}) - a_2a_{12}a_{33},$$
$$\beta_3 = a_{21}(a_1a_{33} + a_3a_{13}) + a_2a_{11}a_{33}$$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = -a_{11}\overline{N_1}u_1 - a_{12}\overline{N_1}u_2 + a_{13}\overline{N_1}u_3 \quad \ldots (4.4.1)$$

$$\frac{du_2}{dt} = -a_{22}\overline{N_2}u_2 + a_{21}\overline{N_2}u_1 \quad \ldots (4.4.2)$$

$$\frac{du_3}{dt} = -a_3u_3 + a_{34}\overline{N_3}u_4 \quad \ldots (4.4.3) \qquad \frac{du_4}{dt} = n_4u_4 \quad \ldots (4.4.4)$$

Here $n_4 = a_4 + a_{43}\dfrac{a_3}{a_{33}}$ ... (4.4.5)

The characteristic equation of which is

$$\left[\lambda^2 + (a_{11}\overline{N_1} + a_{22}\overline{N_2})\lambda + (a_{11}a_{22} - a_{12}a_{21})\overline{N_1}\,\overline{N_2}\right](\lambda - n_4)(\lambda + a_3) = 0 \quad ..(4.4.6)$$

The characteristic roots of (4.4.6) are

$$\lambda = \frac{-(a_{11}\bar{N}_1 + a_{22}\bar{N}_2) \pm \sqrt{(a_{11}\bar{N}_1 - a_{22}\bar{N}_2)^2 - 4a_{12}a_{21}\bar{N}_1\bar{N}_2}}{2}, \lambda = n_4, \lambda = -a_3$$

One root of the characteristic equation $\lambda = n_4$ is positive and the remaining three roots are negative. Hence the equilibrium state is unstable and the solutions are

$$u_1 = \left\{\frac{(u_{10} + u_{20} - Q_1^* - Q_2^*)a_{12}\overline{N_1} - (\lambda_2 + a_{11}\overline{N_1})(w_1 + w_2)}{\lambda_2 - \lambda_1}\right\}e^{\lambda_1 t}$$
$$+\left\{\frac{[u_{10} - (w_1 + w_2)](\lambda_2 - \lambda_1) + (u_{10} + u_{20} - Q_1^* - Q_2^*)a_{12}\overline{N_1} - (\lambda_2 + a_{11}\overline{N_1})(w_1 + w_2)}{\lambda_2 - \lambda_1}\right\}e^{\lambda_2 t}$$
$$+ w_1 e^{-a_3 t} + w_2 e^{n_4 t} \quad \ldots(4.4.7)$$

$$u_2 = \left\{\frac{(u_{10} + u_{20} - Q_1^* - Q_2^*)a_{12}\overline{N_1} - (\lambda_2 + a_{11}\overline{N_1})(w_1 + w_2)}{\lambda_2 - \lambda_1}\right\}\xi_1 e^{\lambda_1 t}$$
$$+\left\{\frac{[u_{10} - (w_1 + w_2)](\lambda_2 - \lambda_1) + (u_{10} + u_{20} - Q_1^* - Q_2^*)a_{12}\overline{N_1} - (\lambda_2 + a_{11}\overline{N_1})(w_1 + w_2)}{\lambda_2 - \lambda_1}\right\}\xi_2 e^{\lambda_2 t} + Q_1^* e^{-a_3 t} + Q_2^* e^{n_4 t}$$
$$\ldots (4.4.8)$$

$$u_3 = \left[u_{30} - \frac{a_3 a_{34} u_{40}}{a_{33}(n_4 + a_3)}\right]e^{-a_3 t} + \frac{a_3 a_{34} u_{40}}{a_{33}(n_4 + a_3)}e^{n_4 t} \quad ...(4.4.9)$$

$$u_4 = u_{40} e^{n_4 t} \quad .. (4.4.10)$$

Where

$$Q_1 = \frac{a_3 a_{34} u_{40}}{a_{33}(n_4 + a_3)}, A = a_{11}\overline{N_1} + a_{22}\overline{N_2}, B = (a_{11}a_{22}\overline{N_1} + a_{21})\overline{N_2}$$

$$w_1 = \frac{(u_{30} - Q_1)(a_{22}\overline{N_2} - a_3)a_{13}\overline{N_1}}{a_3^{\,2} - a_3 A + B}, w_2 = \frac{a_{13}\overline{N_1}Q_1(\lambda_4 + a_{22}\overline{N_2})}{\lambda_4^{\,2} + \lambda_4 A + B}$$

$$Q_1^* = \frac{a_{13}\overline{N_1}(u_{30} - Q_1) + w_1(a_3 - a_{11}\overline{N_1})}{a_{12}\overline{N_1}}, Q_2^* = \frac{a_{13}\overline{N_1}Q_1 - w_2(n_4 + a_{11}\overline{N_1})}{a_{12}\overline{N_1}}$$

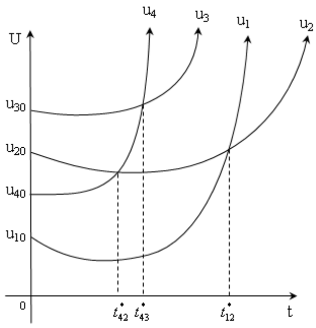
$$\xi_1 = \frac{-(\lambda_1 + p_3)}{a_{12}\overline{N_1}}, \xi_2 = \frac{-(\lambda_2 + p_3)}{a_{12}\overline{N_1}}, p_3 = a_{11}\overline{N_1}$$

The solution curves are exhibited in figures 9 & 10.

Case (i): If $u_{10} < u_{40} < u_{20} < u_{30}$ and $a_2 < a_3 < n_4 < a_1$

In this case initially the host (S_3) of S_1 dominates S_4 till the time instant t_{43}^* and there after the dominance is reversed. Also the predator (S_2) dominates S_4 till the time instant t_{42}^* and there after the dominance is reversed. Similarly the predator (S_2) dominates the prey (S_1) till the time instant t_{12}^* and there after the dominance is reversed.

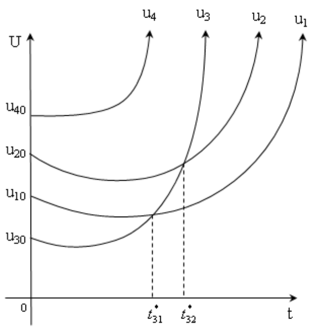
Fig. 9



Case (ii): If $u_{30} < u_{10} < u_{20} < u_{40}$ and $n_4 < a_2 < a_1 << a_3$

In this case initially the predator (S_2) dominates over the host (S_3) of S_1 till the time instant t_{32}^* and there after the dominance is reversed. Also the prey (S_1) dominates the host (S_3) of S_1 till the time instant t_{31}^* and there after the dominance is reversed.

Fig. 10



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