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A Mathematical Model of Four Species Syn-Ecosymbiosis Comprising of Prey-Predation, Mutualism and Commensalisms-III (Two of the Four Species are washed out States)

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ABSTRACT

This investigation deals with a mathematical model of a four species (S_1, S_2, S_3 and S_4) Syn-Ecological system (Two of the four species are washed out states). S_2 is a predator surviving on the prey S_1: the prey is a commensal to the host S_3 which itself is in mutualism with the fourth species S_4. Further S_2 and S_4 are neutral. The model equations of the system constitute a set of four first order non-linear ordinary differential coupled equations. In all, there are sixteen equilibrium points. Criteria for the asymptotic stability of six of the sixteen equilibrium points: Two of the four species are washed out states only are established in this paper. The linearized equations for the perturbations over the equilibrium points are analyzed to establish the criteria for stability and the trajectories illustrated.

Key words: Equilibrium state, stability, Mutualism, Commensalisms.

INTRODUCTION

Mathematical modeling of ecosystems was initiated by Lotka [6] and by Volterra [12]. The general concept of modeling has been presented in the treatises of Meyer [7], Paul Colinvaux [8], Freedman [2], Kapur [3, 4]. The ecological interactions can be broadly classified as prey-predation, competition, mutualism and so on. N.C. Srinivas [11] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Later, Lakshmi Narayan [5] has investigated the two species prey-predator models. Recently stability analysis of competitive species was investigated by Archana Reddy [1]. Local stability analysis for a two-species ecological mutualism model has been investigated by B. Ravindra Reddy et. al [9, 10].

2. Basic equations:
Notation Adopted:

$N_1(t)$: The Population of the Prey (S_1)
$N_2(t)$: The Population of the Predator (S_2)
$N_3(t)$: The Population of the Host (S_3) of the Prey (S_1) and mutual to S_4
$N_4(t)$: The Population of S_4 mutual to S_3
t : Time instant
a_1,a_2,a_3,a_4 : Natural growth rates of S_1, S_2, S_3, S_4
$a_{11},a_{22},a_{33},a_{44}$: Self inhibition coefficients of S_1, S_2, S_3, S_4
a_{12},a_{21} : Interaction (Prey-Predator) coefficients of S_1 due to S_2 and S_2 due to S_1
a_{13} : Coefficient for commensal for S_1 due to the Host S_3
a_{34}, a_{43} : Mutually interaction between S_3 and S_4

$\frac{a_1}{a_{11}}, \frac{a_2}{a_{22}}, \frac{a_3}{a_{33}}, \frac{a_4}{a_{44}}$: Carrying capacities of S_1, S_2, S_3, S_4

Further the variables N_1, N_2, N_3, N_4 are non-negative and the model parameters a_1, a_2, a_3, a_4; a_{11}, a_{22}, a_{33}, a_{44}; a_{12}, a_{21}, a_{13}, a_{24} are assumed to be non-negative constants.

The model equations for the growth rates of S_1, S_2, S_3, S_4 are

$$\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 - a_{12} N_1 N_2 + a_{13} N_1 N_3 \quad \ldots. \quad (2.1)$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{21} N_2 N_1 \quad \ldots. \quad (2.2)$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 + a_{34} N_3 N_4 \quad \ldots. \quad (2.3)$$

$$\frac{dN_4}{dt} = a_4 N_4 - a_{44} N_4^2 + a_{43} N_4 N_3 \quad \ldots. \quad (2.4)$$

3. Equilibrium States:
The system under investigation has sixteen equilibrium states are given by

$$\frac{dN_i}{dt} = 0,\ i = 1, 2, 3, 4 \quad \ldots\ldots \quad (3.1)$$

I. Fully washed out state:

(1) $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$

II. States in which three of the four species are washed out and fourth is surviving

(2) $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$ (3) $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$

(4) $\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = 0$ (5) $\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = 0$

III. States in which two of the four species are washed out while the other two are surviving

(6) $\overline{N_1}=0,\overline{N_2}=0,\overline{N_3}=\dfrac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}},\overline{N_4}=\dfrac{a_3a_{43}+a_4a_{33}}{a_{33}a_{44}-a_{34}a_{43}}$

This state can exist only when $a_{33}a_{44}-a_{34}a_{43}>0$.

(7) $\overline{N_1}=0,\overline{N_2}=\dfrac{a_2}{a_{22}},\overline{N_3}=0,\overline{N_4}=\dfrac{a_4}{a_{44}}$ (8) $\overline{N_1}=0,\overline{N_2}=\dfrac{a_2}{a_{22}},\overline{N_3}=\dfrac{a_3}{a_{33}},\overline{N_4}=0$

(9) $\overline{N_1}=\dfrac{a_1}{a_{11}},\overline{N_2}=0,\overline{N_3}=0,\overline{N_4}=\dfrac{a_4}{a_{44}}$

(10) $\overline{N_1}=\dfrac{a_1a_{33}+a_3a_{13}}{a_{11}a_{33}},\overline{N_2}=0,\overline{N_3}=\dfrac{a_3}{a_{33}},\overline{N_4}=0$

(11) $\overline{N_1}=\dfrac{a_1a_{22}-a_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}},\overline{N_2}=\dfrac{a_1a_{21}+a_2a_{11}}{a_{11}a_{22}+a_{12}a_{21}},\overline{N_3}=0,\overline{N_4}=0$

This state can exist only when $a_1a_{22}-a_2a_{12}>0$

IV. States in which one of the four species is washed out while the other three are surviving

(12) $\overline{N_1}=0,\overline{N_2}=\dfrac{a_2}{a_{22}},\overline{N_3}=\dfrac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}},\overline{N_4}=\dfrac{a_4a_{33}+a_3a_{43}}{a_{33}a_{44}-a_{34}a_{43}}$

(13) $\overline{N_1}=\dfrac{\alpha_1}{\alpha_2},\overline{N_2}=0,\overline{N_3}=\dfrac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}},\overline{N_4}=\dfrac{a_4a_{33}+a_3a_{43}}{a_{33}a_{44}-a_{34}a_{43}}$

Where

$\alpha_1=a_{13}(a_4a_{34}+a_3a_{44})+a_1(a_{33}a_{44}-a_{34}a_{43}),\ \alpha_2=a_{11}(a_{33}a_{44}-a_{34}a_{43})$

(14) $\overline{N_1}=\dfrac{a_1a_{22}-a_2a_{12}}{a_{11}a_{22}+a_{12}a_{21}},\overline{N_2}=\dfrac{a_1a_{21}+a_2a_{11}}{a_{11}a_{22}+a_{12}a_{21}},\overline{N_3}=0,\overline{N_4}=\dfrac{a_4}{a_{44}}$

(15) $\overline{N_1}=\dfrac{\beta_2}{\beta_1},\overline{N_2}=\dfrac{\beta_3}{\beta_1},\overline{N_3}=\dfrac{a_3}{a_{33}},\overline{N_4}=0$

Where

$\beta_1=a_{33}(a_{11}a_{22}+a_{12}a_{21}),\ \beta_2=a_{22}(a_1a_{33}+a_3a_{13})-a_2a_{12}a_{33}$

$\beta_3=a_{21}(a_1a_{33}+a_3a_{13})+a_2a_{11}a_{33}$

V. The co-existent state (or) Normal steady state

(16) $\overline{N_1}=\dfrac{\gamma_1+a_{13}a_{22}\gamma_2}{\gamma_3},\overline{N_2}=\dfrac{\gamma_4+a_{13}a_{21}\gamma_2}{\gamma_3},$ $\overline{N_3}=\dfrac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}},\overline{N_4}=\dfrac{a_4a_{33}+a_3a_{43}}{a_{33}a_{44}-a_{34}a_{43}}$

Where

$\gamma_1=(a_1a_{22}+a_2a_{12})(a_{33}a_{44}-a_{34}a_{43}),\ \gamma_2=a_3a_{44}+a_4a_{34}$

$\gamma_3=(a_{11}a_{22}+a_{12}a_{21})(a_{33}a_{44}-a_{34}a_{43}),\gamma_4=(a_1a_{21}-a_2a_{11})(a_{33}a_{44}-a_{34}a_{43})$

The present paper deals with two of the four species are washed out states only. The stability of the other equilibrium states will be presented in the forth coming communications.

4. Stability of two of the four species washed out equilibrium states: (Sl. Nos 6, 7, 8, 9, 10, 11 in the above Equilibrium states)

4.1 Stability of the Equilibrium State 6

$$\overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=\frac{a_4a_{34}+a_3a_{44}}{a_{33}a_{44}-a_{34}a_{43}}, \overline{N_4}=\frac{a_3a_{43}+a_4a_{33}}{a_{33}a_{44}-a_{34}a_{43}}$$

Let us consider small deviations $u_1(t), u_2(t), u_3(t), u_4(t)$ from the steady state

i.e. $$N_i(t)=\bar{N}_i+u_i(t),\ i=1,2,3,4 \qquad \text{--- (4.1.1)}$$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt}=l_1u_1 \quad \ldots\ldots (4.1.2) \qquad \frac{du_2}{dt}=a_2u_2 \quad \ldots (4.1.3)$$

$$\frac{du_3}{dt}=-a_{33}\overline{N_3}u_3+a_{34}\overline{N_3}u_4 \quad \ldots\ldots (4.1.4) \qquad \frac{du_4}{dt}=a_{43}\overline{N_4}u_3-a_{44}\overline{N_4}u_4 \quad \ldots (4.1.5)$$

Here $l_1=a_1+a_{13}\overline{N_3}$ ... (4.1.6)

The characteristic equation of which is

$$(\lambda-l_1)(\lambda-a_2)\left[\lambda^2-(a_{33}\overline{N_3}+a_{44}\overline{N_4})\lambda+(a_{33}a_{44}-a_{34}a_{43})\overline{N_3}\,\overline{N_4}\right]=0 \quad \ldots (4.1.7)$$

The characteristic roots of (4.1.7) are

$$\lambda=l_1, \lambda=a_2, \lambda=\frac{-(a_{33}\bar{N}_3+a_{44}\bar{N}_4)\pm\sqrt{(a_{33}\bar{N}_3-a_{44}\bar{N}_4)^2+4a_{34}a_{43}\bar{N}_3\bar{N}_4}}{2}$$

Two roots of the equation (4.1.7) are positive and the other two roots are negative.
Hence the equilibrium state is **unstable**.

The solutions of the equations (4.1.2), (4.1.3), (4.1.4), (4.1.5) are

$$u_1=u_{10}e^{l_1t} \quad \ldots (4.1.8) \qquad u_2=u_{20}e^{a_2t} \quad \ldots (4.1.9)$$

$$u_3=\left[\frac{u_{30}(\lambda_3+a_{44}\bar{N}_4)+u_{40}a_{34}\bar{N}_3}{\lambda_3-\lambda_4}\right]e^{\lambda_3t}+\left[\frac{u_{30}(\lambda_4+a_{44}\bar{N}_4)+u_{40}a_{34}\bar{N}_3}{\lambda_4-\lambda_3}\right]e^{\lambda_4t} \quad \ldots (4.1.10)$$

$$u_4=\left[\frac{u_{40}(\lambda_3+a_{33}\bar{N}_3)+u_{30}a_{43}\bar{N}_4}{\lambda_3-\lambda_4}\right]e^{\lambda_3t}+\left[\frac{u_{40}(\lambda_4+a_{33}\bar{N}_3)+u_{30}a_{43}\bar{N}_4}{\lambda_4-\lambda_3}\right]e^{\lambda_4t} \quad \ldots (4.1.11)$$

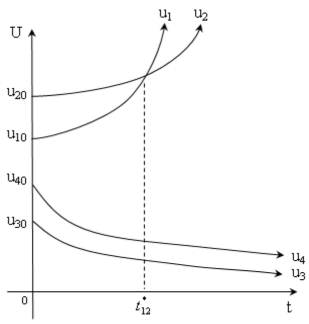
where u_{10}, u_{20}, u_{30}, u_{40} are the initial values of u_1, u_2, u_3, u_4 respectively.

There would arise in all 576 cases depending upon the ordering of the magnitudes of the growth rates a_1, a_2, a_3, a_4 and the initial values of the perturbations $u_{10}(t)$, $u_{20}(t)$, $u_{30}(t)$,$u_{40}(t)$ of the species S_1, S_2, S_3, S_4. Of these 576 situations some typical variations are illustrated through respective solution curves that would facilitate to make some reasonable observations.
The solutions are illustrated in figures 1 & 2.

Case (i) : If $u_{30}<u_{40}<u_{10}<u_{20}$, $a_3 < l_1 < a_4 < a_2$

In this case initially the predator (S_2) dominates over the prey (S_1) till the time instant t_{12}^* and there after the dominance is reversed. It is evident both the species prey and Predator are going away from the equilibrium point while the other two species converge to the equilibrium point. Hence the equilibrium state is unstable.

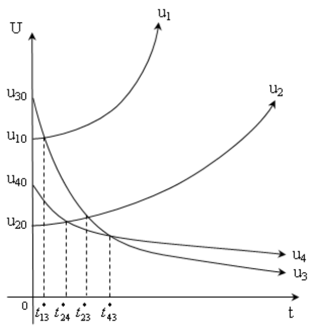
Fig.1



Case (ii) : If $u_{20}<u_{40}<u_{10}<u_{30}$, $a_2 < a_4 < a_3 < l_1$

In this case initially the host (S_3) of S_1 dominates over the prey (S_1), S_4 and the predator (S_2) till the time instant $t_{13}^*, t_{43}^*, t_{23}^*$ respectively and there after the dominance is reversed. Also S_4 dominated over by the predator (S_2) till the time instant t_{24}^* and there after the dominance is reversed.

Fig.2



4.2 Stability of the Equilibrium State 7

$$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = r_1 u_1 \quad \dots (4.2.1) \qquad \frac{du_2}{dt} = -a_2 u_2 + \frac{a_{21} a_2}{a_{22}} u_1 \quad \dots (4.2.2)$$

$$\frac{du_3}{dt} = l_3 u_3 \quad \dots (4.2.3) \qquad \frac{du_4}{dt} = -a_4 u_4 + \frac{a_{43} a_4}{a_{44}} u_3 \quad \dots (4.2.4)$$

Here $r_1 = a_1 - \frac{a_{12}a_2}{a_{22}}$, $l_3 = a_3 + \frac{a_{34}a_4}{a_{44}}$ ... (4.2.5)

The characteristic equation of which is
$(\lambda - r_1)(\lambda + a_2)(\lambda - l_3)(\lambda + a_4) = 0$ ... (4.2.6)

Case (A): When $r_1 < 0$ (i.e., when $a_1 < \frac{a_{12}a_2}{a_{22}}$)

The roots r_1, $-a_2$, $-a_4$ are negative and l_3 is positive.
Hence the equilibrium state is **unstable.**

The solutions of the equations (4.2.1) (4.2.2), (4.2.3), (4.2.4) are

$$u_1 = u_{10}e^{r_1 t} \quad \dots (4.2.7)$$

$$u_2 = [u_{20} - \frac{a_{21}a_2u_{10}}{a_{22}(r_1 + a_2)}]e^{-a_2 t} + \frac{a_{21}a_2u_{10}}{a_{22}(r_1 + a_2)}e^{r_1 t} \quad \dots (4.2.8)$$

$$u_3 = u_{30}e^{l_3 t} \quad \dots (4.2.9)$$

$$u_4 = [u_{40} - \frac{a_{43}a_4u_{30}}{a_{44}(l_3 + a_4)}]e^{-a_4 t} + \frac{a_{43}a_4u_{30}}{a_{44}(l_3 + a_4)}e^{l_3 t} \quad \dots (4.2.10)$$

The solution curves are as shown in figures 3 & 4.
Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $a_4 < l_3 < a_2 < r_1$

In this case initially S_4 dominates over the Host (S_3) of S_1 till the time instant t^*_{34} and there after the dominance is reversed. Also the commensal species is observed to be going away from the equilibrium point while the other three species converge to the equilibrium point. Hence the equilibrium state is unstable.

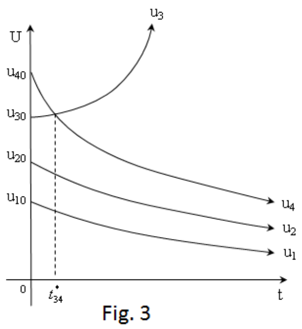


Fig. 3

Case (ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $l_3 < a_4 < r_1 < a_2$.
In this case initially the Prey (S_1) dominates over S_4, the host (S_3) of S_1 and the Predator (S_2) till the time instant $t^*_{41}, t^*_{31}, t^*_{21}$ respectively and there after the dominance is reversed. Also S_4 dominates over the Host (S_3) of S_1 till the time instant t^*_{34} and there after the dominance is reversed.

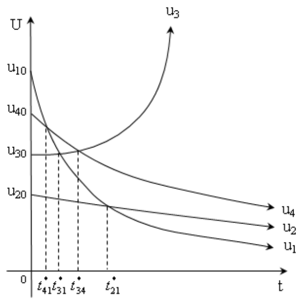


Fig. 4

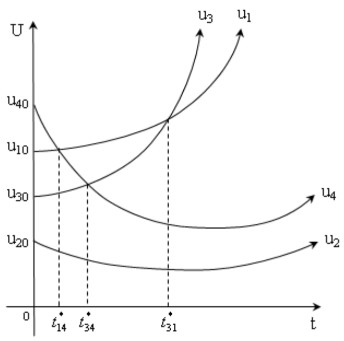
Case (B): When $r_1 > 0$ (i.e., when $a_1 > \frac{a_{12}a_2}{a_{22}}$)

The roots $-a_2$, $-a_4$ are negative and r_1, l_3 are positive.
Hence the equilibrium state is **unstable.**

In this case the solutions are same as in case (A) and the solutions are illustrated in figures 5 & 6.
Case (i): If $u_{20} < u_{30} < u_{10} < u_{40}$ and $r_1 < l_3 < a_2 < a_4$.

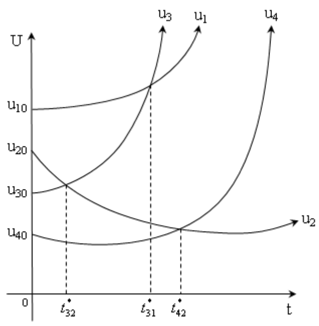
In this case initially S_4 dominates over the Prey (S_1) and the Host (S_3) of S_1 till the time instant t^*_{14}, t^*_{34} respectively and there after the dominance is reversed. Also the Prey (S_1) dominates the Host (S_3) of S_1 till the time instant t^*_{31} and there after the dominance is reversed.

Fig.5



Case (ii): If $u_{40} < u_{30} < u_{20} < u_{10}$ and $a_2 < r_1 < l_3 < a_4$.
In this case initially the Predator (S_2) dominates over the Host (S_3) of S_1 and S_4 till the time instant t^*_{32}, t^*_{42} respectively and the dominance gets reversed there after. Also the Prey (S_1) dominates the Host (S_3) of S_1 till the time instant t^*_{31} and there after the dominance is reversed.

Fig.6



4.3 Stability of the Equilibrium State 8

$\overline{N_1} = 0, \overline{N_2} = \frac{a_2}{a_{22}}, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = s_1 u_1 \quad \ldots (4.3.1) \qquad \frac{du_2}{dt} = -a_2 u_2 + \frac{a_{21}a_2}{a_{22}} u_1 \quad \ldots (4.3.2)$$

$$\frac{du_3}{dt} = -a_3 u_3 + \frac{a_{34}a_3}{a_{33}} u_4 \quad \ldots (4.3.3) \qquad \frac{du_4}{dt} = n_4 u_4 \quad \ldots (4.3.4)$$

Here $s_1 = a_1 + \frac{a_{13}a_3}{a_{33}} - \frac{a_{12}a_2}{a_{22}}$ ... (4.3.5) $\qquad n_4 = a_4 + \frac{a_{43}a_3}{a_{33}}$ ... (4.3.6)

The characteristic equation of which is

$$(\lambda - s_1)(\lambda + a_2)(\lambda + a_3)(\lambda - n_4) = 0 \quad \ldots (4.3.7)$$

Case (A): When $s_1 < 0$ (i.e., when $a_1 + \frac{a_{13}a_3}{a_{33}} < \frac{a_{12}a_2}{a_{22}}$)

The roots $s_1, -a_2, -a_3$ are negative and n_4 is positive.

Hence the equilibrium state is **unstable.**

The solutions of the equations (4.3.1) (4.3.2), (4.3.3), (4.3.4) are

$$u_1 = u_{10} e^{s_1 t} \quad \ldots (4.3.8)$$

$$u_2 = [u_{20} - \frac{a_{21}a_2 u_{10}}{a_{22}(s_1 + a_2)}] e^{-a_2 t} + \frac{a_{21}a_2 u_{10}}{a_{22}(s_1 + a_2)} e^{s_1 t} \quad \ldots (4.3.9)$$

$$u_3 = [u_{30} - \frac{a_3 a_{34} u_{40}}{a_{33}(n_4 + a_3)}] e^{-a_3 t} + \frac{a_3 a_{34} u_{40}}{a_{33}(n_4 + a_3)} e^{n_4 t} \quad \ldots (4.3.10)$$

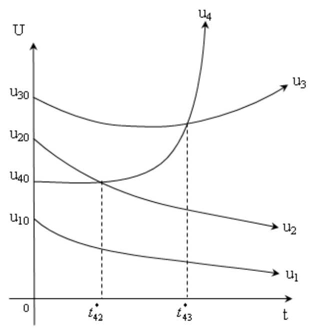
$$u_4 = u_{40} e^{n_4 t} \quad \ldots (4.3.11)$$

The solution curves are exhibited in figures 7 & 8.

Case (i): If $u_{10} < u_{40} < u_{20} < u_{30}$ and $a_2 < a_3 < s_1 < n_4$

In this case initially the Host (S_3) of S_1 dominates over S_4 till the time instant t^*_{43} and there after the dominance is reversed. Also the Predator (S_2) dominates over the S_4 till the time instant t^*_{42} and there after the dominance is reversed.

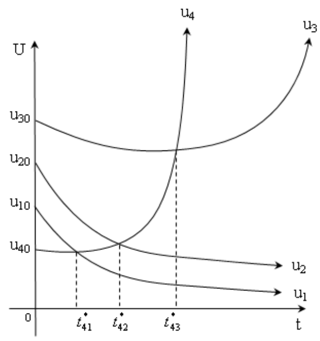
Fig.7



Case (ii): If $u_{40} < u_{10} < u_{20} < u_{30}$ and $a_3 < n_4 < a_2 < s_1$

In this case initially the Prey (S_1) dominates over S_4 till the time instant t^*_{41} and there after the dominance is reversed. Also the Predator (S_2) dominates over the S_4 till the time instant t^*_{42} and there after the dominance is reversed. Similarly the Host (S_3) of S_1 dominates over S_4 till the time instant t^*_{43} and the dominance is gets reversed there after.

Fig.8



Case (B): When $s_1 > 0$ (i.e., when $a_1 + \frac{a_{13}a_3}{a_{33}} > \frac{a_{12}a_2}{a_{22}}$)

The roots $-a_2$, $-a_3$ are negative and s_1, n_4 are positive.

Hence the equilibrium state is **unstable.**

In this case the solutions are same as in case (A) and the solutions are illustrated in figures 9 & 10.

Case (i): If $u_{10} < u_{20} < u_{30} < u_{40}$ and $s_1 < a_2 < a_3 < n_4$

In this case initially the Host (S_3) of S_1 dominates the Predator (S_2) and Prey (S_1) till the time instant t^*_{13}, t^*_{23} respectively and there after the dominance is reversed. Also the Predator (S_2) dominates over the Prey (S_1) till the time instant t^*_{12} and the dominance gets reversed there after. Similarly S_4 dominates over the Predator (S_2) and the Prey (S_1) till the time instant t^*_{24}, t^*_{14} respectively and there after the dominance is reversed.

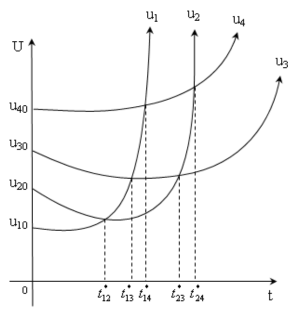
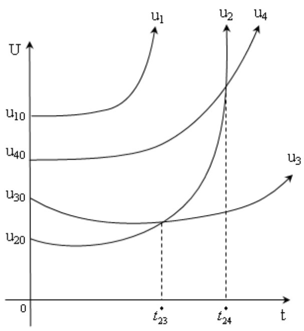


Fig.9

Case (ii): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_3 < n_4 < s_1 < a_2$

In this case initially the Host (S_3) of S_1 dominates the Predator (S_2) till the time instant t_{23}^* and there after the dominance is reversed. Also S_4 dominates over the Predator (S_2) till the time instant t_{24}^* and there after the dominance is reversed.

Fig. 10



4.4 Stability of the Equilibrium State 9

$$\overline{N_1} = \frac{a_1}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = 0, \overline{N_4} = \frac{a_4}{a_{44}}$$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = -a_1 u_1 - \frac{a_{12}a_1}{a_{11}} u_2 + \frac{a_{13}a_1}{a_{11}} u_3 \qquad \ldots (4.4.1)$$

$$\frac{du_2}{dt} = q_2 u_2 \qquad \ldots (4.4.2)$$

$$\frac{du_3}{dt} = l_3 u_3 \qquad \ldots (4.4.3) \qquad \frac{du_4}{dt} = -a_4 u_4 + \frac{a_{43}a_4}{a_{44}} u_3 \qquad \ldots (4.4.4)$$

Here $q_2 = a_2 + \frac{a_{21}a_1}{a_{11}}, l_3 = a_3 + \frac{a_{34}a_4}{a_{44}}$ … (4.4.5)

The characteristic equation of which is

$$(\lambda + a_1)(\lambda - q_2)(\lambda - l_3)(\lambda + a_4) = 0 \qquad \ldots (4.4.6)$$

The roots q_2, l_3 are positive and $-a_1, -a_4$ are negative.
Hence the equilibrium state is **unstable.**

The solutions of the equations (4.4.1) (4.4.2), (4.4.3), (4.4.4) are

$$u_1 = [u_{10} - \frac{a_{13}a_1u_{30}}{a_{11}(l_3 + a_1)} + \frac{a_{12}a_1u_{20}}{a_{11}(q_2 + a_1)}]e^{-a_1 t} + \frac{a_{13}a_1u_{30}}{a_{11}(l_3 + a_1)} e^{l_3 t} - \frac{a_{12}a_1u_{20}}{a_{11}(q_2 + a_1)} e^{q_2 t} \qquad \ldots (4.4.7)$$

$$u_2 = u_{20} e^{q_2 t} \qquad \ldots (4.4.8) \qquad u_3 = u_{30} e^{l_3 t} \qquad \ldots (4.4.9)$$

$$u_4 = [u_{40} - \frac{a_{43}a_4u_{30}}{a_{44}(l_3 + a_4)}]e^{-a_4 t} + \frac{a_{43}a_4u_{30}}{a_{44}(l_3 + a_4)} e^{l_3 t} \qquad \ldots (4.4.10)$$

The solution curves are as shown in figures 11 & 12.

Case (i): If $u_{20} < u_{30} < u_{10} < u_{40}$ and $a_1 < a_4 < l_3 < q_2$
In this case initially the Prey (S_1) dominates over the Host (S_3) of S_1 and the Predator (S_2) till the time instant t^*_{31}, t^*_{21} respectively and there after the dominance is reversed. Also S_4 dominates over the Host (S_3) of S_1 and the Predator (S_2) till the time instant t^*_{34}, t^*_{24} respectively and there after the dominance is reversed. Similarly the host (S_3) of S_1 dominates over the Predator (S_2) till the time instant t^*_{21} and there after the dominance is reversed.

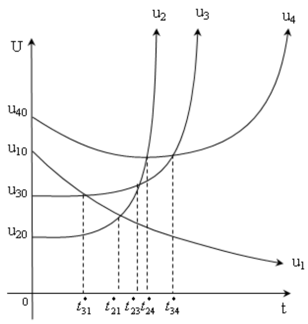
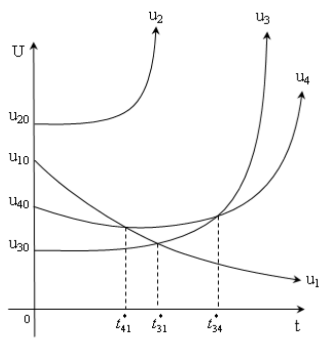


Fig. 11

Case (ii): If $u_{30} < u_{40} < u_{10} < u_{20}$ and $q_2 < l_3 < a_1 < a_4$.
In this case initially the Prey (S_1) dominates over S_4, and the host (S_3) of S_1 till the time instant t^*_{41}, t^*_{31} respectively and there after the dominance is reversed. Also S_4 dominates over the Host (S_3) of S_1 till the time instant t^*_{34} and there after the dominance is reversed.

Fig. 12



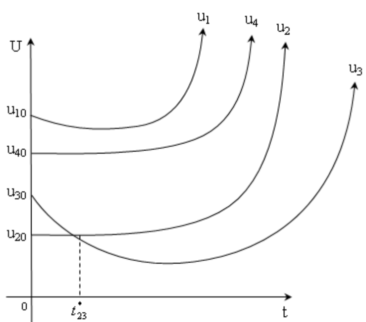


Fig. 13

4.5 Stability of the Equilibrium State 10

$$\overline{N_1} = \frac{a_1 a_{33} + a_3 a_{13}}{a_{11} a_{33}}, \overline{N_2} = 0, \overline{N_3} = \frac{a_3}{a_{33}}, \overline{N_4} = 0$$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = M_1 u_1 - a_{12}\overline{N_1}u_2 + a_{13}\overline{N_1}u_3 \qquad \dots (4.5.1)$$

$$\frac{du_2}{dt} = r_2 u_2 \qquad \dots (4.5.2)$$

$$\frac{du_3}{dt} = -a_3 u_3 + \frac{a_{34}a_3}{a_{33}}u_4 \qquad \dots (4.5.3) \qquad\qquad \frac{du_4}{dt} = n_4 u_4 \qquad \dots (4.5.4)$$

Here $M_1 = -a_1 - \dfrac{a_3 a_{13}}{a_{33}}$ ... (4.5.5)

$$r_2 = a_2 + a_{21}\overline{N_1}, n_4 = a_4 + a_{43}\overline{N_3} \qquad \dots (4.5.6)$$

The characteristic equation of which is

$$(\lambda + M_1)(\lambda - r_2)(\lambda + a_3)(\lambda - n_4) = 0 \qquad \dots (4.5.7)$$

The roots r_2, n_4 are positive and $-M_1, -a_3$ are negative.

Hence the equilibrium state is **unstable.**

The solutions of the equations (4.5.1) (4.5.2), (4.5.3), (4.5.4) are

$$u_1 = \left\{ u_{10} + \left[\frac{a_{12}\overline{N_1}u_{20}(-a_3 + M_1) - a_{13}\overline{N_1}u_{30}(r_2 + M_1)}{(r_2 + M_1)(-a_3 + M_1)} \right] \right\} e^{-M_1 t}$$
$$+ \left[\frac{a_{13}\overline{N_1}\left[(u_{30} - \eta_7)e^{-a_3 t} + \eta_7 e^{n_4 t}\right](r_2 + M_1) - a_{12}\overline{N_1}u_{20}e^{r_2 t}(-a_3 + M_1)}{(r_2 + M_1)(-a_3 + M_1)} \right] \qquad \dots (4.5.8)$$

$$u_2 = u_{20}e^{r_2 t} \qquad \dots (4.5.9)$$

$$u_3 = [u_{30} - \frac{a_{34}a_3 u_{40}}{a_{33}(n_4 + a_3)}]e^{-a_3 t} + \frac{a_{34}a_3 u_{40}}{a_{33}(n_4 + a_3)}e^{n_4 t} \qquad \dots (4.5.10)$$

$$u_4 = u_{40}e^{n_4 t} \qquad \dots (4.5.11)$$

Where $\eta_7 = \dfrac{a_{34}a_3 u_{40}}{a_{33}(n_4 + a_3)}$

The solution curves are exhibited in figures 13 & 14.

Case (i): If $u_{20} < u_{30} < u_{40} < u_{10}$ and $a_3 < M_1 < n_4 < r_2$

In this case initially the Host (S_3) of S_1 dominates over the Predator (S_2) till the time instant t_{23}^* and there after the dominance is reversed.

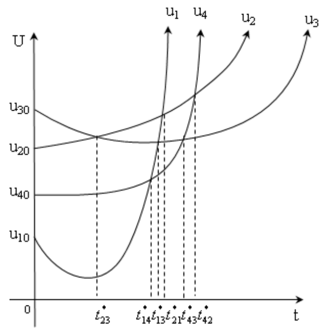


Fig. 14

Case (ii): If $u_{10} < u_{40} < u_{20} < u_{30}$ and $a_3 < M_1 < r_2 < n_4$

In this case initially the Host (S_3) of S_1 dominates over the Predator (S_2), S_4 and the Prey (S_1), till the time instant $t^*_{23}, t^*_{43}, t^*_{13}$ respectively and there after the dominance is reversed. Also the Predator (S_2) dominates over the Prey (S_1) till the time instant t^*_{21} and there after the dominance is reversed. And the Predator (S_2) dominates over S_4 till the time instant t^*_{42} and the dominance is gets reversed there after. Similarly S_4 dominates the Prey (S_1) till the time instant t^*_{14} and there after the dominance is reversed.

4.6 Stability of the Equilibrium State 11

$$\overline{N_1} = \frac{a_1 a_{22} - a_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_2} = \frac{a_1 a_{21} + a_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}}, \overline{N_3} = 0, \overline{N_4} = 0$$

Substituting (4.1.1) in (2.1), (2.2), (2.3), (2.4) and neglecting products and higher powers of u_1, u_2, u_3, u_4, we get

$$\frac{du_1}{dt} = -a_{11}\overline{N}_1 u_1 - a_{12}\overline{N}_1 u_2 + a_{13}\overline{N}_1 u_3 \qquad \text{----- (4.6.1)}$$

$$\frac{du_2}{dt} = a_{21}\overline{N}_2 u_1 - a_{22}\overline{N}_2 u_2 \qquad \text{----- (4.6.2)}$$

$$\frac{du_3}{dt} = a_3 u_3 \qquad \text{------ (4.6.3)}$$

$$\frac{du_4}{dt} = a_4 u_4 \qquad \text{----- (4.6.4)}$$

The characteristic equation of which is

$$\left[\lambda^2 + (a_{11}\overline{N}_1 + a_{22}\overline{N}_2)\lambda + a_{12}a_{21}\overline{N}_1\overline{N}_2\right](\lambda - a_3)(\lambda - a_4) = 0 \qquad \text{----- (4.6.5)}$$

The characteristic roots of (4.6.5) are

$$\lambda = \frac{-(a_{11}\overline{N}_1 + a_{22}\overline{N}_2) \pm \sqrt{(a_{11}\overline{N}_1 + a_{22}\overline{N}_2)^2 - 4a_{12}a_{21}\overline{N}_1\overline{N}_2}}{2}, \ \lambda = a_3, \lambda = a_4 \qquad \text{--- (4.6.6)}$$

Two roots of the equation (4.6.5) are positive and the other two roots are negative. Hence the equilibrium state is unstable.
The trajectories are given by

$$u_1 = \left[\frac{a_{12}\bar{N}_1(u_{10}+u_{20}) - a_{13}\bar{N}_1 u_{30} - \phi_1(\lambda_2 - a_3)}{\lambda_2 - \lambda_1}\right] e^{\lambda_1 t} + \left[\frac{(u_{10}-\phi_1)(\lambda_2-\lambda_1) - a_{12}\bar{N}_1(u_{10}+u_{20}) + a_{13}\bar{N}_1 u_{30} + \phi_1(\lambda_2 - a_3)}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t} + \phi_1 e^{a_3 t} \qquad (4.6.7)$$

$$u_2 = \left[\frac{a_{12}\bar{N}_1(u_{10}+u_{20}) - a_{13}\bar{N}_1 u_{30} - \phi_1(\lambda_2 - a_3)}{\lambda_2 - \lambda_1}\right] \xi_1 e^{\lambda_1 t} + \left[\frac{(u_{10}-\phi_1)(\lambda_2-\lambda_1) - a_{12}\bar{N}_1(u_{10}+u_{20}) + a_{13}\bar{N}_1 u_{30} + \phi_1(\lambda_2 - a_3)}{\lambda_2 - \lambda_1}\right] \xi_2 e^{\lambda_2 t} + \phi_2 e^{a_3 t} \qquad \ldots (4.6.8)$$

$$u_3 = u_{30} e^{a_3 t} \quad \text{---- } (4.6.9) \qquad u_4 = u_{40} e^{a_4 t} \quad \text{--- } (4.6.10)$$

Here

$$\phi_1 = \frac{\beta_2}{a_3^{\,2} + \psi_1 a_3 + \beta_1}, \quad \phi_2 = \frac{a_{13}\bar{N}_1 u_{30} - \phi_1(a_3 + P_3)}{a_{12}\bar{N}_1}, \quad P_3 = a_{11}\bar{N}_1$$

$$\beta_1 = (a_{11}a_{22} + a_{12}a_{21})\bar{N}_1\bar{N}_2, \quad \beta_2 = u_{30}a_{13}\bar{N}_1\left(a_3 + a_{22}\bar{N}_2\right)$$

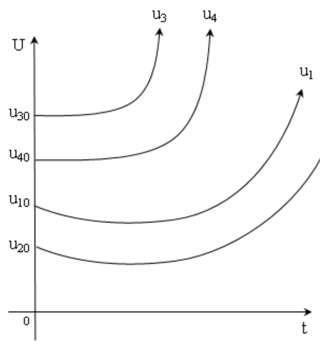
$$\psi_1 = a_{11}\bar{N}_1 + a_{22}\bar{N}_2, \quad \xi_1 = \frac{-(\lambda_1 + P_3)}{a_{12}\bar{N}_1}, \quad \xi_2 = \frac{-(\lambda_2 + P_3)}{a_{12}\bar{N}_1}$$

The solutions are illustrated in figures 15 & 16.

Case (i) : If $u_{20} < u_{10} < u_{40} < u_{30}$ and $a_2 < a_1 < a_4 < a_3$

In this case initially the host (S_3) of S_1 dominates over S_4, the prey (S_1) and the predator (S_2) in natural growth rate as well as in its initial population strength. It is evident that all the four species going away from the equilibrium point. Hence the equilibrium state is unstable as shown in figure.

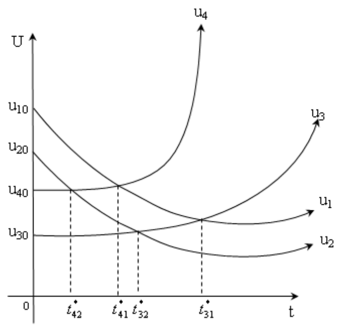
Fig. 15



Case (ii): If $u_{30} < u_{40} < u_{20} < u_{10}$, $a_3 < a_2 < a_1 < a_4$

In this case initially the prey (S_1) dominates over S_4 and the host (S_3) of S_1 till the time instant t_{41}^*, t_{31}^* respectively and there after the dominance is reversed. Also the predator (S_2) dominates over S_4 and the host (S_3) of S_1 till the time instant t_{42}^*, t_{32}^* respectively and there after the dominance is reversed.

Fig. 16



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