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A fixed point theorem on continuous self maps

Rajesh Shrivastava*, Kiran Rathore* and K. Qureshi**

**Department of Mathematics, Govt. Science & Commerce College, Benazir, Bhopal, (MP)*
***Higher Education Department, Govt. of M. P., Bhopal, (MP)*

ABSTRACT

The aim of this paper is to prove a unique common fixed point theorem which generalizes the result of Aage C.T. The conditions continuity and completeness of a metric space are replaced by weaker conditions such as compatible pair of reciprocally continuous self maps.

Keywords: continuous maps, fixed point, self map

INTRODUCTION

According to G. Jungck [4] two self maps P and T of a Metric space compatible mappings if

$\lim_{n\to\infty} d(PTx_n, TPx_n) = 0$ whenever $< x_n >$ is $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Tx_n = t$

for some t belongs to X.

Metric Space (X, d) are said to be a sequence in X such that

Two self maps P and T of a Metric Space (X, d) are said to be Reciprocally continuous if

$\lim_{n\to\infty} PTx_n = Pt$ and $\lim_{n\to\infty} TPx_n = Tt$

whenever $< x >$ is a sequence in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Tx_n = t$

for some t belongs to X.

Definition : A real valued function φ defined on X, R is said to be upper semi continuous if

$\lim_{n\to\infty} \emptyset(t_n) \le \emptyset(t)$, for every sequence $\{t_n\}$, X with $t_n \to t$ as $n \to \infty$.

Aage C.T. and Salunke J.N. [1] proved the following theorem:-

Theorem A: Suppose P,R, T and Q are four self mappings of a complete metric space (X, d) into itself satisfying the conditions

(i) $P(X) \in Q(X), T(X) \in R(X)$

(ii) $d^2\,(Px\,, Ty)\max\Big\{\emptyset\big(d(Rx\,, Qx)\big)\emptyset d(Rx\,, Px)\Big), \emptyset\big(d(Rx, Qy)\big)\emptyset\big(d(Qy, Ty)\big),$

$\emptyset(d(Rx,Px))\emptyset(d(Qy,Ty))\ \varphi(d(Rx,Ty))\varphi(d(Qy,Px))\}$

for all $x, y\ \ X$.

(iii) φ is contractive modulus as in definition (1.2).

(iv) one of P, R, T and Q is continuous.

And if

(v) the pairs (P, R) and (T, Q) are compatible of type

(A)Then P, R, T and Q have a unique common fixed point.

Now we prove the following theorem.

Theorem B. Suppose P, R, T and Q are four self mappings of a metric space (X, d) into itself satisfying the conditions
(i) $P(X)\ \ Q(X), T(X)\ \ R(X)$.
(ii) $d^2(Px,Ty) \le \max\{\varphi(d(Rx,Qy))\varphi(d(Rx,Px)), \varphi(d(Rx,Qy))\varphi(d(Qy,Ty)),$

$\varphi(d(Rx,Px))\varphi(d(Qy,Ty)), \varphi(d(Rx,Ty))\varphi(d(Qy,Px))\},$

for all $x, y \in X$.

Proof: Let x_0 in X be arbitrary. Choose a point x_1 in X such that $Px_0 = Qx_0$.

This can be done

since $P(X) \in Q(X)$. Let x_2 be a point in X such that $Tx_1 = Rx_1$. This can be done since $T(X)\ \ R(X)$.In general we can choose $x_{2n}, x_{2n+1}, x_{2n+2},\cdots$ such that $Px_{2n} = Qx_{2n+1}$and $Tx_{2n+1} = Rx_{2n+2}$,so that we obtain a sequence

$Px_0, Tx_1, Px_2, Tx_3,\cdots.(1)$

Taking condition (i). (ii) and (iii) as in Aage and Salunke [1] $\{Px_{2n}\}$ is a Cauchy sequence and consequently the sequence (1) is a Cauchy. The sequence (1) converges to a limit z in X. Hence the subsequences $\{Px_{2n}\} = \{Qx_{2n+1}\}$ and $\{Tx_{2n-1}\} = \{Rx_{2n}\}$ also converge to the limit point z.

Suppose that the pair (P, R) is compatible pair of reciprocally continuous.By the definition of reciprocally continuous, there is a sequence $< xn >$ in X such that

$Px_{2n} \to z, Rx_{2n} \to z$ then $PRx_{2n} \to Pz, RPx_{2n} \to Rz$ as $n \to \infty$. ... (2)

Since the pair (P, R) is compatible we have $Px_{2n} \to z, Rx_{2n} \to z\ as\ n \to \infty$ and

$\lim_{n\to\infty} d(PRx_{2n}., RPx_{2n}) = 0$....(3)

using (2) and (3) we get

$d(Pz, Rz) = 0\ or\ Pz = Rz$. Since $Pz = Rz$,.

Now by (ii)

$d^2(Pz, Tx_{2n+1}) \le max\{\varphi(d(Rz, Qx_{2n+1}))\varphi(d(Rz, Pz), \varphi(d(Rz, Qx_{2n+1}))$
$\varphi(d(Qx_{2n+1}, Tx_{2n+1})), \varphi(d(Rz, Pz))\varphi(d(Qx_{2n+1}, Tx_{2n+1})),$
$\varphi(d(Rz, Tx_{2n+1}))\varphi(d(Qx_{2n+1}, Pz))\}.$

$Qx_{2n+1} \to z, Tx_{2n+1} \to z$ as $n \to \infty$ and $Rz = Pz$, so letting $n \to \infty$ we get

$d^2(Pz,z) \leq max\{\varphi(d(Pz,z))\varphi(d(Pz,Pz)),$

$\varphi(d(Pz,z))\varphi(d(z,z)),\varphi(d(Pz,Pz))\varphi(d(z,z)),\varphi(d(Pz,z))\varphi(d(z,Pz))\}$

$= \varphi(d(Pz,z))\varphi(d(z,Pz))$

i.e. $d(Pz,z) \leq \varphi(d(Pz,z)) \leq d(Pz,z)$. Hence $\varphi(d(Pz,z)) = 0$ i.e. $Pz = z$ Thus $Pz = Rz = z$

Further

Since $P(X)$ $Q(X)$, there is a point w X such that

$z\,Pz = Qw$.

Now we prove that $Qw = Tw$. Now by (ii)

$d^2(Pz,w) \leq max\{\varphi(d(Rz,Qw))\varphi(d(Rz,Pz)),\varphi(d(Rz,Qw))\varphi(d(Qw,Tw)),$
$\varphi(d(Rz,Pz))\varphi(d(Qw,Tw)),\varphi(d(Rz,Tw))\varphi(d(Qw,Pz))\}$

$= max\{\varphi(d(Qw,Qw))\varphi(d(Qw,Qw)),\varphi(d(Qw,Qw))\varphi(d(Qw,Tw)),$
$\varphi(d(Qw,Qw))\varphi(d(Qw,Tw)),\varphi(d(Qw,Tw))\varphi(d(Qw,Qw))\}$

so $d^2(Qw,Tw) \leq 0$ implies $d(Qw,Tw) = 0$, hence $z = Qw = Tw$.

Since the pair (T,Q) is compatible pair of reciprocally continuous.By the definition of reciprocally continuous, there is a sequence $< x_n >$ in X such that

$Tx_{2n} \rightarrow z, Qx_{2n} \rightarrow z$ then $TQx_{2n} \rightarrow Tz, QTx_{2n} \rightarrow Qz$ as $n \rightarrow \infty$….(4)

Since the pair (T,J) is compatible we have $Tx_{2n} \rightarrow z, Qx_{2n} \rightarrow z$ as $n \rightarrow \infty$ and.

$\lim_{n\to\infty} d(TQx_{2n}., QTx_{2n}) = 0$….(5)

using (4) and (5) we get

$d(Tz,Qz) = 0$, hence $Tz = Qz$. Now $d^2(z,Tz) = d^2Pz,Tz)$

$\leq max\{\varphi(d(Rz,Qz))\varphi(d(Rz,Pz)),\varphi(d(Rz,Qz))\varphi(d(Qz,Tz)),$
$\leq \varphi(d(Rz,Pz))\varphi(d(Qz,Tz)),\varphi(d(Rz,Tz))\varphi(d(Qz,Pz))\}$

$= max\{\varphi(d(z,Tz))\varphi(d(z,z)),\varphi(d(z,Tz))\varphi(d(Tz,Tz)), \varphi(d(z,z))\varphi(d(Tz,Tz)),\varphi(d(z,Tz))\varphi(d(Tz,z))\}$
$= \varphi(d(Tz,z))\varphi(d(Tz,z))$

implies that $d(Tz,z) \leq \varphi(d(Tz,z)) \leq d(Tz,z)$.

Hence $\varphi\ (d(Tz,z)) = 0\ i.e. Tz = z$ and $z = Tz = Qz$. So z is a common fixed point of P, R, Q and T.

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